Non-Simple Forbidden Configurations and Design Theory

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A survey of results about forbidden configurations is available at my web page and I would welcome comments. The results in this paper are examples of some of the exact bounds that have been obtained. A listing of many exact bounds, asymptotic bounds and open problems can be found in:

Survey at www.math.ubc.ca/∼anstee

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e.g. K_m^d is the $m \times \binom{m}{d}$ simple matrix
of all columns of column sum d.

Design Theory

Definition A triple system $S_\lambda(2,3,m)$ is a collection of triples, subsets of $\{1,2,\ldots,m\}$, such that for each pair $i,j \in \{1,2,\ldots,m\}$, there are exactly λ triples containing i,j.

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A triple system is simple if there are no repeated triples

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Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple triple system $S_\lambda(2, 3, m)$.

Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3}\binom{m}{2}$ simple matrix with all columns of column sum 3 and having no submatrix

$$J_{2,\lambda+1} = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}}^{\lambda+1}$$

Theorem Let A be an $m \times n$ simple matrix with no submatrix

$$J_{2,q} = \begin{bmatrix} \overbrace{1 \quad 1 \quad \cdots \quad 1}^{q} \\ 1 \quad 1 \quad \cdots \quad 1 \end{bmatrix}$$

Then

$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-2}{3}\binom{m}{2}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,q-2}]$$

if $m \ge q$ and $m \equiv 1, 3 \pmod 6$.

Theorem Let q be given. Then there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{1,q} \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}}^{q} \\ \begin{matrix} 1 & 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

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with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,q-2} K_m^m]$$

if $m \geq q$ and $m \equiv 1, 3 (\text{mod } 6)$.

Why do we need $m > M$?
If $m = q + 1$, we may take

$$A = [K_m^0 K_m^1 K_m^2 K_m^3 K_m^m]$$

which has

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \binom{m}{m}$$

columns and which has no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{1,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 \quad 1 \quad \cdots \quad 1}^{q} \\ 1 \quad 1 \quad \cdots \quad 1 \\ 0 \quad 0 \quad \cdots \quad 0 \end{bmatrix}$$

but has

$$\frac{(q+1)q}{6}$$

more columns than the previous bound.

Theorem Let q be given. Then there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{2,q} \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q}$$

Then

$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-3}{3}\binom{m}{2} + \binom{m}{m-2} + \binom{m}{m-1} + \binom{m}{m}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,a} T_{m,b}^c K_m^{m-2} K_m^{m-1} K_m^m]$$

(for some choice a, b with $a + b = q - 3$)
if $m \ge q$ and $m \equiv 1, 3 (\text{mod } 6)$.

Proof Ideas

Assume that A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{2,q} \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}}^{q} \\ \begin{matrix} 1 & 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

Assume n is larger than the given bound:

$$n > \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-3}{3}\binom{m}{2} + \binom{m}{m-2} + \binom{m}{m-1} + \binom{m}{m}$$

Pigeonhole Argument

Idea Total number of $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$'s $\leq 6(q-1)\binom{m}{4}$.

A column of sum k yields $\binom{k}{2}\binom{m-k}{2}$ $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$'s.

Let a_i = number of columns of column sum i

Thus

$$\binom{m-2}{2}a_2+\binom{3}{2}\binom{m-3}{2}a_3+\cdots+\binom{m-2}{2}a_{m-2} \leq 6(q-1)\binom{m}{4}$$

(a_i= number of columns of column sum i)

If we assume $a_2 + a_3 + \cdots + a_{m-2} \geq \frac{q+3}{3}\binom{m}{2}$, then we deduce

$$2\binom{m}{2} - c_1 m \leq a_2 + a_{m-2} \leq 2\binom{m}{2}$$

$$a_4 + a_5 + \cdots + a_{m-4} \leq c_2 m$$

$$a_3 + a_{m-3} \geq \frac{q-3}{3}\binom{m}{2} - c_3 m$$

These bounds already bring us to within some constant times m of our desired bound.

Degrees

Consider the columns of column sum 3.

Let $d_1(ij)$ = degree of $\begin{matrix} i \\ j \end{matrix}\begin{bmatrix}1\\1\end{bmatrix}$

$$\begin{array}{c} \\ i \\ j \\ \\ \\ \\ \\ \end{array}\overbrace{\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}}^{d_1(ij)}$$

Thus $d_1(ij) \leq q - 1$.

Consider the columns of column sum $m-3$.

Let $d_0(ij)$ = degree of $\begin{matrix} i \\ j \end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{matrix} & \overbrace{\begin{matrix} & & & \end{matrix}}^{d_0(ij)} \\ i & 0 \quad 0 \quad \cdots \quad 0 \\ j & 0 \quad 0 \quad \cdots \quad 0 \end{matrix}$$

$$\begin{matrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix}$$

Thus $d_0(ij) \le q-1$.

