MATH 340. Explicit Dictionaries vs. Revised Simplex Method. Richard Anstee

We display the dictionary method on the left and the corresponding revised simplex method on the right. We use B^{-1} in the revised simplex method below for convenience. In actual computation one does not explcitly compute B^{-1}. There are several possible questions about updates that can be asked. One could ask to give the new basic feasible solution after the pivot. And one could ask for the eta matrix E that updates the old B to the new B. It is rarely asked to provide the update to B^{-1} itself since it is not used in practice. The update rule is given but you can ignore if you wish. The actual revised simplex does not directly compute B^{-1} since that is numerically more unstable and so one uses a different approach.

Dictionary Format

$$\begin{array}{lrrrcr}
\max & 5x_1 & +x_2 & -2x_3 & & \\
 & -4x_1 & +x_2 & +2x_3 & \leq & 3 \\
 & -x_1 & -3x_2 & +2x_3 & \leq & 3 \\
 & x_1 & +x_2 & -x_3 & \leq & -1
\end{array}$$

Phase One:

$$\begin{array}{rcrrrrr}
x_4 & = & 3 & +4x_1 & -x_2 & -2x_3 & +x_0 \\
x_5 & = & 3 & +x_1 & +3x_2 & -2x_3 & +x_0 \\
x_6 & = & -1 & -x_1 & -x_2 & +x_3 & +x_0 \\
w & = & & & & & -x_0
\end{array}$$

Revised Simplex Method

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_0 \end{array} \\
\begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} -4 & 1 & 2 & 1 & 0 & 0 & -1 \\ -1 & -3 & 2 & 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix}
\end{array} \qquad \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} \end{array}$$

$$\text{Phase One has } w = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_0 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$$\text{basis } \{x_4, x_5, x_6\},\ B^{-1} = \begin{array}{c|c} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\text{special pivot: } B^{-1}\mathbf{b} = \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} \qquad B^{-1}A_0 = A_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} x_0 \geq \mathbf{0}. \text{ So } x_0 \text{ enters } x_6 \text{ leaves}$$

new basis $\{x_4, x_5, x_0\}$

new basic feasible solution $(1, 0, 0, 0, 4, 4, 0)^T$

$$\text{eta matrix update } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Non-standard Pivot to Feasibility: x_0 enters and x_6 leaves

$$\begin{array}{rcrrrrr} x_4 & = & 4 & +5x_1 & & -3x_3 & +x_6 \\ x_5 & = & 4 & +2x_1 & +4x_2 & -3x_3 & +x_6 \\ x_0 & = & 1 & +x_1 & +x_2 & -x_3 & +x_6 \\ w & = & -1 & -x_1 & -x_2 & +x_3 & -x_6 \end{array}$$

$$\text{new } B^{-1} = \begin{array}{c} \\ x_4 \\ x_5 \\ x_0 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \end{array} \text{ from old } B^{-1} \text{ by pivot } \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Now } w \text{ yields } c = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_0 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$$c_N^T - c_B^T B^{-1} A_N =$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_0 \end{array} \\ \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_0 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} -4 & 1 & 2 & 0 \\ -1 & -3 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} -1 & -1 & 1 & -1 \end{pmatrix} \end{array} \quad \text{so } x_3 \text{ enters}$$

$$B^{-1}A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \quad B^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} x_4 \\ x_5 \\ x_0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} x_3 \geq \mathbf{0}. \text{ So } x_0 \text{ leaves}$$

new basis $\{x_4, x_5, x_3\}$

new basic feasible solution $(0,0,0,1,1,1,0)^T$ or $(0,0,1,1,1,0)^T$ ignoring x_0

$$\text{eta matrix update } \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

x_3 enters and x_0 leaves

$$\begin{array}{rcrrrrr} x_4 & = & 1 & +2x_1 & -3x_2 & +3x_0 & -2x_6 \\ x_5 & = & 1 & -x_1 & +x_2 & +3x_0 & -2x_6 \\ x_3 & = & 1 & +x_1 & +x_2 & -x_0 & +x_6 \\ w & = & & & & -x_0 & \end{array}$$

