Forbidden Configurations: A Survey

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Introduction

I have worked with a number of coauthors in this area: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Connor Meehan, U.S.R. Murty, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of *VC-dimension* uses a forbidden configuration.

Survey at www.math.ubc.ca/$\sim$anstee

An Elementary Extremal Problem

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Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is intersecting if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq 1$.

Theorem *If $\mathcal{F} \subseteq 2^{[m]}$ and $\mathcal{F}$ is intersecting, then*

$$|\mathcal{F}| \leq 2^{m-1}.$$

A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G.

Theorem (Erdős, Stone, Simonovits 46, 66) Let G be a simple graph. Then

$$\lim_{m\to\infty} ex(m, G)/\binom{m}{2} = 1 - \frac{1}{\chi(G) - 1}.$$

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The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets results in a graph of many edges that requires k colours. Thus if $\chi(G) = k + 1$, then G is not a subgraph of $T(m, k)$, i.e. $ex(m, G) \geq |E(T(m, k))|$

Let Δ denote the triangle on 3 vertices.

Theorem (Mantel 1907) $ex(m,\Delta) = |T(m,2)| = \lfloor \frac{m^2}{4} \rfloor$

Note $\chi(\Delta) = 3$. If we consider a graph H consisting of two disjoint copies of Δ then $\chi(H) = 3$ and $ex(m,H) > ex(m,\Delta)$. Yet

$$\lim_{m\to\infty} ex(m,H)/\binom{m}{2} = 1 - \frac{1}{3-1} = 1/2.$$

Theorem (Turán 41) Let G denote the clique on k vertices where every pair of vertices are joined. Then
$ex(m,G) = |E(T(m,k-1))|$.

Note $\chi(G) = k$ and

$$\lim_{m\to\infty} |E(T(m,k-1))|/\binom{m}{2} = 1 - \frac{1}{k-1} = \frac{k-2}{k-1}.$$

We consider one possible generalization of the problem:
from graphs to hypergraphs, and subgraphs to subhypergraphs.

Hypergraphs → Simple Matrices

Consider a hypergraph H with vertices $[4] = \{1, 2, 3, 4\}$ and with the following family of subsets as edges :

$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\} \subseteq 2^{[4]}$$

The incidence matrix A of the family $\mathcal{A} \subseteq 2^{[4]}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{A}|$$

Subhypergraphs $\rightarrow$ Configurations

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

$ex(m, G) \rightarrow forb(m, F)$

We consider the property of forbidding a configuration F in A.

Definition Let

$forb(m, F)$= $\max\{\|A\| : A$ m-rowed simple, no configuration $F\}$

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$$\text{e.g. } forb(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$$

Some Main Results

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

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When a matrix A has a copy of K_k on some k-set of rows S, then we say that A shatters S. The results of Vapnik and Chervonenkis were for application in Applied Probability, in *Learning Theory*. One defines A to have VC-dimension k if k is the maximum cardinality of a shattered set in A. There are applications, e.g. a Computational Geometry result of Matoušek for which the geometric construction is verified to have VC-dimension at most 6.

One well used result about VC-dimension involves the following. Let $S \subset [m]$ be a transversal of A if each column of A has at least one 1 in a row of S. Seeking a minimum transversal, we let $\mathbf{x}$ be the (0,1)-incidence vector of S, and compute:

$$\tau = \min\left\{\mathbf{1}\cdot\mathbf{x} \text{ subject to } A^T\mathbf{x} \geq \mathbf{1}, \quad \mathbf{x} \in \{0,1\}^m\right\}.$$

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$$\tau^*=\min\left\{\mathbf{1}\cdot\mathbf{x}\text{ subject to }A^T\mathbf{x}\geq\mathbf{1},\quad \mathbf{x}\geq\mathbf{0}\right\}.$$

Theorem (Haussler and Welzl 87) If A has VC-dimension k then $\tau\leq 16k\tau^*\log(k\tau^*)$.

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

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So $|sh(A)| = 7 \geq 6 = \|A\|$

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Theorem (Pajor 85) $\quad |sh(A)| \geq \|A\|$.

