The Quest of the Perfect Square

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Introduction

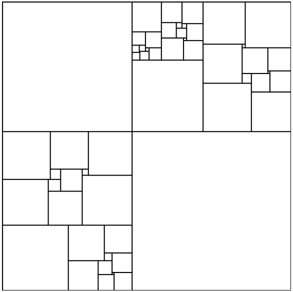
This talk is loosely based on a 1965 AMS paper of W.T. Tutte of the same title. Bill Tutte was first described to me as the ‘King’. Some called him Mr. Graph Theory for his pioneering work. One obituary described Paul Erdős, Claude Berge and Bill Tutte as the three most important figures in Graph Theory in the 20th century.

W.T. Tutte, 1917-2002

He entered Cambridge University in 1935, majoring in Chemistry. He also had an interest in mathematical problems, strong enough to make him join the Trinity Mathematical Society. He formed a close bond with three other members of the Society: Leonard Brooks, Cedric Smith and Arthur Stone. Each was destined to make his mark on Graph Theory. The four of them collaborated on the problem of squaring the square, i.e., partitioning a square into unequal smaller squares, publishing in 1940, 'The Dissection of Rectangles into Squares'.

‘Tutte was a graduate student in Chemistry at Cambridge University in England when, in January 1941, he was asked by his Tutor to go to Bletchley Park, the now legendary organization of code-breakers of Britain. Many have read of the successes which they had there in deciphering the codes produced by the machines called Enigma. In fact, that success was with the naval and air force versions; the army version of Enigma proved to be more resistant to analysis. Since they could not always read army Enigma, they tried to read the machine-cipher named FISH, which was used only by the Army High Command. Tutte’s great contribution was to uncover, from samples of the messages alone, the structure of the machines which generated these FISH ciphers. This led to the decipherment of these codes on a regular basis.’

The problem the undergraduates considered was whether you could dissect a square into smaller squares all of different sizes. The problem came from an 1931 edition of 'The Canterbury Puzzles and other curious problems' by Dudeney. The undergraduates worked for a period 1934-1938 on the problem and came out with a solution in their 1940 paper 'Dissections of Rectangles into Squares', published in the Duke Mathematical Journal. They were scooped by Sprague by 1 year but their paper had a multitude of new results that led to much later work.



Sprague's squared square

I was introduced to some aspects of this problem in Grade 11 at a Math contest lecture! That started me down the path to research in Discrete Mathematics.

some transparencies

Given the graph one might hope to discover the sizes of squares the edge correspond to by having flow in equal flow out at every node except the top and bottom nodes. Tutte and his fellow undergrads used the following ideas, perhaps from their classes. Electricity obeys the equation $V = IR$ where V is the voltage drop and I the current and R the resistance. If we assume the resistance is 1, then the current is the voltage drop. Thus if we think of an edge $i \rightarrow j$ as a wire having resistance 1 with voltage v_i at i and voltage v_j at j, then the current from i to j is $v_i - v_j$.

We let $A = (a_{ij})$ denote the $n \times n$ matrix with

$$a_{ij} = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } i \neq j, \quad i \text{ and } j \text{ are joined} \\ 0 & \text{if } i \neq j, \quad i \text{ and } j \text{ are not joined} \end{cases}$$

We note that A is a symmetric matrix and more importantly each row and column sum is 0 so that, for example, the sum of the columns is the zero vector. (It is sometimes called the *Laplacian*). For our example

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

We imagine a battery attached to our network of unit resistance wires with a potential introduced across the two nodes $1, n$. Let v_i denote the unknown potential at node i. The net flow of electricity into a node i can be computed as

$$\sum_j a_{ij}(v_j - v_i) = \sum_j a_{ij} v_j \quad \text{using} \ \sum_j a_{ij} = 0.$$

Imagining that the first node 1 is the top node or source of the electricity and node n is the bottom node or sink for the electricity, Kirchoff's laws give us

$$\sum_j a_{ij} v_j = \begin{cases} 0 & m \notin \{i,j\} \\ I & j = 1 \\ -I & j = n \end{cases}$$

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This is n equations in n unknown potentials, but one equation is redundant and we can drop the first equation and we can also add a constant to the potentials so that $v_q = 0$ and hence drop the variable v_q. We then have $n-1$ equations in $n-1$ variables. Then Cramer's rule lets us solve for the currents (the square sizes) using

$$v_p - v_q = v_p = \frac{\det(A_{1n} \text{ with } p\text{th column replaced by column } \alpha)}{\det(A_{1q})}$$

$$\text{where } \alpha = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -I \end{bmatrix}$$

Hence $$v_p - v_q = v_p = \frac{(-1)^{f(p,q)} I \det((A_{1q})_{np})}{\det(A_{1q})}$$

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Now in order for Cramer's rule to work you need $\det(A_{1n}) \neq 0$ and then we can set $I = \det(A_{1n})$ in order to end up with square sizes being integers.

