The Higher Direct Images of a Coherent Sheaf under a Proper Morphism are Coherent

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In this note we prove that for a proper morphism of noetherian schemes $f : X \to S$ and a coherent sheaf $\mathcal{F}$ on X, the higher direct images $R^i f_* \mathcal{F}$ are also coherent for every $i \geq 0$. This will be an application of the technique of devissage and obviously, Chow's lemma. I learnt of this proof from EGA and that's the only reference, I think.

1 The Big Theorems

Theorem 1. *(Chow's Lemma) Let $f : X \to S$ be a proper morphism of noetherian schemes. Then there is a projective S-scheme X' and a surjective S-morphism $\phi : X' \to X$ which induces an isomorphism $\phi^{-1}(U) \to U$ for a dense open subset U of X.*

Theorem 2. *(Grothendieck's Devissage) Let X be a Noetherian scheme and let $\mathcal{P}$ be a property of coherent sheaves on X. Assume that the following hold:*

- *(Two-out-of-three) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent sheaves on X. If any two of the $\mathcal{F}_i$ have property $\mathcal{P}$, then so does the third.*
- *(Devissage) For every closed integral subscheme $Y \subset X$ with generic point η, there is a coherent sheaf $\mathcal{F}$ on X with the property $\mathcal{P}$ satisfying*
  1. *Supp$(\mathcal{F}) = Y$.*
  2. *$\mathfrak{m}_\eta \mathcal{F}_\eta = 0$ where $\mathfrak{m}_\eta$ is the maximal ideal of the local ring $\mathcal{O}_{X,\eta}$.*
  3. *$\dim_{k(\eta)} \mathcal{F}_\eta = 1$ where $k(\eta) = \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta \cong \mathcal{O}_{Y,\eta}$ is the residue field.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

We will also need the Leray/Grothendieck spectral sequence of a composed functor. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories with enough injectives and

$$F : \mathcal{A} \to \mathcal{B}, \;\; G : \mathcal{B} \to \mathcal{C}$$

are left exact functors satisfying the property that F maps the injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$, then there is a spectral sequence computing the right derived functors of $G \circ F$. For any object A in $\mathcal{A}$,

$$E_2^{p,q} = R^p G(R^q F(A)) \Rightarrow E_\infty^{p,q} = Gr_L^p R^{p+q}(G \circ F)(A)$$

where L is the natural filtration.

Apply this to the case when U, V, W are schemes with maps $\psi : U \to V$ and $\phi : V \to W$. If $\mathfrak{Sh}(X)$ denotes the category of sheaves of $\mathcal{O}_X$-modules (this has enough injectives), then there are the pushforward functors

$$F = \psi_* : \mathfrak{Sh}(U) \to \mathfrak{Sh}(V), \; G = \phi_* : \mathfrak{Sh}(V) \to \mathfrak{Sh}(W)$$

and $G \circ F = (\phi \circ \psi)_* : \mathfrak{Sh}(U) \to \mathfrak{Sh}(W)$. Any injective object is flasque and the pushforward of any flasque sheaf is flasque. Since flasque sheaves are acyclic objects for a pushforward functor, there is a spectral sequence for any $\mathcal{O}_U$-module $\mathcal{F}$:

$$E_2^{p,q} = R^p \phi_*(R^q \psi_* \mathcal{F}) \Rightarrow E_\infty^{p,q} = Gr_L^p R^{p+q}(\phi \circ \psi)_* \mathcal{F} \tag{1}$$

2 Proof of the Main Theorem

Theorem 3. *Let X, S be noetherian schemes, $\mathcal{F}$ a coherent sheaf on X and $f : X \to S$ a proper morphism. Then all higher direct image sheaves $R^i f_* \mathcal{F}$, $i \geq 0$ are coherent on S.*

Proof. Verifying 'two-out-of-three' is easy: If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of coherent sheaves on X, the associated long exact sequence runs as

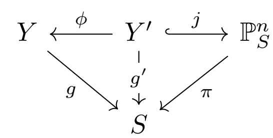
$$\cdots \to R^p f_* \mathcal{F}_3 \to R^{p+1} f_* \mathcal{F}_1 \to R^{p+1} f_* \mathcal{F}_2 \to R^{p+1} f_* \mathcal{F}_3 \to R^{p+2} f_* \mathcal{F}_1 \to \cdots$$

If the theorem holds for two of the $\mathcal{F}_i$, then the higher direct images of the third are sandwiched in the long exact sequence between coherent sheaves, and are hence, coherent by the two-out-of-three lemma for coherent sheaves.

