FLAT MORPHISMS ARE OPEN

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The goal of this note is to prove that a flat finite-type morphism between Noetherian schemes is open. A crucial ingredient in the proof is that the going-down property holds for a flat extension of rings, which really, is some tricky commutative algebra.

1 Faithfully Flat Modules

Throughtout, let A be a commutative ring.

Definition 1.1. An A-module M is called faithfully flat if the following holds: The sequence of A-modules $N' \to N \to N''$ is exact iff the sequence

$$N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$$

is exact.

A faithfully flat module is obviously also flat. The word 'faithfully flat' is because the definition implies that $- \otimes_A M : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is a fully faithful functor.

Lemma 1.2. The following are equivalent:

1. M is faithfully flat.
2. M is flat and $N \otimes_A M \neq 0$ whenever $N \neq 0$.
3. M is flat and ${}^{A}\!/_{\mathfrak{m}} \otimes_A M \neq 0$ for all maximal ideals $\mathfrak{m}$ of A.

Proof.

- (1) $\implies$ (2) : If $N \otimes_A M \neq 0$ for $N \neq 0$, then $0 \to 0 = N \otimes_A M \to 0$ is exact but $0 \to N \to 0$ is not.

- (2) $\implies$ (1) : Given a sequence $N' \xrightarrow{f} N \xrightarrow{g} N''$, suppose

$$N' \otimes_A M \xrightarrow{f\otimes 1} N \otimes_A M \xrightarrow{g\otimes 1} N'' \otimes_A M$$

is exact. To show that $\operatorname{im} f \subset \ker g$, let $x \in N'$ and $g(f(x)) = z \in N''$ with $z \neq 0$. For any $m \in M$,

$$0 = (g \otimes 1)(f \otimes 1)(x \otimes m) = z \otimes m$$

in $N'' \otimes_A M$. Let $E \subset N''$ be the submodule generated by z, so that $E \neq 0$ and $E \otimes_A M$ is a submodule of $N'' \otimes_A M$ by the flatness of M. However, $E \otimes_A M \neq 0$ by hypothesis but the above computation shows that $E \otimes_A M$ is the zero submodule of $N'' \otimes_A M$, a contradiction. So $z = 0$ and $\operatorname{im} f \subset \ker g$. From the exact sequence

$$0 \to \operatorname{im} f \to \ker g \to {}^{\ker g}\!/_{\operatorname{im} f} \to 0$$

we get on tensoring with M,

$$\left({}^{\ker g}\!/_{\operatorname{im} f}\right) \otimes_A M \cong {}^{(\ker g \otimes_A M)}\!/_{(\operatorname{im} f \otimes_A M)}$$

Now, $\ker g \otimes_A M$ is a submodule of $N \otimes_A M$ and is contained in $\ker(g \otimes 1)$. But $\ker(g \otimes 1) = \operatorname{im}(f \otimes 1) = \operatorname{im} f \otimes_A M$. Since $\operatorname{im} f \subset \ker g$, $\operatorname{im} f \otimes_A M$ is a submodule of $\ker g \otimes_A M$, hence $\ker(g \otimes 1) \subset \ker g \otimes_A M$. Putting this together,

$${}^{(\ker g \otimes_A M)}\!/_{(\operatorname{im} f \otimes_A M)} = {}^{\ker(g \otimes 1)}\!/_{\operatorname{im}(f \otimes 1)} = 0$$

and this implies ${}^{\ker g}\!/_{\operatorname{im} f} = 0$, proving that $N' \xrightarrow{f} N \xrightarrow{g} N''$ is exact.

- (2) $\implies$ (3) : Obvious.
- (3) $\implies$ (2) : Let n be a non-zero element of N. The map of A-modules $A \to N$ given by $a \to an$ has non-zero image so the kernel is a proper ideal $\mathfrak{I}$ of A. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{I}$. Since $A/\mathfrak{I} \subset N$, $A/\mathfrak{I} \otimes_A M$ is a submodule of $N \otimes_A M$. But from the surjection $A/\mathfrak{m} \to A/\mathfrak{I} \to 0$,

$$A/\mathfrak{m} \otimes_A M \to A/\mathfrak{I} \otimes_A M \to 0$$

so $A/\mathfrak{I} \otimes_A M \neq 0$. This implies $N \otimes_A M \neq 0$ as well.

□

Corollary 1.3. Let $\phi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local ring homomorphism and M a finitely generated non-zero B-module. Then, as an A-module, M is flat iff M is faithfully flat.

