Universal Ext

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For a variety X over an algebraically closed field k and coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, the space $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$ classifies upto isomorphism, extensions of $\mathcal{F}$ by $\mathcal{G}$ i.e. coherent sheaves $\mathcal{E}$ fitting into a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

If X is projective or more generally proper, a standard spectral sequence argument shows that $V = \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$ is a finite dimensional k-vector space. Let $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}^\bullet V^\vee)$ be the affine space associated to V. Assume now that $\mathcal{F}$ is a vector bundle. We construct a 'universal' extension on $Y = X \times \mathbb{A}(V)$ - by this we mean a coherent sheaf $\mathcal{E}^{uni}$ on Y and a 'universal' short exact sequence

$$0 \to p^*\mathcal{G} \to \mathcal{E}^{uni} \to p^*\mathcal{F} \to 0$$

such that for every closed point $v \in V$, restricting the universal sequence to $Y_v = X$ gives the extension $\mathcal{E}^{uni}_v \in \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}) = V$ associated to v. Here $p : Y \to X$ is the projection to X.

$$\begin{array}{ccc} Y & \xrightarrow{\ p\ } & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{A}(V) & \xrightarrow[\ t\ ]{} & \mathrm{Spec}\, k \end{array}$$

To see the existence of $\mathcal{E}^{uni}$, we compute

$$\begin{aligned} \mathrm{Ext}^1_Y(p^*\mathcal{F}, p^*\mathcal{G}) &= \mathrm{Ext}^1_Y(\mathcal{O}_Y, p^*(\mathcal{F}^\vee \otimes \mathcal{G})) \\ &= H^1(Y, p^*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= H^0(\mathbb{A}(V), R^1 q_* p^*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= H^0(\mathbb{A}(V), t^* R^1 s_* \mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= H^0(\mathbb{A}(V), t^* H^1(\widetilde{X, \mathcal{F}^\vee} \otimes \mathcal{G})) \\ &= H^0(\mathbb{A}(V), t^* \widetilde{\mathrm{Ext}^1_X}(\mathcal{F}, \mathcal{G})) \\ &= H^0(\mathbb{A}(V), \mathcal{O}_{\mathbb{A}(V)} \otimes V) \\ &= \mathrm{Sym}^\bullet V^\vee \otimes V \end{aligned}$$

where we use flat base change in going from equality 3 to 4 and a spectral sequence argument in equality 2 to 3. So the extensions of $p^*\mathcal{F}$ by $p^*\mathcal{G}$ on Y are in one-to-one correspondence with polynomial maps $V \to V$. The correspondence is given as follows: For an extension

$$0 \to p^*\mathcal{G} \to \mathcal{H} \to p^*\mathcal{F} \to 0$$

the corresponding polynomial map is the map $v \to \mathcal{H}_v \in V$. Take the identity map on V to get the universal extension $\mathcal{E}^{uni}$.

Example: Let $X = \mathbb{P}^1$ with $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(-2)$ so that $V = \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$ is a one-dimensional vector space with nontrivial extension given by the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

As advertised before, there is a universal extension $\mathcal{E}^{uni}$ on $\mathbb{P}^1 \times \mathbb{A}^1_v$ such that

$$\mathcal{E}^{uni}_v \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } v \neq 0 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \text{if } v = 0 \end{cases}$$

Infact $\mathcal{E}^{uni}$ is a rank 2 vector bundle and can be described more explicitly. If the homogenous coordinates on $\mathbb{P}^1$ are z_0, z_1 with $z = \frac{z_1}{z_0}$ the coordinate on $U_0 = D(z_0)$ and $w = \frac{z_0}{z_1}$ that for $U_1 = D(z_1)$, then the transition function on $U_0 \cap U_1$ for $\mathcal{O}_{\mathbb{P}^1}(-1)$ in going from from U_0 to U_1 is just multiplication by z. If v is the coordinate on $\mathbb{A}^1$, to describe a rank 2 vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ is to give the 2×2 transition matrix, relating $\mathbb{A}^1_z \times \mathbb{A}^1_v$ to $\mathbb{A}^1_w \times \mathbb{A}^1_v$. Consider the matrix

$$\begin{bmatrix} vz & (v-1)z \\ (v+1)z & vz \end{bmatrix}$$

By changing this matrix by invertible matrices in $GL_2(\mathbb{C}[z,v])$ and $GL_2(\mathbb{C}[w,v])$ we see for $v \neq 0$ that this is the transition matrix of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by observing that

$$\begin{bmatrix} 1 & -(v-1)v^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} vz & (v-1)z \\ (v+1)z & vz \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(v+1)v^{-1} & 1 \end{bmatrix} = \begin{bmatrix} v^{-1}z & 0 \\ 0 & vz \end{bmatrix}$$

and for $t = 0$, this is the transition matrix of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ because

$$\begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} = \begin{bmatrix} z^2 & z \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$$

The purpose of such an example is to illustrate bad behavior for families of sheaves: If $p : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$ is the projection and $\mathcal{E} := p^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*\mathcal{O}_{\mathbb{P}^1}(-1)$ then $\mathcal{E}^{uni}_v \cong \mathcal{E}_v$ for all $v \neq 0$ but $\mathcal{E}^{uni}_0$ is not isomorphic to $\mathcal{E}_0$.

If we perform a naive attempt to define a moduli of vector bundles (with some invariants - rank, first Chern class, etc) on a projective scheme X, say $\mathcal{M}$ is the functor assigning to a scheme S a vector bundle E on $X \times S$ such that

$E_s := E|_{X_s}$ is a vector bundle on $X_s = X$ with those invariants. The example shows that such naive moduli functors are not representable by separated coarse moduli spaces - indeed if a separated coarse moduli space M were to exist for rank 2 vector bundles on $\mathbb{P}^1$ of determinant $\mathcal{O}_{\mathbb{P}^1}(-2)$, the families above induce two morphisms $\mathbb{A}^1 \to M$ which agree on $\mathbb{A}^1 - \{0\}$, so are equal, a contradiction.

The way to 'fix' this is by introducing notions of stability. If X is a projective variety with a fixed very ample line bundle H, one has the notion of slope or Gieseker (semi)stability of coherent sheaves on X with respect to H. For a numerical polynomial $p(t)$, let $\mathcal{M}^{ss}_{X,p(t)}$ denote the moduli functor parametrizing semistable sheaves of Hilbert polynomial $p(t)$ on X i.e. for a k-scheme S, $\mathcal{M}^{ss}_{X,p(t)}(S)$ is the set of coherent sheaves $\mathcal{F}$ on $X \times S$ flat over S such that for every closed point $s \in S$ the sheaf $\mathcal{F}_s := \mathcal{F}|_{X_s}$ on $X_s = X$ is semistable of Hilbert polynomial $p(t)$, upto twist by a line bundle of the form $p_S^* L$ where L is a line bundle on S.

Notice that examples like the above do not show up as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ is not semistable on $\mathbb{P}^1$. In flat families, the property of semistability is an open property so this is already a better behaved notion. With GIT, one constructs a fine moduli space of stable sheaves but we do not take it up here.