Proof: Decompose A as follows:

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Hence $|sh(A)| \geq \|A\|$.

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

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Then $sh(A)$ can only contain sets of size $k-1$ or smaller.
Then

$$\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \geq |sh(A)| \geq \|A\|.$$

Some Main Results

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Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

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Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then* $forb(m, F) = O(m^k)$.

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Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

Problem Given F, can we predict the behaviour of $forb(m, F)$?

A result for simple F

Definition Let F be a k-rowed configuration and let α be a k-rowed column vector. Define $[F|\alpha]$ to be the concatenation of F and α.

Theorem (A, Fleming 10) *Let F be a $k \times \ell$ simple matrix satisfying . . . various conditions. . .*
Then forb(m, F) is $O(m^{k-2})$ (instead of $\Theta(m^{k-1})$).
Moreover if F satisfies . . . various additional conditions. . . then for any k-rowed column α not in F, we have forb$(m, [F|\alpha])$ is $\Theta(m^{k-1})$.

A result for simple F

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming 10) *Let F be a $k \times \ell$ simple matrix such that there is a pair of rows with no configuration E_1 and there is a pair of rows with no configuration E_2 and there is a pair of rows with no configuration E_3. Then forb(m, F) is $O(m^{k-2})$.*

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Moreover if we select three subsets $I_1, I_2, I_3 \subseteq [k]$ with $|I_1| = |I_2| = |I_3| = 2$ and F is the column maximal k-rowed simple matrix that has no configuration E_1 on rows I_1, no configuration E_2 on rows I_2 and no configuration E_3 on rows I_3, then for any k-rowed column α not in F', we have forb$(m, [F|\alpha])$ is $\Theta(m^{k-1})$.

Example

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad forb(m, F) = 2m$$

A Product Construction

The building blocks of our product constructions are I, I^c and T, e.g:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \; T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Conjecture

We conjecture that product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

Definition Given two matrices A, B, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of A on top of a column of B in all possible ways. (A, Griggs, Sali 97)

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \times \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.

Examples

$$[01] \times [01] = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] = K_2$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2,m/2} = T(m,2)$

We conjecture that product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

Definition Let F be given. Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Conjecture (A, Sali 05) *forb*(m, F) *is* $\Theta(m^{x(F)})$.

The Conjecture $\chi(G) \rightarrow x(F)$?

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 11) plus other cases.

6-rowed Configurations with Quadratic Bounds

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem (A,Raggi,Sali 11) *forb*$(m, G_{6\times 3})$ *is* $\Theta(m^2)$. *Moreover* $G_{6\times 3}$ *is a boundary case, namely for any column* α, *then* *forb*$(m, [G_{6\times 3}|\alpha])$ *is* $\Omega(m^3)$. *In fact if* F *is* 6*-rowed and not a configuration in* $G_{6\times 3}$, *then* *forb*(m, F) *is* $\Omega(m^3)$.

The proof uses induction to reduce to a 5-rowed case which is then established by a quite complicated induction.

Repeated Columns

Definition Let $t \cdot M$ be the matrix $[M \mid M \mid \cdots \mid M]$ consisting of t copies of M placed side by side.

Repeated Columns

Definition Let $t \cdot M$ be the matrix $[M \,|\, M \,|\, \cdots \,|\, M]$ consisting of t copies of M placed side by side.

Theorem (A, Füredi 86) *Let t, k be given.*

$$forb(m, t \cdot K_k) \leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

with equality if a certain k-design exists.

A 4-rowed F with a quadratic bound

Using the result of A and Fleming 10, there are three simple column-maximal 4-rowed F for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

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Can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed t, let

$$F_8(t) = \begin{bmatrix} \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} & t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

A 4-rowed F with a quadratic bound

$$F_8(t) = \left[\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

Theorem (A, Raggi, Sali 09) *Let t be given. Then forb$(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column α not already present t times in $F_8(t)$, then forb$(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.*

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

We have been able to determine an 'easy' criteria for k-rowed F for which $forb(m, F)$ is $O(m^{k-1})$ as opposed to $\Theta(m^k)$.

