Talk 1:

What is Gromov-Witten theory?
Short answer: A (virtual) count of curves inside a smooth projective variety.

Two examples of enumerative problems:

(1) Let $p_1, \dots, p_5$ be five points in general position in $\mathbb{P}^2$. How many conics pass through these points?
Answer: A conic in $\mathbb{P}^2$ is defined by a homogenous quadratic polynomial in three variables, X_0, X_1, X_2, i.e. conics correspond to global sections $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = V$ upto multiplication by scalars.

$$\text{Conics in } \mathbb{P}^2 \longleftrightarrow \mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}^\bullet V^\vee)$$

(Projective space of lines in V)

For any point $p = (p_0, p_1, p_2) \in \mathbb{C}^3$ there is an evaluation morphism

$$e_p : V \longrightarrow \mathbb{C}$$
$$e_p(f) = f(p)$$

so that $e_p \in V^\vee$ defines a global section $s_p \in H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$.

The zero locus of s_p is precisely the hyperplane $H_p := \{[f] \in \mathbb{P}(V) \mid f(p) = 0\} \subseteq \mathbb{P}(V)$.

Then the conics we want are in the intersection

$$\bigcap_{i=1}^{5} H_{p_i}$$

and if the points p_i are in general position, the intersection is exactly one point. So there is exactly one conic passing through all p_i.

In this example, there is a vector bundle

$$\underset{\text{rank } 5}{\mathcal{O}_{\mathbb{P}(V)}(1)^{\oplus 5}} \longrightarrow \underset{\text{dim } 5}{\mathbb{P}(V)}$$

and a global section $(e_{p_1}, \ldots, e_{p_5})$ of $\mathcal{O}_{\mathbb{P}(V)}(1)^{\oplus 5}$ whose zero locus is the answer to the problem.

(2) Let $Z \subset \mathbb{P}^3$ be a smooth cubic surface. One knows that Z contains exactly 27 lines. Let us try to see this fact for a general cubic surface.

Let $G = Gr(2,4)$ be the Grassmanian of lines in $\mathbb{P}^3$, with tautological bundle S. Similar to (1) there is a correspondence

$$\begin{matrix}\text{Homogenous degree 3} \\ \text{polynomials in } x_0, x_1, x_2, x_3\end{matrix} \longleftrightarrow \begin{matrix}\text{Global sections} \\ H^0(G, \mathrm{Sym}^3(S^\vee))\end{matrix}$$

i.e. cubic surfaces in $\mathbb{P}^3$ correspond to sections of $\mathrm{Sym}^3(S^\vee)$ and the zero locus in G of a section f is precisely the lines $L \subseteq \mathbb{P}^3$ on which f vanishes, i.e. $L \subset Z(f)$.

So we are reduced to computing

$$\int_{Gr(2,4)} c_4(\mathrm{Sym}^3(S^\vee))$$

(Top Chern class of a complex v.b. is the Euler class)

The splitting principle shows that

$$c_4(\mathrm{Sym}^3(S^\vee)) = 9c_2(S^\vee)\left(2c_1^2(S^\vee) + c_2(S^\vee)\right)$$

and the Chern classes of $S^\vee$ are represented by Schubert varieties

$$c_1(S^\vee) = \Pi_\ell := \{ L \subseteq \mathbb{P}^3 \mid L \cap \ell \neq \emptyset \}. \quad \ell \subseteq \mathbb{P}^3 \text{ a line}$$

$$c_2(S^\vee) = \Pi_h := \{ L \subseteq \mathbb{P}^3 \mid L \subseteq h \} \quad h \subseteq \mathbb{P}^3 \text{ a plane}$$

with computations

$$\Pi_\ell^2 = \Pi_h + \Pi_p \left(:= \{ L \subseteq \mathbb{P}^3 \mid p \in L \} \right) \quad p \in \mathbb{P}^3 \text{ a point}$$

$$\Pi_h \Pi_p = 0 \quad , \quad \Pi_h^2 = 1$$

So

$$\begin{aligned} c_4(\mathrm{Sym}^3(S^\vee)) &= 9\Pi_h\left(2\Pi_\ell^2 + \Pi_h\right) \\ &= 9\Pi_h\left(3\Pi_h + 2\Pi_p\right) \\ &= 27\Pi_h^2 + 18\Pi_h\Pi_p \\ &= \underline{\underline{27}} \end{aligned}$$

In this example too, there is a vector bundle

$$\mathrm{Sym}^3(S^\vee) \longrightarrow Gr(2,4)$$

rank 4 $\qquad\qquad\qquad$ dim 4

the zero locus of a general section giving the answer.

