Forbidden Configurations

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Joint work with Connor Meehan, Miguel Raggi, Attila Sali
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I have worked with a number of coauthors in this area: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Connor Meehan, U.S.R. Murty, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of *VC-dimension* uses a forbidden configuration.

Survey at www.math.ubc.ca/∼anstee

Consider the following family of subsets of $\{1,2,3,4\}$:

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

The incidence matrix A of the family $\mathcal{A}$ of subsets of $\{1,2,3,4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{A}|$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

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We consider the property of forbidding a configuration F in A.

Definition Let

forb$(m, F)= \max_A\{\|A\| : A$ m-rowed simple, no configuration $F\}$

Some Main Results

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

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When a matrix A has a copy of K_k on some k-set of rows S, then we say that A shatters S. The results of Vapnik and Chervonenkis were for application in Applied Probability, in *Learning Theory*. One defines A to have VC-dimension k if k is the maximum cardinality of a shattered set in A. There are applications, e.g. a Computational Geometry result of Matoušek for which the geometric construction is verified to have VC-dimension at most 6.

One well used result about VC-dimension involves the following. Let $S \subset [m]$ be a transversal of A if each column of A has at least one 1 in a row of S. Seeking a minimum transversal, we let $\mathbf{x}$ be the (0,1)-incidence vector of S, and compute:

$$\tau = \min\left\{\mathbf{1}\cdot\mathbf{x} \text{ subject to } A^T\mathbf{x} \geq \mathbf{1}, \quad \mathbf{x} \in \{0,1\}^m\right\}.$$

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Theorem (Haussler and Welzl 87) If A has VC-dimension k then
$\tau \leq 16 k \tau^* \log(k\tau^*)$.

Let $sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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So $|sh(A)| = 7 \geq 6 = \|A\|$

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Theorem (Pajor 85) Let A be simple. Then $|sh(A)| \geq \|A\|$.

Proof: Decompose A as follows:

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$|sh(A)| \geq |sh(A_0)| + |sh(A_1)|$.
Hence $|sh(A)| \geq \|A\|$.

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

Proof: Let A be m-rowed and have no K_k.
Then A shatters no k-set.

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Then A shatters no k-set.
Then $sh(A)$ can only contain sets of size $k-1$ or smaller.
Then

$$\|A\| \le |sh(A)| \le \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Main Bounds

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$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

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Problem Given F, can we predict the behaviour of $forb(m, F)$?

Quadratic Bounds

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What F have the property that $forb(m, F)$ is $\Theta(m^2)$? The minimal F which yield quadratic bounds are the following:

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \text{ or } \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \text{ or } \begin{bmatrix}0 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 1 & 0 & 0 & 1\end{bmatrix} \text{ or }$$

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$$\text{or } \begin{bmatrix}0 & 1\\0 & 1\\1 & 0\\1 & 0\end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \notin I \times I^c$$

Product Construction: Building Blocks

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A Product Construction

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \times \underset{I_3^c}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{I_3} \times \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}_{I_3^c} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\0&1&1&0&1&1&0&1&1\\1&0&1&1&0&1&1&0&1\\1&1&0&1&1&0&1&1&0\end{bmatrix}$$

e.g. $F = \begin{bmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\end{bmatrix} \notin I \times I^c$.

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$I_{m/2} \times I^c_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding F,
so $forb(m, F)$ is $\Omega(m^2)$.

(A, Ferguson, Sali 01 $forb(m, F) = \lfloor \frac{m^2}{4} \rfloor + \binom{m}{1} + \binom{m}{0}$)

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$I_{m/2} \times I_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding F,
so $forb(m, F)$ is $\Omega(m^2)$.
$(forb(m, F) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0})$

The result of Füredi ensures all 2-rowed F have $\mathit{forb}(m, F)$ being $O(m^2)$.