Proof. From lemma 1.2, it only suffices to show that $A/\mathfrak{m} \otimes_A M \neq 0$. As $\phi(\mathfrak{m}) \subset \mathfrak{n}$, there is a surjection

$$A/\mathfrak{m} \otimes_A M = M/\phi(\mathfrak{m})M \to M/\mathfrak{n}M$$

So it is enough to show that $M/\mathfrak{n}M \neq 0$ but this is just Nakayama's lemma. □

Corollary 1.4. Let $\phi : A \to B$ be a flat map of rings. Then for every prime $\mathfrak{q} \subset B$ and $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \subset A$, $B_\mathfrak{q}$ is a faithfully flat $A_\mathfrak{p}$-module.

Proof. Since ϕ is flat, the induced map on localizations, $A_\mathfrak{p} \to B_\mathfrak{q}$ is also flat. Now corollary 1.3 applies, with the module being $B_\mathfrak{q}$. □

Corollary 1.5. Let $\phi : A \to B$ be a faithfully flat map of rings. Then the induced map $\widetilde{\phi} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof. For $\mathfrak{p} \in \mathrm{Spec}(A)$, we need to show that the fiber $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ is non-empty, that is, $B \otimes_A k(\mathfrak{p})$ is non-zero. This follows from (2) of lemma 1.2 as $k(\mathfrak{p}) \neq 0$. □

Theorem 1.6 (Going-down)**.** If $A \subset B$ is a flat extension of rings, then the going-down property holds.

Proof. Suppose $\mathfrak{p}' \subset \mathfrak{p}$ are primes in A and $\mathfrak{q}$ is a prime in B lying over $\mathfrak{p}$. Then $A_\mathfrak{p} \subset B_\mathfrak{q}$ is a flat extension of local rings, so is faithfully flat by corollary 1.4. By corollary 1.5, $\mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective, i.e., there is a prime $\mathfrak{q}'$ of B such that $\mathfrak{q}' \subset \mathfrak{q}$ and $\mathfrak{q}' B_\mathfrak{q} \cap A_\mathfrak{p} = \mathfrak{p}' A_\mathfrak{p}$. So $\mathfrak{q}' \cap A = \mathfrak{p}'$ which proves the theorem. □

2 Openness of Flat Morphisms

Recall that a morphism $f : X \to Y$ of schemes is called flat if for every $x \in X$, the induced morphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Corollary 1.3 implies that $\mathcal{O}_{X,x}$ is a faithfully flat $\mathcal{O}_{Y,f(x)}$-module and 1.5 then implies that $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,f(x)})$ is surjective. Recalling that $\mathrm{Spec}(\mathcal{O}_{X,x})$ consists of precisely all the generizations of x, we see that the image of a flat morphism is closed under generizations. (A point y is called a generization of x if $x \in \overline{\{y\}}$.)

We will also need the theorem of Chevalley about the images of morphisms. A reference is Hartshorne, ch. II, exercise 3.19.

Theorem 2.1 (Chevalley)**.** Let $f : X \to Y$ be a morphism of finite-type of Noetherian schemes. If E is a constructible subset of X, the image $f(E)$ is also constructible in Y.

Theorem 2.2. Let $f : X \to Y$ be a flat, finite-type morphism of Noetherian schemes. Then f is open.

Proof. Let $F = Y - f(X)$ and we need to show that F is closed. Let $\overline{F}$ be the closure of F and give $\overline{F}$ some subscheme structure. Let $\{F_i\}_{i=1}^n$ be the irreducible components of $\overline{F}$, having generic points $\{y_i\}_{i=1}^n$ respectively. We claim that every $y_i \in F$. Suppose that $y_i \in f(X)$ for some i. By Chevalley's theorem, $y_i \in U \cap V \subset f(X)$ for some U open and V closed in Y. Since V is closed and $y_i \in V$, $F_i \subset V$ so we may assume that $V = F_i$. Let $U' = U \cap (X - \cup_{j \neq i} F_j)$. Then U' is an open set contained in $f(X)$ and $y_i \in U'$. But $F \subset X - U' \implies \overline{F} \subset X - U'$ and $y_i \in \overline{F}$, i.e., $y_i \in X - U'$, a contradiction. This proves the claim.

Now if $z \in \overline{F} - F$, z is in the image $f(X)$ and also is in some $F_i = \overline{\{y_i\}}$. Since y_i is a generization of z, we see that $y_i \in f(X)$ as f is flat, a contradiction. Therefore $\overline{F} = F$, which proves that $f(X)$ is open. □

Corollary 2.3. Let $f : X \to Y$ be a finite flat morphism of Noetherian schemes. If Y is irreducible, f is surjective.

Proof. From theorem 2.2, the image $f(X)$ is open. Since finite morphisms are closed, $f(X)$ is closed. The assertion now follows as Y is irreducible, hence connected. □

References

[1] Robin Hartshorne, *Algebraic Geometry*, Springer, New York, 1977.

[2] James Milne, *Etale Cohomology*, Princeton University Press, New Jersey, 1980.