Vague moral : Moduli spaces should be obtained, atleast locally, by the following recipe -

X : a smooth scheme
E : a vector bundle on X
$s \neq 0 \in \Gamma(X, E)$ a nonzero global section
$\mathcal{M}$ = zero locus of s, the moduli space.

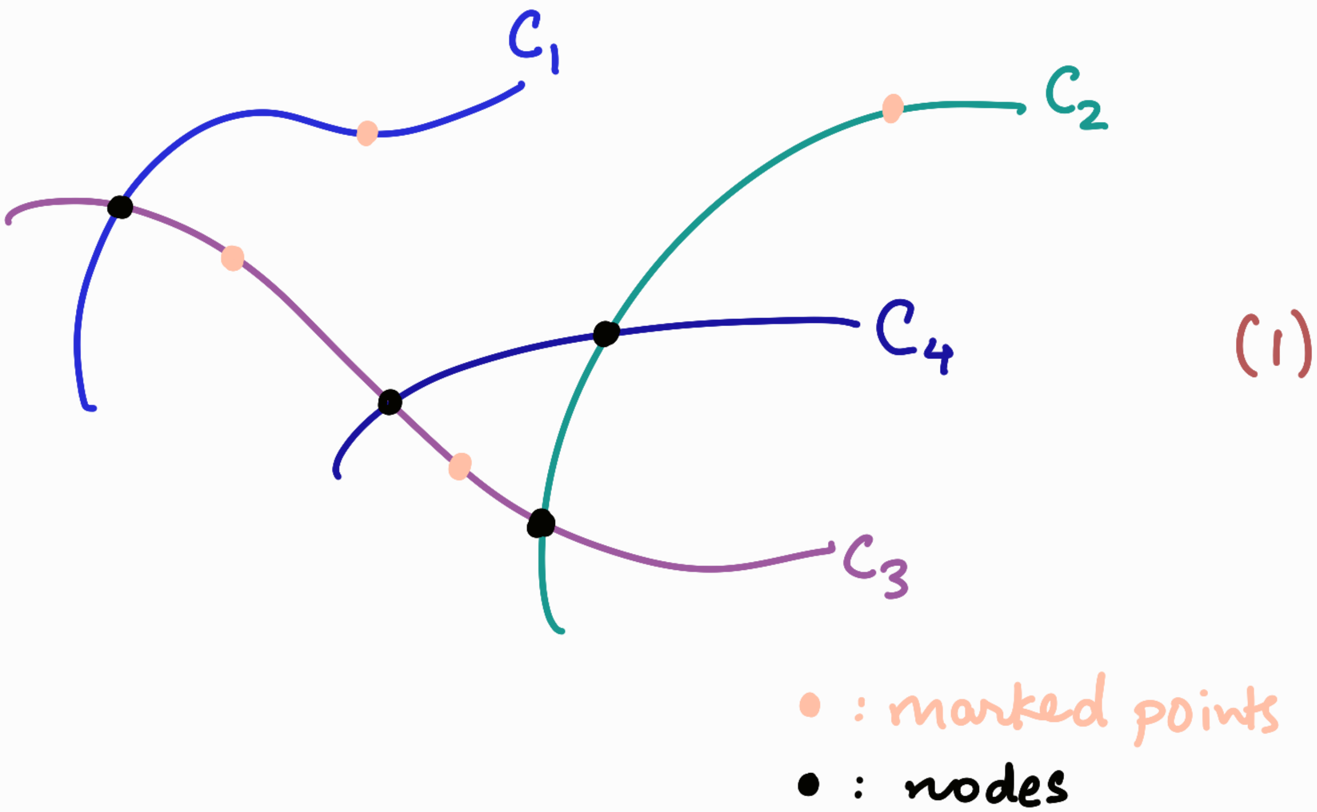
$$\begin{tikzcd} & E \arrow[d, shift right] \\ \mathcal{M} = s^{-1}(0) \arrow[r, hook] & X \arrow[u, shift right, bend right, "s"'] \end{tikzcd}$$

(Here will arise transversality issues : $\mathrm{codim}(\mathcal{M}, X)$ need not be equal to rank E. An easy example is taking two functions $xz, yz \in \mathbb{C}[x,y,z]$ whose zero locus isn't equidimensional)

Counting curves in a projective variety X: Embedded curves in X have as bad singularities as possible. Kontsevich's idea is to redefine a 'curve' as (the image of) a map $f : C \to X$ where C is now a curve with prescribed singularities only + extra conditions to have nice moduli spaces.