$$\text{new } B^{-1} = \begin{array}{c} \\ x_4 \\ x_5 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \end{array} \text{ from old } B^{-1} \text{ by pivot } \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

x_0 removed from basis End of Phase One

End of Phase One

Phase Two Begins

$z = 5x_1 + x_2 - 2x_3 = -2 + 3x_1 - x_2 - 2x_6.$

$$\begin{array}{ccccccc} x_4 & = & 1 & +2x_1 & -3x_2 & -2x_6 \\ x_5 & = & 1 & -x_1 & +x_2 & -2x_6 \\ x_3 & = & 1 & +x_1 & +x_2 & +x_6 \\ z & = & -2 & +3x_1 & -x_2 & -2x_6 \end{array}$$

$$(c \quad 0) = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 5 & 1 & -2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} 5 & 1 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_3 \end{array} \\ \begin{pmatrix} 0 & 0 & -2 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} -4 & 1 & 0 \\ -1 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} 5 & 1 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -2 \end{pmatrix} \end{array} \quad \text{so } x_1 \text{ enters}$$

$$B^{-1}A_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad B^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} x_4 \\ x_5 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} x_1 \geq \mathbf{0}. \quad \text{So } x_5 \text{ leaves}$$

new basis $\{x_4, x_1, x_3\}$

new basic feasible solution $(1, 0, 2, 3, 0, 0)^T$

eta matrix update $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

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x_1 enters and x_5 leaves

$$\begin{array}{ccccccc} x_4 & = & 3 & -2x_5 & -x_2 & -6x_6 \\ x_1 & = & 1 & -x_5 & +x_2 & -2x_6 \\ x_3 & = & 2 & -x_5 & +2x_2 & -x_6 \\ z & = & 1 & -3x_5 & +2x_2 & -8x_6 \end{array}$$

$$\text{new } B^{-1} = \begin{array}{c} \\ x_4 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \text{ from old } B^{-1} \text{ by pivot } \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{new basis } \{x_4, x_1, x_3\}$$

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_4 & x_1 & x_3 \end{array} \\ \begin{pmatrix} 0 & 5 & -2 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -1 & 3 & 8 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 2 & -3 & -8 \end{pmatrix} \end{array} \quad \text{so } x_2 \text{ enters}$$

$$B^{-1}A_2 = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \ B^{-1}\mathbf{b} = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} x_4 \\ x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} x_2 \geq \mathbf{0}. \quad \text{So } x_4 \text{ leaves}$$

new basis $\{x_2, x_1, x_3\}$

new basic feasible solution $(4, 3, 8, 0, 0, 0)^T$

$$\text{eta matrix update} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

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x_2 enters and x_4 leaves

$$\begin{array}{rclccc} x_2 & = & 3 & -2x_5 & -x_4 & -6x_6 \\ x_1 & = & 4 & -3x_5 & -x_4 & -8x_6 \\ x_3 & = & 8 & -5x_5 & -2x_4 & -13x_6 \\ z & = & 7 & -7x_5 & -2x_4 & -20x_6 \end{array}$$

Optimal solution: $(4, 3, 8, 0, 0, 0)$ with $z = 7$.

$$\text{new } B^{-1} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_2 \\ x_1 \\ x_3 \end{array} \begin{pmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 2 & 5 & 13 \end{pmatrix} \end{array} \text{ from old } B^{-1} \text{ by pivot } \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{new basis } \{x_2, x_1, x_3\}$$

$$c_N^T - c_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_2 & x_1 & x_3 \end{array} \\ \begin{pmatrix} 1 & 5 & -2 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_2 \\ x_1 \\ x_3 \end{array} \begin{pmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 2 & 5 & 13 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 2 & 7 & 20 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -2 & -7 & -20 \end{pmatrix} \end{array}$$

So we are at optimality with:

$$\begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_2 \\ x_1 \\ x_3 \end{array} \begin{pmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 2 & 5 & 13 \end{pmatrix} \end{array} \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} \end{array} = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix}, \quad z = c_B^T B^{-1}\mathbf{b} = \begin{pmatrix} 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} = 7.$$