Forbidden Configurations: Critical Substructures

Richard Anstee
UBC, Vancouver

Discrete Math Seminar, Sep. 29, 2009

Introduction

Let F be a kxl (0,1)-matrix. We say that a (0,1)-matrix A has F as a *configuration* if some row and column permutation of F is a submatrix of A. Our extremal problem is given m,F to determine the maximum number of columns $\text{forb}(m, F)$ in an m-rowed (0,1)-matrix A with no repeated columns which has no configuration F.

A *critical substructure* of F is a configuration F' which is contained in F and such that $\text{forb}(m, F') = \text{forb}(m, F)$. We give some examples to demonstrate how this idea often helps in determining $\text{forb}(m, F)$.

This talk is mainly based on joint work with Steven Karp.

Survey at www.math.ubc.ca/∼anstee

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.
i.e. if A is m-rowed then A is the incidence matrix of some $\mathcal{F} \subseteq 2^{[m]}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{F} = \big\{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\big\}$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

Definition Let $\text{forb}(m, F)$ be the largest function of m and F so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration F. Thus if A is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then A contains F as a configuration.

For example, $\quad \mathrm{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

For example, $\text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m - 1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For example, $\quad \text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m - 1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Each pair of rows of A can have at most 2 configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ else it has 2 in the same orientation and the forbidden configuration.

For example, $\text{forb}(m, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) = m + 2$

Ignoring the column of 0's and the column of 1's, each remaining column has at least $m-1$ configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Each pair of rows of A can have at most 2 configurations $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ else it has 2 in the same orientation and the forbidden configuration. Thus if n denotes the number of columns not all 0's or all 1's, then

$$(m-1)n \leq 2\binom{m}{2}$$

from which we deduce $n \leq m$ and hence the bound.

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows (i.e. incidence matrix of $2^{[k]}$).

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0} = \Theta(m^{k-1})$$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows (i.e. incidence matrix of $2^{[k]}$).

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0} = \Theta(m^{k-1})$$

Theorem (Füredi 83). *Let F be a $k \times l$ matrix. Then* $forb(m, F) = O(m^k)$

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with k 1's on top of ℓ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with k 1's on top of ℓ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

Definition Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of t copies of M placed side by side.

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with k 1's on top of ℓ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

Definition Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of t copies of M placed side by side.

Theorem (A, Füredi 86)

$$forb(m, t \cdot K_k) = forb(m, t \cdot \mathbf{1}_k)$$

$$\leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

with equality if a certain k-design exists.

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column with k 1's on top of ℓ 0's. Then let $\mathbf{1}_k = \mathbf{1}_k\mathbf{0}_0$.

Definition Let $t \cdot M$ be the matrix $[M\, M \cdots M]$ consisting of t copies of M placed side by side.

Theorem (A, Füredi 86)

$$forb(m, t \cdot K_k) = forb(m, t \cdot \mathbf{1}_k)$$

$$\leq \frac{t-2}{k+1}\binom{m}{k} + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

with equality if a certain k-design exists.

Definition Let K_k^ℓ denote the $k \times \binom{k}{\ell}$ simple matrix of all possible columns of sum ℓ on k rows.

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration F' contained in F such that

$$\text{forb}(m, F) = \text{forb}(m, F')$$

A critical substructure is what drives the construction yielding a lower bound $\text{forb}(m, F)$ where some other argument provides the upper bound for $\text{forb}(m, F)$.
A consequence is that for a configuration F'' which contains F' and is contained in F, we deduce that

$$\text{forb}(m, F) = \text{forb}(m, F'') = \text{forb}(m, F')$$

Critical Substructures for K_3

$$K_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{|c|ccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since
$\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{c|ccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since
$\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{cccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2\cdot\mathbf{1}_2$, $2\cdot\mathbf{0}_2$ since
$\text{forb}(m,\mathbf{1}_3)=\text{forb}(m,K_3^1)=\text{forb}(m,K_3^2)=\text{forb}(m,\mathbf{0}_3)$
$=\text{forb}(m,2\cdot\mathbf{1}_2)=\text{forb}(m,2\cdot\mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{ccccccc|c|} 1 & 1 & 1 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\color{red}{\mathbf{0}_3}$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $2 \cdot \mathbf{0}_2$ since
$\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Critical Substructures for K_3

$$K_3 = \left[\begin{array}{cccccc|cc|} 1 & 1 & 1 & 0 & 1 & 0 & \color{red}{0} & \color{red}{0} \\ 1 & 1 & 0 & 1 & 0 & 1 & \color{red}{0} & \color{red}{0} \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_3$, K_3^2, K_3^1, $\mathbf{0}_3$, $2 \cdot \mathbf{1}_2$, $\color{red}{2 \cdot \mathbf{0}_2}$ since $\text{forb}(m, \mathbf{1}_3) = \text{forb}(m, K_3^1) = \text{forb}(m, K_3^2) = \text{forb}(m, \mathbf{0}_3)$ $= \text{forb}(m, 2 \cdot \mathbf{1}_2) = \text{forb}(m, 2 \cdot \mathbf{0}_2)$.