Moduli spaces of stable maps

• Definition: A n-pointed genus g quasi-stable curve $(C, \{p_i\}_{i=1}^n)$ is a projective, connected, reduced curve C of arithmetic genus g with atworst nodal singularities and n distinct nonsingular marked points $p_1, \ldots, p_n \in C$.



Picture of a quasi-stable curve.

Important example: Ignoring marked points, C is a quasi-stable curve of genus 0 iff C is a tree of smooth rational curves.

Proof is an exercise! (Hint - Normalization)

For genus 1, there are several possibilities: Smooth cubic curve in $\mathbb{P}^2$ (elliptic curve), nodal cubic curve, a cycle of rational curves etc.

- Definition (Family of quasi-stable curves)
  Let S be a scheme. A family of n-pointed quasi-stable curves of genus g over S is a flat, projective morphism $\pi: \mathcal{C} \longrightarrow S$ such that for every geometric point $s: \operatorname{spec} \Omega \longrightarrow S$ the geometric fiber $\mathcal{C}_s$ is a genus g quasi-stable curve and there are n disjoint sections $p_i: S \longrightarrow \mathcal{C}$ of π, $i=1,\dots,n$.

So for every s as above, $(\mathcal{C}_s, p_1(s), \dots, p_n(s))$ is a n-pointed genus g quasi-stable curve, varying as $s \in S$ varies in the base.

One is interested in morphisms (or families of morphisms) of a quasi-stable curve to a target scheme X, in line with the idea.

- Definition (Family of morphisms)
  Let S be a scheme and X a target scheme. A family of morphisms of n-pointed genus g quasi-stable curves to X is the data

$$\left(\pi: \mathcal{C} \longrightarrow S, \{p_i: S \to \mathcal{C}\}_{i=1}^{n}, \mu: \mathcal{C} \longrightarrow X \right)$$

with $(\pi: \mathcal{C} \longrightarrow S, \{p_i\}_{i=1}^{n})$ being a family of n-pointed genus g quasi-stable curves over S and μ a morphism.

There is an obvious notion of 'isomorphism of families of morphisms' - in which the obvious diagrams commute for a second S-family

$$\left(\pi' : \mathcal{C}' \to S, \ \{p_i'\}_{i=1}^{n}, \ \mu' : \mathcal{C}' \longrightarrow X \right)$$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "J"] \arrow[dr, "\pi"'] & & \mathcal{C}' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "J"] & & \mathcal{C}' \\ & S \arrow[ul, "p_i"] \arrow[ur, "p_i'"'] & \end{tikzcd} \ \forall i \qquad \begin{tikzcd} \mathcal{C} \arrow[drr, "\mu"] \arrow[d, "J"'] & & \\ \mathcal{C}' \arrow[rr, "\mu'"'] & & X \end{tikzcd}$$

where $J : \mathcal{C} \to \mathcal{C}'$ is an isomorphism over S.

One would like to have conditions ensuring 'stability' as in the case of moduli of pointed curves $\overline{\mathcal{M}_{g,n}}$. For a quasi-stable curve $(C, p_1, \ldots, p_n)$ and an irreducible component $E \subset C$, the special points on E are the marked points on E and the nodal points of intersection $E \cap E'$ for $E' \subset C$ another irreducible component

- **Definition**: Let $(C, p_1, \ldots, p_n)$ be a quasi-stable curve and $\mu : C \longrightarrow X$ a map. The map μ is called stable if for an irreducible component $E \subset C$, the following hold
  - (a) If $g_a(E) = 0$ and $\mu(E) =$ point, then E has atleast three special points.
  - (b) If $g_a(E) = 1$ and $\mu(E) =$ point, then E has atleast one special point.

The stability condition ensures that the morphisms $\mu : C \to X$ have only finitely many automorphisms.

• Definition: A family of stable maps parameterized by S

$$\{\pi : \mathcal{C} \longrightarrow S,\ (p_i)_{i=1}^{n},\ \mu : \mathcal{C} \longrightarrow X\}$$

is a family of maps such that for every geometric fiber $s : \operatorname{spec} \Omega \longrightarrow S$, the map

$$\mu_s : (\mathcal{C}_s,\ p_1(s), \dots, p_n(s)) \longrightarrow X$$

is stable.

