THE THEOREM OF HIGHEST WEIGHT

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The goal of this note is to prove that the finite dimensional irreducible representations of a complex, semisimple (finite dimensional) Lie algebra are classified by their highest weight, which we shall explain in more detail below.

The following notation will be used: Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Φ the set of roots of $\mathfrak{g}$ wrt $\mathfrak{h}$, Δ a basis for Φ consisting of the simple roots $\{\alpha_1, \cdots, \alpha_l\}$ and Φ^+, Φ^- the subsets of positive, resp. negative roots wrt Δ. A partial order on $\mathfrak{h}^*$ is defined as follows: for $\lambda, \mu \in \mathfrak{h}^*$, $\mu \leq \lambda$ iff $\lambda - \mu = \sum_{\alpha \in \Phi^+} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_{\geq 0}$ $\forall\, \alpha \in \Phi^+$.

1 Some definitions and the theorem

If V is a representation of $\mathfrak{h}$, then for $\lambda \in \mathfrak{h}^*$, the λ-weight space of V, V_λ is defined by:

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v, \forall\, h \in \mathfrak{h}\}$$

If $V_\lambda \neq 0$, λ is called a weight of V.

Definition 1.1. A highest weight for a representation V of $\mathfrak{g}$ is a weight λ such that for all weights μ of V, we have $\mu \leq \lambda$.

It is clear from the definition that a highest weight, if exists, is unique, so we may speak of 'the' highest weight of a representation. If λ is the highest weight of V, then any nonzero v in V_λ is called a highest weight vector of V. Further, if V is generated as a $\mathfrak{g}$-module by a highest weight vector, then V is called a highest weight module.

Definition 1.2. A weight vector $w \in V$ (of weight λ, say) such that $\mathfrak{g}_\alpha \cdot w = 0$ for all $\alpha \in \Phi^+$ is called a maximal vector and λ will be called a maximal weight.

Definition 1.3. Let $\lambda \in \mathfrak{h}^*$. λ is called integral if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$, $\forall\, \alpha \in \Delta$. λ is called dominant if $\langle \lambda, \alpha \rangle \geq 0$, $\forall$ $\alpha \in \Phi^+$ or equivalently, $\forall\, \alpha \in \Delta$.

We can now state the theorem:

Theorem 1.4. (i) If V is a finite dimensional irreducible $\mathfrak{g}$-module, then V has a highest weight, which is dominant and integral. In this case, V is a highest weight module.

(ii) If V, V' are finite dimensional irreducible $\mathfrak{g}$-modules with the same highest weight, then they are isomorphic.

(iii) Each dominant integral element of $\mathfrak{h}^*$ is the highest weight of a finite dimensional irreducible $\mathfrak{g}$-module.

This explains the statement that the irreducible finite dimensional $\mathfrak{g}$-modules are classified according to their highest weight. Of course, it should be remembered that all of these definitions depend on our initial choices of $\mathfrak{h}$ and Φ^+.

We end the section with a simple proposition (without proof) and a useful corollary:

Proposition 1.5. For a $\mathfrak{g}$-module V,

(i) For all $\beta, \lambda \in \mathfrak{h}^*$, $\mathfrak{g}_\beta \cdot V_\lambda \subset V_{\beta+\lambda}$.

(ii) The sum $\sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ is direct and is an $\mathfrak{g}$-submodule of V.

(iii) If $\dim V < \infty$, then $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$.

Note that (i) implies that a highest weight vector is maximal but the converse is not true. But the weaker notion of maximal vector is useful and we will use the following corollary.

Corollary 1.6. If $\dim V < \infty$, then V has a maximal weight.

Proof. By (iii) of 1.5, there are only finitely many weights in the direct sum decomposition, so there exists a maximal weight. □

2 Verma Modules

In this section, we define for each $\lambda \in \mathfrak{h}^*$ the highest weight modules $\Delta(\lambda)$, called the Verma modules (named after Dayanand Verma who introduced them in his thesis). These will be defined by an induced module construction.

Let U be the functor from the category of Lie algebras to the category of associative algebras, taking a Lie algebra to its universal enveloping algebra. Let $\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha\right)$ and $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}$ be the Borel and nilpotent subalgebras of $\mathfrak{g}$ associated to Φ^+. Hence as vector spaces, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{b}$. By the Poincare-Birkhoff-Witt (PBW) theorem, the induced maps $U(\mathfrak{b}) \to U(\mathfrak{g})$ and $U(\mathfrak{n}) \to U(\mathfrak{g})$ are injective.