Designs and Forbidden Configurations

A 2-design $S_\lambda(2, 3, v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v] = \{1, 2, \ldots, v\}$ such that for each pair $i, j \in \binom{[v]}{2}$, there are exactly λ triples containing i, j. If we encode the triple system as a v-rowed (0,1)-matrix A such that the columns are the incidence vectors of the triples, then A has no $2 \times (\lambda + 1)$ submatrix of 1's.

Designs and Forbidden Configurations

A 2-design $S_\lambda(2,3,v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v]=\{1,2,\ldots,v\}$ such that for each pair $i,j\in\binom{[v]}{2}$, there are exactly λ triples containing i,j. If we encode the triple system as a v-rowed (0,1)-matrix A such that the columns are the incidence vectors of the triples, then A has no $2\times(\lambda+1)$ submatrix of 1's.

Remark If A is a $v\times n$ (0,1)-matrix with column sums 3 and A has no $2\times(\lambda+1)$ submatrix of 1's then $n\le\frac{\lambda}{3}\binom{v}{2}$ with equality if and only if the columns of A correspond to the triples of a 2-design $S_\lambda(2,3,v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-1\}$ and A has no $3 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

and we have equality if and only if the columns of A correspond to the triples of a 2-design $S_{\lambda}(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-1\}$ and A has no $3 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{matrix}} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

and we have equality if and only if the columns of A correspond to the triples of a 2-design $S_\lambda(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and A has no $4 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers a, b with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of A of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of A of column sum $v - 3$ corresponding to $(v - 3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and A has no $4 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{matrix}} \\ \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers a, b with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of A of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of A of column sum $v - 3$ corresponding to $(v - 3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and A has no $4 \times (\lambda + 1)$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers a, b with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of A of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of A of column sum $v - 3$ corresponding to $(v - 3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

Theorem (N. Balachandran 09) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{4, 5, \ldots, v-1\}$ and A has no 4×2 configuration

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$n \leq \frac{1}{4}\binom{v}{3}$$

with equality only if there is 3-design $S_1(3, 4, v)$ corresponding to $(v-3)$ - sets whose complements (in $[v]$) corresponding to the quadruples of a 3-design $S_1(3, 4, v)$.

Naranjan Balachandran has indicated that he has made further progress on this problem

Exact Bounds

A, Barekat 09

Configuration F	Exact Bound forb(m, F)
$\overbrace{\begin{bmatrix} 1\,1\cdots 1 \\ 1\,1\cdots 1 \\ 0\,0\cdots 0 \end{bmatrix}}^{p}$	$\frac{p+1}{3}\binom{m}{2} + \binom{m}{1} + 2\binom{m}{0}$ for m large, $m \equiv 1, 3 (\text{mod} 6)$
$\overbrace{\begin{bmatrix} 1\,1\cdots 1 \\ 1\,1\cdots 1 \\ 0\,0\cdots 0 \\ 0\,0\cdots 0 \end{bmatrix}}^{p}$	$\frac{p+3}{3}\binom{m}{2} + 2\binom{m}{1} + 2\binom{m}{0}$ for m large, $m \equiv 1, 3 (\text{mod} 6)$

Another Example of Critical Substructures

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

Another Example of Critical Substructures

$$F_1 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Theorem (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2) = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

Another Example of Critical Substructures

$$F_1 = \left[\begin{array}{cc|cc|} 1 & 1 & \color{red}{1} & \color{red}{1} \\ 1 & 1 & \color{red}{0} & \color{red}{0} \\ 1 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{0} & \color{red}{0} \end{array}\right]$$

Theorem (A, Karp 09) For $m \geq 3$ we have

$$\text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1) = \color{red}{\text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)} = \binom{m}{2} + m + 2.$$

Thus for

$$F_2 = \left[\begin{array}{cc|cc|} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we deduce that $\text{forb}(m, F_2) = \text{forb}(m, F_1) = \text{forb}(m, 2 \cdot \mathbf{1}_2\mathbf{0}_1)$
$= \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$.

$k \times 2$ Forbidden Configurations

Let $F_{abcd} = \begin{matrix} a\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ b\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ c\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ d\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \end{matrix} \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$

For the purposes of forbidden configurations we may assume that $a \geq d$ and $b \geq c$.

The following result used a difficult 'stability' result and the resulting constants in the bounds were unrealistic but the asymptotics agree with a general conjecture.

Theorem (A-Keevash 06) *Assume a,b,c,d are given with* $a \geq d$ *and* $b \geq c$. *If* $b > c$ *or* $a, b \geq 1$, *then*

$$forb(m, F_{abcd}) = \Theta(m^{a+b-1}).$$

Also $forb(m, F_{0bb0}) = \Theta(m^b)$ *and* $forb(m, F_{a00d}) = \Theta(m^a)$.