• Describing the moduli functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$:

Let X be a target scheme and $\beta \in A_1(X)$ be a cycle class. A stable map $\mu : (C, (p_i)_{i=1}^{n}) \longrightarrow X$ is said to represent β if $\mu_*[C] = \beta$.

The (contravariant) functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$ maps a scheme S to S-families of stable maps to X representing β, i.e.

$$\overline{\mathcal{M}}_{g,n}(X,\beta) : (Sch/\mathbb{C})^{op} \longmapsto (\text{Groupoids})$$

$$\overline{\mathcal{M}}_{g,n}(X,\beta)(S) := \left\{ \begin{array}{l} \pi : \mathcal{C} \to S,\ (p_i)_{i=1}^{n},\ \mu : \mathcal{C} \longrightarrow X \\ \text{families of stable maps} \\ \text{such that } \mu_s \text{ represents } \beta \\ \text{for every geometric pt } s \in S \end{array} \right\}$$

and the morphisms in $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ are S-isomorphisms of families of maps

Note that the points of this 'space' $\overline{\mathcal{M}}_{g,n}(X,\beta)(\mathbb{C})$ are just stable maps alongwith an extra finite group attached to it - the automorphism group of the stable map.

A detour - heuristic computation of the expected dimension of $\overline{M}_{g,n}(X,\beta)$: We're looking at a space whose points correspond to stable maps $\mu : C \longrightarrow X$. By a little deformation theory, deformations of the map μ are given by

$$H^0(C, \mu^* \mathcal{H}om(\Omega_X, \mathcal{O}_X))$$

(this is the tangent space at μ in the scheme of morphisms $\mathrm{Hom}(C,X)$)

and the obstruction space is

$$H^1(C, \mu^* \mathcal{H}om(\Omega_X, \mathcal{O}_X))$$

So if X is smooth, which we assume from now, the expected dimension of deformations of μ is

$$h^0(C, \mu^* TX) - h^1(C, \mu^* TX)$$

$$= \chi(C, \mu^* TX) = \deg(\mu^* TX) + rk(\mu^* TX)(1-g)$$

by Riemann-Roch $(g = p_a(C))$

The last number is $\deg(\mu^* - K_X) + \dim X (1-g)$
$= -K_X \cdot \beta + \dim X (1-g)$

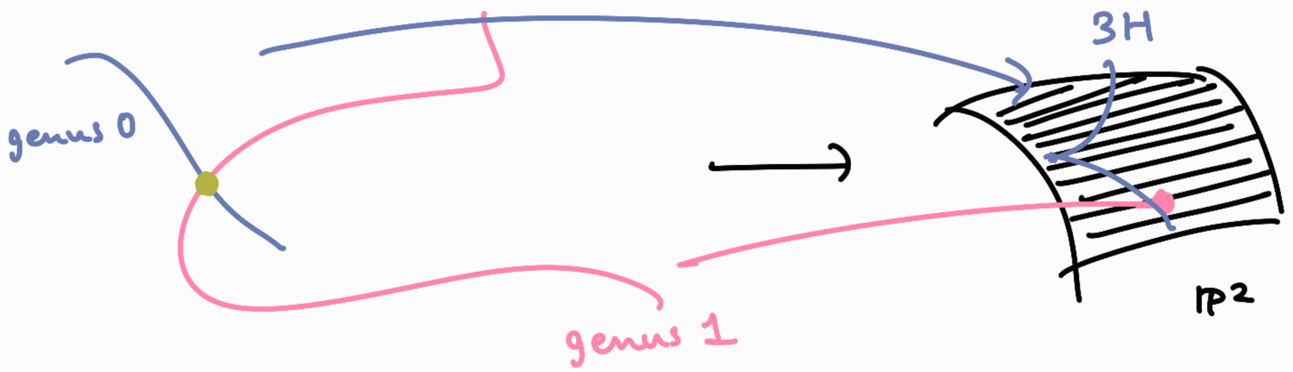
Accounting for deformations of the original curve C adds an extra $3g-3+n$ to the above, so the expected dim (also virtual dimension) is

$$\boxed{\mathrm{vdim}\, \overline{M}_{g,n}(X,\beta) = (\dim X - 3)(1-g) + n - K_X \cdot \beta}$$

Some examples to show that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is badly behaved.