Remarkably one can show that

$$(-1)^{i+j} \det(A_{ij}) = (-1)^{r+s} \det(A_{rs})$$

for any choices i, j, r, s which follows from the fact that the row and column sums of A are 0. In addition $(-1)^{ij} \det(A_{ij})$ counts the number of spanning trees in the graph and hence is nonzero.

For our example

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\det(A_{35}) = \det \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

add column 2 to column 1

$$\det(A_{35}) = \det \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ -1 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

add columns 2,3 to column 1

$$\det(A_{35}) = \det \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & -1 & -1 & 0 \\ 2 & 0 & 3 & -1 & 0 & -1 \\ -2 & -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

add columns 2,3,4 to column 1

$$\det(A_{35}) = \det \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 3 & 0 & -1 & -1 & 0 \\ 1 & 0 & 3 & -1 & 0 & -1 \\ 2 & -1 & -1 & 4 & -1 & -1 \\ -2 & -1 & 0 & -1 & 3 & -1 \\ -2 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

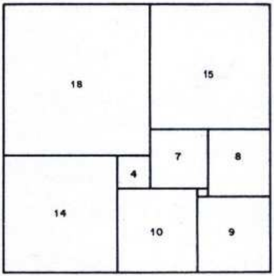
add columns 2,3,4,6 to column 1

$$\det(A_{35}) = \det \left[\begin{array}{cccc|c|c} 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 3 & 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 3 & -1 & 0 & -1 \\ \hline 1 & -1 & -1 & 4 & -1 & -1 \\ -3 & -1 & 0 & -1 & 3 & -1 \\ 1 & 0 & -1 & -1 & -1 & 3 \end{array}\right]$$

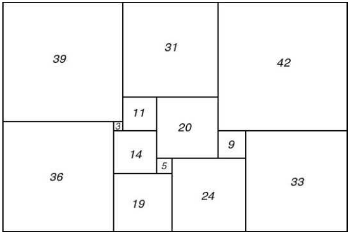
Column 1 is now the negative of column 5 so
$(-1)^{3+5}\det(A_{35}) = (-1)^{3+1}\det(A_{31})$

$$A_{31} = \left[\begin{array}{c|ccccc} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 & 0 \\ \hline -1 & 0 & 3 & -1 & 0 & -1 \\ \hline 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{array}\right]$$

For this example $(-1)^{i+j}\det(A_{ij}) = 66$ and Cramer's rule, with $I = 66$, yields square sizes 36,30,28,20,18,16,14,8,2. We of course divide by the gcd to get the square sizes.

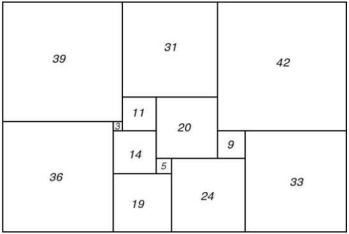


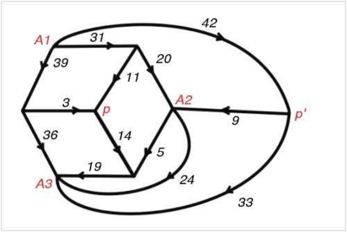
A 13 square example made into a jigsaw puzzle by Brooks.



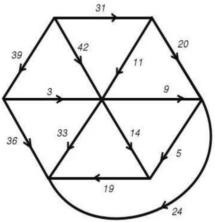
A 13 square example assembled by Brooks' mother from the pieces!

First 13 square example and its electrical network.



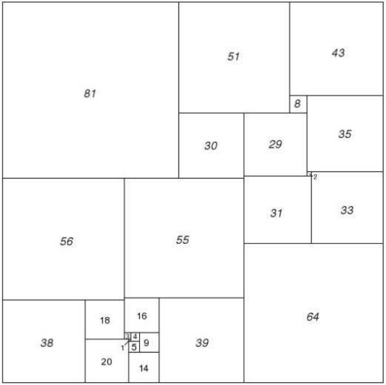


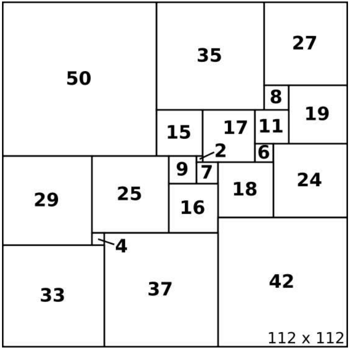
Second 13 square example and its electrical network.



some examples of squares found by the four students

Willcocks Square 1948. For a number of years this was the squared square of the fewest number of squares but was viewed as slightly flawed because in contained a smaller squared rectangle (a compound square).





The perfect square was discovered by Duijvestijn in 1978. This was the result of much effort over the years to catalog all perfect rectangles of up to so many squares (and hence graphs on up to so many edges). Duijvestijn verified that this was the unique perfect square on up to 21 squares. Bouwkamp and Duijvestijn (1994) have actually catalogued all perfect squares of up to 26 squares. It is amazing that they are rare and yet not so rare that you have a hope of finding them.