One can show much more:

$$d_0(ij) + d_1(ij) \le q-1$$

$$
\begin{array}{c}
\\ i \\ j \\ \\ \\ \\ \\ \\ \\ \\ k \\ l
\end{array}
\begin{array}{cccccccc}
\multicolumn{4}{c}{\overbrace{\hphantom{1\quad 1\quad \cdots\quad 1}}^{d_1(ij)}} & & & & \\
1 & 1 & \cdots & 1 & ? & ? & \cdots & ? \\
1 & 1 & \cdots & 1 & ? & ? & \cdots & ? \\
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & 0 & & & \\
 & & & 1 & & 0 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 0 \\
* & * & \cdots & * & 0 & 0 & \cdots & 0 \\
* & * & \cdots & * & 0 & 0 & \cdots & 0 \\
 & & & & \multicolumn{4}{c}{\underbrace{\hphantom{0\quad 0\quad \cdots\quad 0}}_{d_0(kl)}}
\end{array}
$$

Given i, j, for most choices k, l, we have that the entries '*' are 0's and the entries '?' are 1's and so, assuming $a_2 = a_{m-2} = \binom{m}{2}$

$$d_1(ij) + d_0(kl) \le q - 3$$

Fact: $\sum_{ij} d_0(ij) + d_1(ij) = 3a_3 + 3a_{m-3} \approx (q-3)\binom{m}{2}$

We have

$$d_1(ij) + d_0(kl) \leq q - 3$$
$$d_0(ij) + d_1(kl) \leq q - 3$$

Idea: if for any choice i, j we have

$$d_0(ij) + d_1(ij) \geq q - 2$$

then for 'most' pairs k, l we have

$$d_0(kl) + d_1(kl) \leq q - 4$$

which then violates our estimate

$$\sum_{ij} d_0(ij) + d_1(ij) \approx (q-3)\binom{m}{2}$$

Turán's Bound

We can show that the number of pairs i, j with $d_0(ij) + d_1(ij) = q - 3$ is greater than $\binom{m}{2} - c_4 m$.

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Idea: By Turán's argument, there is a large 'clique' of rows B so that given any c_5, if m is large enough

$$i, j \in B \Rightarrow d_0(ij) + d_1(ij) = q - 3$$

$$|B| \geq c_5 \sqrt{m}$$

We can deduce that in the columns of sum $4, 5, \ldots, m-4$, the rows indexed by B avoid certain structures

$$A_{4,5,\ldots,m-4} = \begin{array}{c} \\ \text{rows } B \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \left[\begin{array}{c} \\ \text{no configuration } \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ \\ \end{array} \right]$$

The size of B allows us to select from up to $\binom{c_5\sqrt{m}}{2} \approx \frac{c_5^2}{2} m$ pairs i, j with $d_0(ij) + d_1(ij) = q - 3$.

We finally argue that $a_2 = a_{m-2} = \binom{m}{2}$, $a_4 = a_5 = \cdots = a_{m-4} = 0$ and for all pairs $i, j \in \{1, 2, \ldots, m\}$

$$d_0(ij) + d_1(ij) = q - 3$$

We then can also argue that there exists positive integers a, b with $a + b = q - 3$ so that for all pairs i, j

$$d_1(ij) = a, \qquad d_0(ij) = b$$

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Columns of column sum 3 yield simple triple system with $\lambda = a$

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Columns of column sum 3 yield simple triple system with $\lambda = a$

Columns of column sum $m - 3$ yield complement of simple triple system with $\lambda = b$.

Problem Can we reduce M to something close to $q+2$?

Problem Let q be given. Does there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no $4 \times q$ submatrix which is a row permutation of

$$\begin{bmatrix} J_{3,q} \\ 0_{1,q} \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q}$$

Then

$$n \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \frac{q-3}{4}\binom{m}{3} + \binom{m}{m}$$

with equality only if there exists a simple 3-design $S_\lambda(3,4,m)$ with $\lambda = q - 2$?

Problem Let q be given. Let A is an $m \times n$ simple matrix with no $4 \times 2q$ submatrix which is a row and column permutation of

$$\begin{bmatrix} J_{2,q} & 0_{2,q} \\ 0_{2,q} & J_{2,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 \quad \cdots \quad 1}^{q} & \overbrace{0 \quad \cdots \quad 0}^{q} \\ 1 \quad \cdots \quad 1 & 0 \quad \cdots \quad 0 \\ 0 \quad \cdots \quad 0 & 1 \quad \cdots \quad 1 \\ 0 \quad \cdots \quad 0 & 1 \quad \cdots \quad 1 \end{bmatrix}$$

Can we show that n is $O(m^2)$?

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