EXAMPLE (1) : Consider $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3H)$ where H is the class of a line in $\mathbb{P}^2$. The expected dim of $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3H)$ is $3H \cdot 3H = 9$ but there is a 'boundary component' with bigger dimension described as follows -

Consider source curves to be a genus 1 curve with a genus 0 tail and the morphism to $\mathbb{P}^2$ contracts the genus 1 curve (and maps the genus 0 curve to the class 3H necessarily)



Then these maps are parametrized by

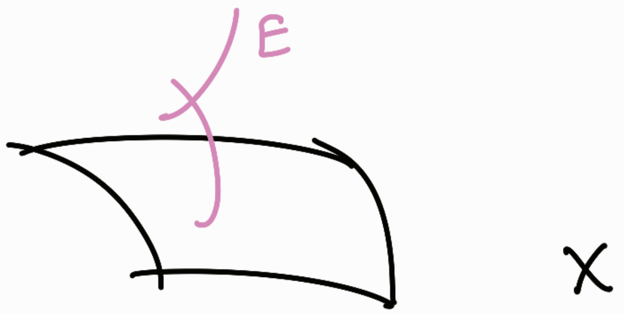
$$\overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, 3H) \times \mathcal{M}_{1,1}$$

the marked point in each factor being the point of intersection. The above space has dimension = expected dimension (this is a fact, coming later) and equals 10 by the formula.

So the boundary is badly behaved.

EXAMPLE (2) : It is not even obvious if $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has nonempty interior, i.e. maps with irreducible source curves. The following 'stupid' example shows that an empty interior may be the case.

Take $\mathbb{P}^2$, blowup a point and then blow up a point on the exceptional divisor to get X.



Let β be the class of the exceptional divisor in X. (Here E is two rational curves intersecting in a node) Then $\mathcal{M}_{0,n}(X,\beta)$ (the interior) is empty as irreducible source curves cannot map to E and E is the only curve representing β. The compactification $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is obviously nonempty.

EXAMPLE (3) : (Calabi-Yau threefolds) Let X be a CY3, i.e. a simply connected Kähler threefold with trivial canonical bundle (such an X is automatically projective). The virtual dimension formula shows that $\operatorname{vdim} \overline{\mathcal{M}}_{g,0}(X,\beta) = 0$ so we may expect a moduli space being a finite set of points.

If Q is a general quintic threefold in $\mathbb{P}^4$ (famously CY3 by the adjunction formula) the homology $H_2(Q,\mathbb{Z}) \cong \mathbb{Z}$ with a line in Q being a generator.

It is also famously known that for a general Q, there are exactly 2875 lines in Q. Then the moduli space $\overline{\mathcal{M}}_{0,0}(Q,2)$ has those many components of dimension 2 corresponding to

$$\mathbb{P}^1 \xrightarrow{(\cdot)^2} \mathbb{P}^1 \hookrightarrow Q$$

the first map being a double cover - the dimension of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,2)$ is 2, computing this obviously doesn't need any formula and the second map embedding $\mathbb{P}^1$ as a line in Q - 2875 possible ways.

- The main theorems

<u>Theorem</u> (Kontsevich) For X a projective scheme and $\beta \in A_1(X)$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a Deligne-Mumford stack and admits a projective coarse moduli space $\overline{M}_{g,n}(X,\beta)$

The situation for GW theory in genus 0 is better than higher genera.

Assume that X is nonsingular and convex (in addition to being projective).

By convexity is meant that for any morphism $\mu : \mathbb{P}^1 \longrightarrow X$, $H^1(\mathbb{P}^1, \mu^* TX) = 0$ (no obstructions). The reason for naming the property 'convex' is probably this -

The vector bundle $\mu^* TX$ on $\mathbb{P}^1$ splits into a sum of line bundles $\bigoplus_{i=1}^{\dim X} \mathcal{O}(a_i)$

The vanishing of H^1 implies that $H^1(X, \mathcal{O}(a_i)) = 0$ $\forall i$ i.e. $a_i \geq -1$ $\forall i$. But if $a_i = -1$ for some i then one can apply the double cover precomposition trick (see EXAMPLE 3) to infact conclude $a_i = -1$ is not possible so $a_i \geq 0$ $\forall i$.

<u>Theorem</u> (Behrend - Manin) Under the above assumptions on X, $\overline{M}_{0,n}(X, \beta)$ is a normal pure dimensional variety with dimension equal to the expected dimension $\dim X - 3 + n - K_X \cdot \beta$