Proposition 2.1. The injections above give an isomoprhism

$$U(\mathfrak{g}) \cong U(\mathfrak{n}) \otimes_{\mathbb{C}} U(\mathfrak{b})$$

as vector spaces as well as $(U(\mathfrak{n}), U(\mathfrak{b}))$ bimodules.

Since $\mathfrak{b}$ is solvable, any finite dimensional irreducible $\mathfrak{b}$-module must be one dimensional (by Lie's theorem), so equals a single weight space of weight λ say, $\mathbb{C}_\lambda$ with action $h \cdot 1 = \lambda(h)$ and $\mathfrak{g}_\alpha \cdot \mathbb{C}_\lambda = 0$ for $\alpha \in \Phi^+$. Conversely, for any $\lambda \in \mathfrak{h}^*$, we define the one dimensional $\mathfrak{b}$-module $\mathbb{C}$ by $h \cdot 1 = \lambda(h)$ and $\mathfrak{g}_\alpha \cdot \mathbb{C} = 0$ for $\alpha \in \Phi^+$. It is easy to check that this defines a $\mathfrak{b}$-module structure on $\mathbb{C}$ and hence, is $\mathbb{C}_\lambda$.

Definition 2.2. The Verma module $\Delta(\lambda)$ associated to λ is defined by:

$$\Delta(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

Set $v_\lambda = 1 \otimes 1$. It is called the canonical generator of $\Delta(\lambda)$. The following proposition explains why.

Proposition 2.3. $\mathrm{Res}_{\mathfrak{n}}^{\mathfrak{g}} \Delta(\lambda)$ is a free $U(\mathfrak{n})$-module of rank 1 by the isomorphism $n \to nv_\lambda$. In particular, $\Delta(\lambda)$ is infinte dimensional.

Proof. As a $U(\mathfrak{n})$-module, we have the isomorphisms

$$\Delta(\lambda) \cong (U(\mathfrak{n}) \otimes_{\mathbb{C}} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \cong U(\mathfrak{n}) \otimes_{\mathbb{C}} \left(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda\right) \cong U(\mathfrak{n}) \otimes_{\mathbb{C}} \mathbb{C}_\lambda \cong U(\mathfrak{n})$$

where we have used the associativity of the tensor product and the isomorphism is clear from the above steps. □

Proposition 2.4. $\Delta(\lambda)$ has the weight space decomposition

$$\Delta(\lambda) = \bigoplus_{\mu\leq\lambda} \Delta(\lambda)_\mu$$

Further, all the weight spaces are finite dimensional and the λ-weight space is the one dimensional space $\mathbb{C}v_\lambda$. $\Delta(\lambda)$ is a highest weight module of weight λ.

Proof. Fix an ordering of the positive roots $\alpha_1, \alpha_2, \cdots, \alpha_r$ and for each i pick nonzero $y_i \in \mathfrak{g}_{-\alpha_i}$. By the PBW theorem, a basis for $U(\mathfrak{n})$ is given by monomials of the form $y_1^{a_1} y_2^{a_2} \ldots y_r^{a_r}$ with $a_i \in \mathbb{Z}_{\geq 0}$. So by 2.3, a basis for $\Delta(\lambda)$ is given by the elements $(y_1^{a_1} y_2^{a_2} \ldots y_r^{a_r})v_\lambda$.

For $h \in \mathfrak{h}$,

$$hv_\lambda = h(1\otimes 1) = h \otimes 1 = 1 \otimes (h.1) = 1 \otimes \lambda(h) = \lambda(h)(1 \otimes 1) = \lambda(h)v_\lambda$$

where the equality $h \otimes 1 = 1 \otimes (h.1)$ is because $U(\mathfrak{h})$ injects into $U(\mathfrak{b})$. Hence, $v_\lambda \in \Delta(\lambda)_\lambda$. By (i) of 1.5, the basis element $(y_1^{a_1} y_2^{a_2} \ldots y_r^{a_r})v_\lambda$ belongs to $\Delta(\lambda)_{\lambda-\sum_{i=1}^r a_i\alpha_i}$. This shows that all weights are $\leq \lambda$ and also shows that the λ-weight space is one dimensional. In general, all weight spaces are finite dimensional, as for a fixed $\mu \leq \lambda$, there are only finitely many ways to reach μ from λ by subtracting positive roots. □

Verma modules have a universal property. Let V be a $\mathfrak{g}$-module with a maximal vector v^+ of weight λ. The map from $\mathbb{C}_\lambda$ to V given by $1 \to v^+$ is a $\mathfrak{b}$-module homomorphism by the maximality of λ. By the universal property of the tensor product (Frobenius reciprocity), we get a $\mathfrak{g}$-module homomorphism from $\Delta(\lambda)$ to V such that v_λ maps to v^+. If V happens to be a highest weight module of weight λ, then the above map from $\Delta(\lambda)$ to V is surjective, i.e., V is a quotient of $\Delta(\lambda)$. To summarize, highest weight modules are quotients of Verma modules.