A and Sali 05 conjecture that, for any F, we can determine $\mathit{forb}(m, F)$ asymptotically by using the asymptotically best product construction whose product terms are I,I^c or T.

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \notin I \times T$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{I_3} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}_{T_3} = \left[\begin{array}{ccc|ccc|ccc}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\\hline 1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{array}\right]$$

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(Frankl, Furedi, Pach 87 $forb(m, F) = \binom{m}{2} + 2m - 1$)

A 3-rowed F with a quadratic bound

Definition Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of t copies of M placed side by side.

$$\text{Let } F_3(t) = \begin{bmatrix} 0 & & \\ 0 & t \cdot & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ 0 & & \end{bmatrix}.$$

Theorem (A, Sali 05) *forb*$(m, F_3(t))$ *is* $\Theta(m^2)$.

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The proof uses the standard directed graph decomposition and clever arguments.
The proof for the general k-rowed generalization of $F_3(t)$ uses Linear Algebra (A and Fleming 11).

A 4-rowed F with a quadratic bound

Using a result of A and Fleming 10, there are three simple column-maximal 4-rowed F for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

A 4-rowed F with a quadratic bound

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How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed t, let

$$F_8(t) = \left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

$$F_8(t) = \begin{bmatrix} 1 & 0 & 1 & 0 & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \\ 0 & 1 & 0 & 1 & t \cdot & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & & \end{bmatrix}$$

Theorem (A, Raggi, Sali 12) *Let t be given. Then forb$(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column α not already present t times in $F_8(t)$, then forb$(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.*

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

A 5-rowed Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices F to be boundary cases, namely $forb(m, F)$ is predicted to be $\Theta(m^2)$ and for any column α we have $forb(m, [F|\alpha])$ being $\Omega(m^3)$. We have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem (A, Raggi, Sali 11) *$forb(m, F_7)$ is $\Theta(m^2)$. Moreover F_7 is a boundary case, namely for any column α, then $forb(m, [F_7|\alpha])$ is $\Omega(m^3)$.*

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The Conjecture predicts nine 5-rowed simple matrices F to be boundary cases, namely $\textit{forb}(m, F)$ is predicted to be $\Theta(m^2)$ and for any column α we have $\textit{forb}(m, [F|\alpha])$ being $\Omega(m^3)$. We have handled the following case.

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Theorem (A, Raggi, Sali 11) *$\textit{forb}(m, F_7)$ is $\Theta(m^2)$. Moreover F_7 is a boundary case, namely for any column α, then $\textit{forb}(m, [F_7|\alpha])$ is $\Omega(m^3)$.*

The proof is currently a rather complicated induction.

All 6-rowed Configurations with Quadratic Bounds

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem (A,Raggi,Sali 11) *$forb(m, G_{6\times 3})$ is $\Theta(m^2)$. Moreover $G_{6\times 3}$ is a boundary case, namely for any column α, then $forb(m, [G_{6\times 3}|\alpha])$ is $\Omega(m^3)$. In fact if F is not a configuration in $G_{6\times 3}$, then $forb(m, F)$ is $\Omega(m^3)$.*

Proof: We use induction and the bound for F_7.

Covering Arrays (in transposed notation)

Let

$$\binom{[m]}{k} \text{ denote all } k\text{-sets of an } m\text{-set:} \quad [m] = \{1, 2, \ldots m\}$$

A covering array of strength k on m elements is an m-rowed matrix A so that

$$\binom{[m]}{k} \subseteq sh(A).$$

Alternatively

$$\text{for every } S \subseteq \binom{[m]}{k}, \quad A|_S \text{ has } K_k.$$

The typical question would be to minimize the number of columns required (given m rows) in order to have a covering array of strength k.

CAN(m, k) would be the notation for this minimum which is usually given as CAN(k, t) with different choice of variables.