Theorem (A.,Fleming, Sali, Füredi 05, A., Fleming 11)
There is an . . . easy to describe list . . . of various k-rowed F for which $forb(m, F)$ is $O(m^{k-1})$.

Moreover if a k-rowed F is not a configuration in one of the . . . easy to describe list . . . then $forb(m, F)$ is $\Theta(m^k)$.

We have been able to determine an 'easy' criteria for k-rowed F for which $forb(m, F)$ is $O(m^{k-1})$ as opposed to $\Theta(m^k)$.

Theorem (A.,Fleming, Sali, Füredi 05, A., Fleming 11)
Let $D(k)$ denote the matrix with all columns of sum at least 1 except those columns with 1's on both rows 1 and 2.
Then $forb(m, [\mathbf{0}_k \mid t \cdot D(k)])$ is $O(m^{k-1})$.

Let B be an $k \times (k+1)$ matrix with one column of each column sum. Then $forb(m, [K_k \mid t \cdot [K_k \backslash B]])$ is $O(m^{k-1})$.

Moreover if F is a k-rowed configuration not a configuration in either $[\mathbf{0} \mid t \cdot D(k)]$) or in $[K_k \mid t \cdot [K_k \backslash B]]$, for some $k \times (k+1)$ matrix B with one column of each column sum, then $forb(m, F)$ is $\Theta(m^k)$.

Exact Bounds

Determining exact bounds $forb(m, F)$ can be very challenging requiring much more understanding than is required for asymptotic bounds. The proofs often depend heavily on the structures of $m \times forb(m, F)$ simple matrices that avoid F; these are called extremal matrices. Despite the challenge (and fun!) of obtaining exact bounds, I suspect the asymptotic results are more important.

Exact Bounds

Theorem (A., Kamoosi 07)

$$forb(m, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}) = \left\lfloor \frac{10}{3}m - \frac{4}{3} \right\rfloor.$$

Exact Bounds

Theorem (A., Karp 10)

$$forb(m, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}) = \frac{4}{3}\binom{m}{2} + m + 1$$

for $m \equiv 1, 3 (\text{mod } 6)$.

A simple design theoretic construction (Steiner Triple System) yields the lower bound $forb(m, \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]) \geq \frac{4}{3}\binom{m}{2} + m + 1$ while a pigeonhole argument yields the upper bound $forb(m, \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]) \leq \frac{4}{3}\binom{m}{2} + m + 1$. Extending this upper bound to the 3×3 matrix requires a careful matching argument.

Definition Let K_k^ℓ denote the $k \times \binom{k}{\ell}$ simple matrix of all possible columns of sum ℓ on k rows.

Theorem (A., Barekat) Let q be given and let F be the following $4 \times q$ matrix

$$F = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}}^{q} \\ \begin{matrix} 1 & 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

Then

$$forb(m, F) \leq \binom{m}{0} + \binom{m}{1} + \frac{q+3}{3}\binom{m}{2} + \binom{m}{m-1} + \binom{m}{m}$$

with equality if $m \geq q$ and $m \equiv 1, 3 (\text{mod } 6)$ and

$$A = [K_m^0 K_m^1 K_m^2 T_{m,a} T_{m,b}^c K_m^{m-2} K_m^{m-1} K_m^m]$$

(for some choice a, b with $a + b = q - 3$) .

Exact Bounds

The following result, found with the help of Genetic algorithms, has a bound just 2 larger than the one we initially expected.