Remark: The Verma modules may be described in an alternative way as follows. For $\lambda \in \mathfrak{h}^*$, let $I_\lambda \subset U(\mathfrak{g})$ be the left ideal generated by the elements of $\mathfrak{b}$ together with $h - \lambda(h)$, $h \in \mathfrak{h}$ (The generators of I_λ are elements which kill v_λ). Then $\Delta(\lambda) \cong {U(\mathfrak{g})}/{I_\lambda}$. It is easy to show this isomorphism from the universal property described above.

3 Irreducible modules from the Verma modules

We will now show that an irreducible module of highest weight λ can be constructed as a quotient of the Verma module $\Delta(\lambda)$. The following lemma is the crucial step.

Lemma 3.1. Let M be a proper submodule of $\Delta(\lambda)$. Then $M = \bigoplus_{\mu<\lambda} M_\mu$ (Note the absence of the weight λ). In particular, $M_\lambda = 0$ and $M \subset \bigoplus_{\mu<\lambda} \Delta(\lambda)_\mu$.

Proof. Let $x \in M$. Then $x = x_1 + x_2 + \cdots x_n$ with $x_i \in \Delta(\lambda)_{\mu_i}$ for some pairwise distinct weights μ_i. We need to show that the x_i are infact in M_{μ_i}. Suppose not, that is there is an x in M with some x_i in its decomposition not in M_{μ_i}. We may assume that the length n of the decompostion is minimal. So $n \geq 2$ and x_i is not in M for all i. Applying $h \in \mathfrak{h}$, we get

$$hx = \mu_1(h)x_1 + \cdots \mu_n(h)x_n$$

and hence,

$$hx - \mu_1(h)x = (\mu_2(h) - \mu_1(h))x_2 + \cdots + (\mu_n(h) - \mu_1(h))x_n$$

Since $hx - \mu_1(h)x$ is in M, by the minimality of n, we have $(\mu_2(h) - \mu_1(h))x_2 \in M$ for all $h \in \mathfrak{h}$. Choosing h such that $\mu_2(h) \neq \mu_1(h)$ gives $x_2 \in M$, a contradiction. This proves that M is a direct sum of its weight spaces. The inclusions

$$(0) \subseteq M_\lambda \subseteq \Delta(\lambda)_\lambda = \mathbb{C}v_\lambda$$

imply that if $M_\lambda \neq (0)$, $v_\lambda \in M_\lambda \subset M \implies U(\mathfrak{g})v_\lambda = \Delta(\lambda) \subset M$, a contradiction as M is proper. This proves the lemma. □

Theorem 3.2. $\Delta(\lambda)$ has a unique maximal submodule $\mathrm{rad}(\Delta(\lambda))$. The quotient

$$L(\lambda) := {}^{\Delta(\lambda)}\!/_{\mathrm{rad}(\Delta(\lambda))}$$

is an irreducible highest weight module of weight λ.

Proof. Let $\mathrm{rad}(\Delta(\lambda))$ be the sum of all proper submodules of $\Delta(\lambda)$. By 3.1, $\mathrm{rad}(\Delta(\lambda)) \subset \bigoplus_{\mu<\lambda} \Delta(\lambda)_\mu$ and it is clear that this is the unique maximal submodule of $\Delta(\lambda)$. In $L(\lambda)$, the image $\overline{v_\lambda}$ of v_λ is nonzero and hence $L(\lambda)$ is an irreducible highest weight module of weight λ. □

Theorem 3.3. If V is an irreducible $\mathfrak{g}$-module with a maximal weight λ, then $V \cong L(\lambda)$. In particular, a maximal weight of an irreducible module, if it exists, is unique.

Proof. There is a nonzero homomorphism of modules $\Pi : \Delta(\lambda) \to V$ from the universal property of $\Delta(\lambda)$. Π is surjective as V is simple. So V is isomorphic to a quotient of $\Delta(\lambda)$ by a maximal submodule (because V is simple). Now 3.2 shows that $V \cong L(\lambda)$. □

Theorem 3.3 proves part (ii) of 1.4 without the finite dimensionality assumption. We also deduce from 1.6 and 3.3 that a finite dimensional irreducible module is a highest weight module.