We can generalize slightly:

$$req(m,F) = \min_A\Big\{\|A\| \,:\, A \text{ is } m\text{-rowed and simple;}$$
$$\text{for all } S \in \binom{[m]}{k},\ A|_S \text{ has } F\Big\}.$$

So for covering arrays in general we are interested in $\mathrm{CAN}(m,k) = req(m, K_k)$.

Theorem (Kleitman and Spencer 73) Let k be given. Then $req(m, K_k) = \Theta(\log m)$.

We can generalize slightly:

$$req(m,F) = \min_A \Big\{ \|A\| \;:\; A \text{ is } m\text{-rowed and simple;} \quad \text{for all } S \in \tbinom{[m]}{k},\; A|_S \text{ has } F \Big\}.$$

So for covering arrays in general we are interested in $\mathrm{CAN}(m,k) = req(m, K_k)$.

Theorem (Kleitman and Spencer 73) Let k be given. Then $req(m,K_k) = \Theta(\log m)$.

i.e. for t fixed, $CAN(k,t)$ is $\Theta(\log k)$

Lucia Moura and Brett Stevens have done substantial investigations of Covering Arrays. Obtaining exact or near exact bounds, rather than asymptotic bounds, is one goal. Fundamental Extremal Set theory results such as Erdős-Ko-Rado are helpful. Applications to fault testing exist. There are variants where we choose a different family of subsets than than $\binom{[m]}{k}$ to be contained in $sh(A)$. Also one might consider forbidding certain things on some subsets of rows.

Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the $(k+\ell)\times 1$ column of k 1's on top of ℓ 0's.

An application for forbidden configurations is the following.

$$\text{Let} \quad F = \mathbf{1}_1\mathbf{0}_1 \times K_{k-2} = \begin{bmatrix} 1\,1\cdots 1 \\ 0\,0\cdots 0 \\ K_{k-2} \end{bmatrix}.$$

Let A be a k-rowed simple matrix with no configuration F. Consider $K_k\backslash A$, namely what we must delete in order to avoid F. Then for every 2-set of rows $S\subseteq\binom{[k]}{2}$, I_2 is a configuration in the set of rows S of $K_k\backslash A$. Thus

$forb(k,\mathbf{1}_1\mathbf{0}_1\times K_{k-2}) = 2^k - req(k,I_2)$.

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$forb(k, \mathbf{1}_1\mathbf{0}_1 \times K_{k-2}) = 2^k - req(k, I_2)$.

Lemma (A and Meehan 11) Let k, p, q be given with $p+q \leq k$. Then $forb(k, \mathbf{1}_p\mathbf{0}_q \times K_{k-(p+q)}) = 2^k - req(k, K^p_{p+q})$.

A and Meehan use this lemma to establish a base case for an induction argument that establishes $forb(m, \mathbf{1}_p\mathbf{0}_q \times K_{k-(p+q)})$ for all m.

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration F' contained in F such that

$$forb(m, F') = forb(m, F).$$

A critical substructure F' has an associated construction avoiding it that yields a lower bound on $forb(m, F')$.
Some other argument provides the upper bound for $forb(m, F)$.
When F' is a configuration in F'' and F'' is a configuration in F, we deduce that

$$forb(m, F') = forb(m, F'') = forb(m, F).$$

Critical Substructures for K_4 (A, Raggi 11)

$$K_4 = \begin{bmatrix} 1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0 \\ 1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0 \\ 1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&0 \\ 1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2\cdot\mathbf{1}_3$, $2\cdot\mathbf{0}_3$.
Note that $forb(m,\mathbf{1}_4) = forb(m,K_4^3) = forb(m,K_4^2) = forb(m,K_4^1)$
$= forb(m,\mathbf{0}_4) = forb(m,2\cdot\mathbf{1}_3) = forb(m,2\cdot\mathbf{0}_3)$.