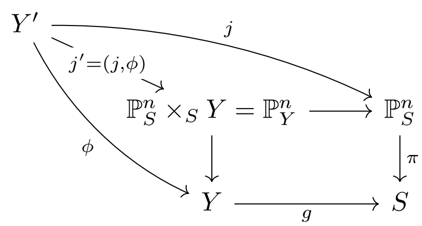
For the 'devissage' step, begin by constructing for a closed integral subscheme $i : Y \to X$ with generic point η, a coherent sheaf $\mathcal{G}$ on Y satisfying the properties

- $\dim_{k(\eta)} \mathcal{G}_\eta = 1$
- If we denote $g = f \circ i : Y \to S$, $R^p g_* \mathcal{G} = 0$ for all $p > 0$
- $R^0 g_* \mathcal{G} = g_* \mathcal{G}$ is coherent

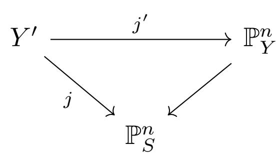
Note that g is proper. Apply Chow's lemma to g to get a diagram

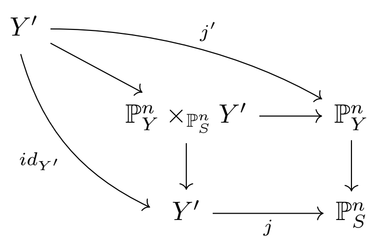


with ϕ surjective, g' projective and a (dense) open subset $U \subset Y$ such that $\phi : \phi^{-1}(U) \to U$ is an isomorphism. The induced map $j' := (j, \phi) : Y' \to \mathbb{P}^n_Y$ is a closed immersion.



(This is really some abstract nonsense: From the diagrams





since $g : Y \to S$ is proper, in particular separated, so is $\mathbb{P}^n_Y \to \mathbb{P}^n_S$ and so is $\mathbb{P}^n_Y \times_{\mathbb{P}^n_S} Y' \to Y'$ by base change. Then $Y' \to \mathbb{P}^n_Y \times_{\mathbb{P}^n_S} Y'$ is a closed immersion, being a section of a separated morphism. The morphism j is a closed immersion hence, so is $\mathbb{P}^n_Y \times_{\mathbb{P}^n_S} Y' \to \mathbb{P}^n_Y$ by base change. Then j' is the composition of two closed immersions, so is one itself.)

So $\mathcal{L} := (j')^* \mathcal{O}_{\mathbb{P}^n_Y}(1) = (j)^* \mathcal{O}_{\mathbb{P}^n_S}(1)$ is a very ample line bundle on Y' wrt ϕ and g' respectively. By Serre's vanishing theorem, for a large enough integer m, all higher direct images $R^p \phi_* \mathcal{L}^{\otimes m}$ and $R^p (g')_* \mathcal{L}^{\otimes m}$ vanish for all $p > 0$.

We claim that $\mathcal{G} := \phi_* \mathcal{L}^{\otimes m}$ works. That $\dim_{k(\eta)} \mathcal{G}_\eta = 1$ is clear because $\mathcal{L}^{\otimes m}$ is a line bundle and ϕ is an isomorphism between $\phi^{-1}(U)$ and U. That $g_* \mathcal{G} = (g')_* \mathcal{L}^{\otimes m}$ is coherent is also clear because we know the theorem for projective morphisms, in this case g'. To show the vanishing of higher direct images, the spectral sequence (1) gives

$$E_2^{p,q} = R^p g_* (R^q \phi_* \mathcal{L}^{\otimes m}) \Rightarrow E_\infty^{p,q} = Gr_L^p R^{p+q} (g')_* \mathcal{L}^{\otimes m}$$

The only nonzero term in E_∞ is at $(0,0)$ while the only nonzero terms in E_2 have $q = 0$. So $E_2^{p,q} = 0$ for all $p > 0$ as well, meaning $R^p g_*(R^0\phi_*\mathcal{L}^{\otimes m}) = R^p g_*\mathcal{G} = 0$ for all $p > 0$. This proves the claim.

To end the devissage argument, $i_*\mathcal{G} := \mathcal{F}$ works. Since i is a closed immersion, $\mathcal{F}$ is coherent. All conditions of devissage are clear except the fact that $\mathcal{F}$ satisfies the theorem. As was proved in the previous construction, $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. For the higher direct images, using (1) again gives a spectral sequence

$$E_2^{p,q} = R^p f_*(R^q i_*\mathcal{G}) \Rightarrow E_\infty^{p,q} = Gr_L^p R^{p+q} g_*\mathcal{G}$$

The same argument as before applies: the only nonzero term in E_∞ is at $(0,0)$ and on the E_2 page the only nonzero terms have $q = 0$ because i is a closed immersion, in particular, affine. So $E_2^{p,0} = E_\infty^{p,0}$, i.e. $R^p f_*\mathcal{F} = Gr_L^p R^p g^*\mathcal{G} = R^p g^*\mathcal{G} = 0$ for all $p > 0$ and the proof is complete.

□

Corollary 1. *If X is a noetherian scheme, A is a noetherian ring and $f : X \to SpecA$ is proper, all cohomologies $H^i(X,\mathcal{F})$ are finitely generated A-modules for any coherent sheaf $\mathcal{F}$ on X.*

Proof. Immediate from theorem 3. □