4 Finite dimensional irreducible modules

The task now is to determine which of the irreducibles, $L(\lambda)$ constructed in Section 3 are finite dimensional. Representation theory of $\mathfrak{sl}(2, \mathbb{C})$ will play an important role in this.

For each positive root α, we know that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional and that one can choose $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ so that $\alpha(h_\alpha) = 2$. Then $\mathfrak{g}_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. It will be denoted by $\mathfrak{sl}(\alpha)$.

Theorem 4.1. If V is a finite dimensional irreducible $\mathfrak{g}$-module, then V has a highest weight, which is dominant and integral.

Proof. From 1.6, V has a maximal weight, say λ and 3.3 shows that $V \cong L(\lambda)$. For each positive root α, V is also an $\mathfrak{sl}(\alpha)$-module and λ remains a maximal weight. By representation theory of $\mathfrak{sl}(2, \mathbb{C})$, a maximal vector w has non-negative integral weight: $h_\alpha w = mw$ for some $m \in \mathbb{Z}_{\geq 0}$. But

$$h_\alpha w = \lambda(h_\alpha)w = \langle \lambda, \alpha \rangle w \implies \langle \lambda, \alpha \rangle = m \in \mathbb{Z}_{\geq 0}$$

This proves the theorem. □

The converse direction is more involved.

Theorem 4.2. If $\lambda \in \mathfrak{h}^*$ is dominant and integral, $L(\lambda)$ is finite dimensional.

Proof. Note that $L(\lambda)$ is a direct sum of finite dimensional weight spaces (since $\Delta(\lambda)$ is). The strategy will be to show that the set of weights of $L(\lambda)$ is finite and this will prove the theorem.

Let $\alpha_1, \cdots, \alpha_l$ be the simple roots as before and for each i, let $\{x_i, y_i, h_i\}$ be an $\mathfrak{sl}(\alpha_i)$ triple. We'll use the following identities in $U(\mathfrak{g})$ for $k \in \mathbb{Z}_{\geq 0}$:
(i) $[x_j, y_i^{k+1}] = 0$ for $i \neq j$ (ii) $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k - h_i)$

Step I : For each i, $L(\lambda)$ contains a nonzero finite dimensional $\mathfrak{sl}(\alpha_i)$-module.
Proof. Let w be a maximal vector in $L(\lambda)$. Note that $m_i = \langle \lambda, \alpha_i \rangle = \lambda(h_i)$ is a positive integer as λ is dominant integral. We claim that $u = y_i^{m_i+1}w = 0$. This is because by identity (i), for $j \neq i$,

$$x_j u = y_i^{m_i+1}(x_j w) + [x_j, y_i^{m_i+1}]w = 0$$

and by identity (ii),

$$x_i u = y_i^{m_i+1}(x_i w) + [x_i, y_i^{m_i+1}]w = -(m_i+1)y_i^{m_i}(m_i - h_i)w = 0$$

as $h_i w = m_i w$. If $u \neq 0$, then u would be a maximal vector of weight $\lambda - (m_i+1)\alpha_i < \lambda$, a contradiction because the maximal weight is unique (Theorem 3.3). So $W = \text{span}\{w, y_i w, \cdots, y_i^{m_i} w\}$ is a nonzero finite dimensional $\mathfrak{sl}(\alpha_i)$-module in $L(\lambda)$ (The containments $h_i W \subseteq W$, $y_i W \subseteq W$ are obvious and $x_i W \subseteq W$ follows from (ii) again).

Step II : For each fixed i, $L(\lambda)$ is the sum of all finite dimensional $\mathfrak{sl}(\alpha_i)$-modules contained in it.
Proof. Let E be the sum of all finite dimensional $\mathfrak{sl}(\alpha_i)$-modules contained in $L(\lambda)$. We show that E is a $\mathfrak{g}$-submodule of $L(\lambda)$. Along with step I that $E \neq 0$, this forces $E = L(\lambda)$. So, for $x \in \mathfrak{g}$ and $w \in E$, we need to show that $xw \in E$. Since $w \in E$, $w \in F$ for some finite dimensional $\mathfrak{sl}(\alpha_i)$-module F. If $x = \sum_{\beta \in \phi \cup \{0\}} x_\beta$, $x_\beta \in \mathfrak{g}_\beta$, then $xw \in \text{span}_\beta\{x_\beta F\} = K$. But K is finite dimensional and easily seen to be $\mathfrak{sl}(\alpha_i)$-invariant ($x_i K \subset \text{span}_\beta\{x_i x_\beta F\}$ and $x_i x_\beta F = x_\beta(x_i F) + [x_i, x_\beta]F \subset x_\beta F + \mathfrak{g}_{\alpha_i+\beta} \cdot F \subset K$, similarly one shows this for y_i, h_i). So, $xw \in K \subset E$ which ends the proof.