Theorem (A., Raggi 11) Assume $m \geq 6$. Then

$$forb(m, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}) = \binom{m}{2} + m + 4.$$

The construction was relatively complicated:

$$A = \begin{matrix} U \\ \\ L \end{matrix} \left[\begin{array}{c|c|c|c|c|c|c|c|c} \mathbf{0}_u & \mathbf{0}_u & \mathbf{0}_u & \mathbf{0}_u & K_u^1 & \mathbf{1}_u & K_u^1 & K_u^2 & \mathbf{1}_u \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \mathbf{0}_\ell & \mathbf{1}_\ell & K_\ell^{\ell-1} & K_\ell^{\ell-2} & K_\ell^{\ell-1} & \mathbf{0}_\ell & \mathbf{1}_\ell & \mathbf{1}_\ell & \mathbf{1}_\ell \end{array} \right].$$

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration F' contained in F such that

$$forb(m, F') = forb(m, F).$$

A critical substructure F' has an associated construction avoiding it that yields a lower bound on $forb(m, F')$.
Some other argument provides the upper bound for $forb(m, F)$.
When F' is a configuration in F'' and F'' is a configuration in F, we deduce that

$$forb(m, F') = forb(m, F'') = forb(m, F).$$

Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the $(k+\ell)\times 1$ column of k 1's on top of ℓ 0's.

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$ $= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

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$$K_4 = \left[\begin{array}{|c|ccccccccccccccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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$$K_4 = \left[\begin{array}{ccccccccccccccc|c|} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

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Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $\color{red}{2 \cdot \mathbf{0}_3}$.
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$= forb(m, \mathbf{0}_4) = forb(m, 2 \cdot \mathbf{1}_3) = forb(m, 2 \cdot \mathbf{0}_3)$.

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan) For $m \geq 5$, we have $forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

We can extend K_4 and yet have the same bound

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Theorem (A., Meehan) For $m \geq 5$, we have
$forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.
We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

$$F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Not all F are likely to yield simple exact bounds:

Theorem *Let c be a positive real number. Let A be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no F_{2110}. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no F_{2110}.*

Theorem (P. Dukes 09) $forb(m, F_{2110}) \leq .691m^2$

The proof used inequalities and linear programming

Open Problems

Determine the asymptotics for $forb(m, F)$ for

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \qquad (\textit{Hard}!)$$

Open Problems

Determine an exact bound for $forb(m, F)$ for

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Show that $2 \cdot \mathbf{1}_4$ (the 4×2 matrix of 1's) and $2 \cdot \mathbf{0}_4$ are the only 4-rowed critical substructures of K_5 by showing that for m large,

$$forb(m, [\mathbf{0}_4 \,|\, 2 \cdot K_4^1 \,|\, 2 \cdot K_4^2 \,|\, 2 \cdot K_4^3 \,|\, \mathbf{1}_4]) < forb(m, K_5).$$

THANKS to the University of Regina for the invitation!

5×6 Simple Configuration with Quadratic bound

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem (A, Raggi, Sali) *forb*(m, F_7) *is* $\Theta(m^2)$. *Moreover* F_7 *is a boundary case, namely for any column* α, *then forb*$(m, [F_7|\alpha])$ *is* $\Omega(m^3)$.

The Conjecture predicts nine 5-rowed simple matrices F to be boundary cases of which this is one.

Induction

Let A be an $m \times forb(m, F_7)$ simple matrix with no configuration F_7. We can select a row r and reorder rows and columns to obtain

$$A = \begin{matrix} \text{row } r \\ \\ \end{matrix} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

Induction

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Now $[B_r C_r D_r]$ is an $(m-1)$-rowed simple matrix with no configuration F_7. Also C_r is an $(m-1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from F_7.

C_r has no F in

$$\mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \right\}$$

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Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

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Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

To show $forb(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of r.

Repeated Induction

Let C_r be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row s_i and reorder rows and columns to obtain

$$C_r = {}^{\text{row } s_i}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & & G_i & G_i & & H_i \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of s_i. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices s_i results in a contradiction.

Repeated Induction

Let C_r be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row s_i and reorder rows and columns to obtain

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To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of s_i. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices s_i results in a contradiction.

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We may choose s_1 and form L_1.
Then choose $s_2 \in L_1$ and form L_2.
Then choose $s_3 \in L_2$ and form L_3.
etc.
We can show the sets $L_1 \backslash s_2, L_2 \backslash s_3, L_3 \backslash s_4, \ldots$ are disjoint.
Assuming $\|G_i\| \geq 8$ for all choices s_i results in $|L_i \backslash s_{i+1}| \geq 3$ which yields a contradiction.