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$$K_4 = \left[\begin{array}{|c|cccccccccccccccc}1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0\end{array}\right]$$

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Critical Substructures for K_4 (A, Raggi 11)

$$K_4 = \left[\begin{array}{c|cccc|ccccccccccc} 1 & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{0} & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{1} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{1} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & \color{red}{0} & \color{red}{1} & \color{red}{1} & \color{red}{1} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, $\color{red}{K_4^3}$, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathit{forb}(m, \mathbf{1}_4) = \mathit{forb}(m, K_4^3) = \mathit{forb}(m, K_4^2) = \mathit{forb}(m, K_4^1)$
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$$K_4 = \left[\begin{array}{ccccc|cccccc|ccccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 11) For $m \geq 5$, we have
$\mathit{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \mathit{forb}(m, K_4)$.

We can extend K_4 and yet have the same bound

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 11) For $m \geq 5$, we have
$forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.
We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

We have a conjecture which asserts that among the asymptotically best constructions are those from our products of I, I^c, T. We can determine the asymptotic behaviour of $forb(m, F)$ by searching our product constructions for those which avoid F.
To complete the quadratic bounds there are ten more cases.
Two are similar to $F_8(t)$ and eight are similar to F_7.
We need more general arguments!

THANKS to the organizers, Daniel Panario, Lucia Moura, and Brett Stevens!

Theorem (A, Füredi 86) *Let k, t be given.*

$$forb(m, t \cdot K_k) = forb(m, t \cdot \mathbf{1}_k)$$

$$\leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

with equality if a certain k-design with blocksize $k+1$ exists.

Let $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem (A, Fleming 10) *Let F be a $k \times \ell$ simple matrix such that there is a pair of rows with no configuration E_1 and there is a pair of rows with no configuration E_2 and there is a pair of rows with no configuration E_3. Then forb(m, F) is $O(m^{k-2})$.*

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Note that $F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has no E_1 on rows 1,3, no E_2 on rows 1,2 and no E_3 on rows 2,3. Thus *forb*(m, F_1) is $O(m)$.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming 10) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times \ell$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 1,2.

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$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 2,3.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

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$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 1,3.

Note that F_2 has E_3 on every pair of rows hence *forb*(m, F_2) is $\Theta(m^2)$ (A, Griggs, Sali 97).

In particular, this means $F_2 \notin T \times T$ which is the construction to achieve the bound.

Boundary cases k-rowed F with bounds $\theta(m^{k-1})$

Let B be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices C, D, let $C \backslash D$ denote the matrix obtained from C by deleting any columns of D that are in C (i.e. set difference). Let

$F_B(t) = [K_k | t \cdot [K_k \backslash B]]$.

Theorem (A, Griggs, Sali 97, A, Sali 05,
A, Fleming, Füredi, Sali 05)
Let t, B be given. Then forb$(m, F_B(t))$ is $\Theta(m^{k-1})$.

The difficult problem here was the bound although induction works.

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \,|\, (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \quad (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right]\right]$$

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \,|\, (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{cc} 0 & \\ 0 & \\ 0 & (t+1) \cdot \\ 0 & \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \right]$$

Theorem (A, Sali 05 (for $k = 3$), A, Fleming 11)
Let t be given. Then forb$(m, F_D(t))$ is $\Theta(m^{k-1})$.

The argument used standard results for directed graphs, indicator polynomials and a linear algebra rank argument.

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \,|\, (t+1)\cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \;\; (t+1)\cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right]\right]$$

Theorem (A, Sali 05 (for $k = 3$), A, Fleming 11)
Let t be given. Then forb$(m, F_D(t))$ is $\Theta(m^{k-1})$.

The argument used standard results for directed graphs, indicator polynomials and a linear algebra rank argument.

Theorem *Let k be given and assume F is a k-rowed configuration which is not a configuration in $F_B(t)$ for any choice of B as a $k \times (k+1)$ simple matrix with one column of each column sum and not in $F_D(t)$ or $F_D(t)^c$, for any t. Then forb(m, F) is $\Theta(m^k)$.*