If $\Pi(\lambda)$ is the set of weights of $L(\lambda)$, we now show that the Weyl group $\mathcal{W}$ acts on $\Pi(\lambda)$ by permutations. If this is known, $\Pi(\lambda)$ decomposes into a disjoint union of $\mathcal{W}$-orbits. Then it is enough to show that there are only finitely many orbits: this coupled with the fact that $\mathcal{W}$ is finite will show that $\Pi(\lambda)$ is finite.

Step III : $\mu \in \Pi(\lambda) \implies \sigma_i\mu \in \Pi(\lambda)$ where σ_i is the reflection corresponding to α_i. Furthermore, $\dim L(\lambda)_\mu = \dim L(\lambda)_{\sigma_i\mu}$.
Proof. Since $L(\lambda)_\mu$ is finite dimensional, by Step II, there is a finite dimensional $\mathfrak{sl}(\alpha_i)$-module F containing $L(\lambda)_\mu$. For $0 \neq w \in L(\lambda)_\mu$, on one hand, we have $h_i w = \mu(h_i)w$. But $w \in F$ is then a weight vector for h_i so by representation theory for $\mathfrak{sl}(2, \mathbb{C})$, $\mu(h_i) = \langle \mu, \alpha_i \rangle = m$ is an integer (This also proves that all weights are integral, as the preceeding argument holds for any i). Because m occurs as a weight of F, so does $-m$ and $\dim F_m = \dim F_{-m}$. If m is nonnegative, then $y_i^m w \neq 0$ and belongs to F_{-m}. But $y_i^m w \in L(\lambda)_{\mu - \langle \mu, \alpha_i \rangle \alpha_i}$ as well by (i) of 1.5, i.e, $\sigma_i\mu$ is also a weight. If m is negative, take $x_i^{-m}w$, the argument is similar. The equality of dimensions follows because if $\{w_1, \cdots, w_k\}$ is a basis for $L(\lambda)_\mu$, then $\{w_1, \cdots, w_k\}$ are linearly independent in F_m, so applying y_i^m or x_i^{-m} leaves them linearly independent and sends them to $L(\lambda)_{\mu - \langle \mu, \alpha_i \rangle \alpha_i}$ so $\dim L(\lambda)_\mu \leq \dim L(\lambda)_{\sigma_i\mu}$. Applying the same to $\sigma_i\mu$ gives $\dim L(\lambda)_{\sigma_i\mu} \leq \dim L(\lambda)_{\sigma_i^2\mu} = \dim L(\lambda)_\mu$ which gives the equality.

Step IV : For $\mu \in \Pi(\lambda)$, its Weyl orbit $\mathcal{W}\mu$ contains a dominant weight.
Proof. The orbit $\mathcal{W}\mu$ is finite, so we can choose $\eta \in \mathcal{W}\mu$ maximal wrt the ordering $\leq$. Then η is dominant. For if not, $\langle \eta, \alpha_i \rangle \in \mathbb{Z}_{<0}$ for some i. Then $\sigma_i\eta \in \mathcal{W}\mu$ and $\sigma_i\eta = \eta - \langle \eta, \alpha_i \rangle \alpha_i > \eta$, contradicting the choice of η.

Step V : The set $S = \{\eta \mid \eta \text{ is dominant}, \eta \leq \lambda\}$ is finite.
Proof. If η is in the above set, $\lambda - \eta$ is a sum of positive roots with nonnegative integer coefficients, hence η lies in a discrete set. Also, $\lambda + \eta$ is dominant, so $\langle \lambda + \eta, \alpha_i \rangle \geq 0$ for all i, in particular $(\lambda + \eta, \lambda - \eta) \geq 0 \implies (\lambda, \lambda) \geq (\eta, \eta)$. So η lies in a compact set as well (in the closed ball of radius $\sqrt{(\lambda, \lambda)}$). So S is in the intersection of a discrete set and a compact set, a finite set.

From Step IV, any $\mathcal{W}$-orbit of $\Pi(\lambda)$ contains an element of S. Step V implies that the number of orbits is finite and this finishes the proof. $\square$

References

[1] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1980.