Note that the first column of F_{abcd} is $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$.

Theorem (A, Karp 09) Let $a, b \geq 2$. Then

$$\text{forb}(m, F_{ab01}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab10}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m}^{m} \binom{m}{j}$$

$$\text{forb}(m, F_{ab11}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_2) = \sum_{j=0}^{a+b-1} \binom{m}{j} + \sum_{j=m-1}^{m} \binom{m}{j}$$

Problem (A, Karp 09). Let a, b, c, d be given with a, b *much larger* than c, d. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

Problem (A, Karp 09). Let a, b, c, d be given with a, b *much larger* than c, d. Is it true that $\text{forb}(m, F_{abcd}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$?

We are asking when we can make the first column with $a + b$ 1's and $c + d$ 0's dominate the bound.

Exact Bounds

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Theorem (A, Karp 09)

$$\text{forb}(m, F) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \leq \frac{4}{3}\binom{m}{2} + m + 1$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

Exact Bounds

$$F_3 = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

Theorem (A, Karp 09)

$$\text{forb}(m, F) = \text{forb}(m, 3 \cdot \mathbf{1}_2) \leq \frac{4}{3}\binom{m}{2} + m + 1$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

Pseudo-Exact Bounds

When determining forb(m, F) it is possible that there is a subconfiguration that dominates the bound but does not yield the exact bound? This is typically the case (when the bound is known) but the following result sharpens the typical results.

Theorem (A, Raggi 09) Let $t, q \geq 1$ be given. Let

$$F_4(t,q) = \left[t \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} q \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right].$$

Then forb$(m, F_4(t,q))$ is forb$(m, t \cdot \mathbf{1}_4)$ plus $O(qm^2)$.

$$F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

Theorem *Let c be a positive real number. Let A be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no F_{2110}. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no F_{2110}.*

Theorem (P. Dukes 09) $\text{forb}(m, F_{2,1,1,0}) \leq .691m^2$

The proof used inequalities and linear programming

End of slides

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

Note that $\text{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F)+1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Conjecture (A, Sali 05) *forb*(m, F) *is* $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Conjecture (A, Sali 05) *forb*(m, F) *is* $\Theta(m^{x(F)})$.

In other words, our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

The conjecture has been verified for $k \times l$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

Refinements of the Sauer Bound

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*(m, K_k) *is* $\Theta(m^{k-1})$.

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let* F *be a* $k \times l$ *simple matrix such that there is a pair of rows with no configuration* E_1 *and there is a pair of rows with no configuration* E_2 *and there is a pair of rows with no configuration* E_3*. Then* *forb*(m, F) *is* $O(m^{k-2})$.

Refinements of the Sauer Bound

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*(m, K_k) *is* $\Theta(m^{k-1})$.

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $E_2 = \begin{bmatrix}0\\0\end{bmatrix}$, $E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let F be a $k \times l$ simple matrix such that there is a pair of rows with no configuration E_1 and there is a pair of rows with no configuration E_2 and there is a pair of rows with no configuration E_3. Then forb(m, F) is $O(m^{k-2})$.*

Note that $F_7 = \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 0\\0 & 1 & 0 & 1 & 0 & 1\\1 & 1 & 0 & 0 & 1 & 1\\1 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$ has no E_1 and no E_2 on rows 1,2 and no E_3 on rows 3,4. Thus forb(m, F_7) is $O(m^2)$.

New Result

$$F_7(t) = \left[\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

Theorem (A, Raggi, Sali 09) Let t be given. Then $\text{forb}(m, F_7(t))$ is $O(m^2)$.

Note that $F_7 = F_7(1)$. We cannot maintain the quadratic bound and repeat any other columns of F_7 since repeating columns of sum 1 or 3 in F_7 will yield constructions of $\Theta(m^3)$ columns avoiding them.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \left[\begin{array}{|cc|cc}\hline 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ \hline 0 & 0 & 1 & 1\end{array}\right]$$ has E_3 on rows 1,2.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \boxed{1 \quad 0} & & 1 \\ 0 & \boxed{0 \quad 1} & & 1 \end{bmatrix}$$ has E_3 on rows 2,3.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} \boxed{1} & 0 & 1 & \boxed{0} \\ 0 & 1 & 0 & 1 \\ \boxed{0} & 0 & 1 & \boxed{1} \end{bmatrix}$$ has E_3 on rows 1,3.

Definition $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_6 = \begin{bmatrix} \boxed{1} & 0 & 1 & \boxed{0} \\ 0 & 1 & 0 & 1 \\ \boxed{0} & 0 & 1 & \boxed{1} \end{bmatrix}$$ has E_3 on rows 1,3.

Note that F_6 has E_3 on every pair of rows hence forb(m, F_6) is $\Theta(m^2)$ (A, Griggs, Sali 97).