Bott-Samelson varieties

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1 The flag manifold

1.1 G/B

Let $G = GL_n(\mathbb{C}), B, B^- =$ upper/lower triangular matrices in G. Instead of matrix Schubert varieties

$$\overline{B^- \cdot \Pi \cdot B} \subseteq \mathrm{Mat}_{n\times n}$$

(where Π is some partial permutation matrix), today we'll think about *(opposite) Schubert varieties*

$$\overline{BwB/B} \subseteq G/B.$$

It might not seem like a big deal to change the perspective this way, but it should be pointed out that G/B is *much* better for many purposes than $\mathrm{Mat}_{n\times n}$, e.g. it is projective, it knows about all other projective homogeneous spaces of G, it can easily be generalized to other G's, studying its geometry is intimately connected to the representation theory of G, and so on. The only real argument for working in $\mathrm{Mat}_{n\times n}$ is that it is $\mathbb{C}^{n^2}$, so we can write down equations for stuff, and has an enormous torus acting on it, so we can Gröbner degenerate matrix Schubert varieties.

1.2 (One) Motivation for studying G/B and its Schubert varieties

Let $G \to GL(V)$ be an irrep (since G is reductive, all reps are direct sums of these). Then G acts on $\mathbb{P}(V)$. So B acts on $\mathbb{P}(V)$ via the G-action. Since $\mathbb{P}(V)$ is projective, it is complete. As B is solvable, by Borel's fixed point theorem, there is a fixed point $[v] \in \mathbb{P}(V)$. We may pick a representative $v \in V$ for $[v]$, which is going to be a (highest) weight vector for B, i.e. $\forall b \in B, b \cdot v = \lambda(b)v$, where $\lambda : B \to \mathbb{C}^\times$ is a group homomorphism. That, and the fact that $\mathbb{C}^\times$ is abelian, means that λ factors through the abelianization $T = B/[B,B]$, so in fact,

$$\lambda \in T^* = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^\times).$$

Now, if we have any weight $\mu \in T^*$, as $B = T \ltimes [B,B]$, $\forall b \in B, \exists!(t,n) \in T \times [B,B]$ s.t. $b = tn$. Now we can just define $\mu(b) = \mu(t)$, and the 1-dimensional B-representation $\mathbb{C}_\mu$, where $\forall b \in B, z \in \mathbb{C},\ b \cdot z = \mu(b)z$. Then the space

$$L_\mu = G \times_B \mathbb{C}_{-\mu} = (G \times \mathbb{C}_{-\mu})/\{(g,v) \sim (gb, b^{-1}\cdot v)\ \forall b \in B\}$$

is the total space of a line bundle over G/B. I cannot stress enough how wonderful the following theorem really is:

Theorem 1.1. *(Borel-Weil) Let V be an irrep of G with highest weight λ. Then*

$$V \cong H^0(G/B, L_{w_0\cdot\lambda})$$

as representations of G.

To motivate the study of Schubert varieties $X^w \subseteq G/B$, we remark that if $X^{w_0 s_i}$ is a Schubert divisor, and L_i is the line bundle associated to it, then

$$L_i = L_{\omega_i}$$

where ω_i is the i-th fundamental weight. This means that if λ is any weight, then $\lambda = \sum_{i=1}^{l} \langle \lambda, \alpha_i \rangle \omega_i$, and

$$L_\lambda = \bigotimes_{i=1}^{l} L_{\omega_i}^{\langle \lambda, \alpha_i \rangle}$$

i.e. Schubert varieties are super-important.

1.3 Flags in $\mathbb{C}^n$

So let's try to find a more tangible interpretation for G/B when $G = GL_n$. Note that an invertible $n \times n$ matrix g is basically an ordered basis $\{v_1 \ldots, v_n\}$ (v_i is the i-th column of g) for $\mathbb{C}^n$. When we have an ordered basis, we can consider the *flag* associated to it:

$$\mathcal{F}_g = \{0\} \subset \mathrm{Span}(v_1) \subset \mathrm{Span}(v_1, v_2) \subset \ldots \subset \mathrm{Span}(v_1, \ldots, v_n) = \mathbb{C}^n.$$

Now a question you might ask is what ordered bases give you the same flag? Since G acts on itself transitively, this is the same as computing $\mathrm{Stab}_G(\mathcal{F})$ for any flag, so we can take

$$\mathcal{F} = \{0\} \subset \mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \subset \ldots \subset \mathrm{Span}(e_1, \ldots, e_n) = \mathbb{C}^n$$

where $e_1, \ldots, e_n$ are the standard basis vectors for $\mathbb{C}^n$. It is not hard to see that $\mathrm{Stab}_G(\mathcal{F}) = B$. Also, we can certainly pick a basis for any flag and use it as columns of an invertible matrix so we have a bijection

$$\mathrm{Flags}(\mathbb{C}^n) \leftrightarrow G/B.$$

2 Schubert varieties

2.1 In $\mathrm{Flags}(\mathbb{C}^n)$

Now we will look at the analogs of matrix Schubert varieties in G/B. We define (for a permutation matrix w)

$$X^w = \overline{BwB/B} \subseteq G/B.$$

Let's try to see what this is in $\mathrm{Flags}(\mathbb{C}^n)$. For any permutation w, we have a special flag, namely

$$\mathcal{F}_w = \{0\} \subset \mathrm{Span}(e_{w(1)}) \subset \mathrm{Span}(e_{w(1)}, e_{w(2)}) \subset \ldots \subset \mathrm{Span}(e_{w(1)}, \ldots, e_{w(n)}) = \mathbb{C}^n.$$

By the same trick (upward row and rightward column operations) as with the matrix Schubert varieties, we see that a general flag $\mathcal{F} = (\{0\} \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n)$ lies in the B-orbit of exactly one $\mathcal{F}_w$, and the B-orbit of a $\mathcal{F}_w$ is the Bruhat cell X_o^w. Now how do we identify which Bruhat cell our flag $\mathcal{F}$ lies in? Using the definition of $\mathcal{F}_w$, we see that $\mathcal{F} \in X_o^w$ if and only if

$$\dim(V_i \cap \mathbb{C}^k) = |\{w(1), w(2), \ldots, w(i)\} \cap \{1, 2, \ldots k\}|.$$

And just like with the matrix Schubert varieties, $\mathcal{F} \in X^w$ if and only if

$$\dim(V_i \cap \mathbb{C}^k) \leq |\{w(1), w(2), \ldots, w(i)\} \cap \{1, 2, \ldots k\}|.$$

2.2 Singularities

Schubert varieties are closures of affine spaces in a projective space, there is no reason why they should be smooth, and:

Proposition 2.1. *Let $G = GL_n(\mathbb{C})$. The Schubert variety X^w (for $w \neq w_0$) is smooth if and only if w avoids 3412 and 4231. i.e. $\nexists(1 \leq i < j < k < l \leq n)$ s.t.*

$$w(k) < w(l) < w(i) < w(j) \text{ or } w(l) < w(j) < w(k) < w(i)$$

So they are all smooth for $n \leq 3$, there are only a couple singular ones in $GL_4(\mathbb{C})$, but for larger n, they are mostly singular. Their singularities have been subject to extensive study, to the extent that there is a (very good) book titled "Singular Loci of Schubert Varieties" by Billey and Lakshmibai.

3 The Bott-Samelson(-Demazure-Hansen) resolution

3.1 Definition

One thing that geometers like to do with singular varieties is to desingularize them. In general, if we have a singular variety X, and a smooth variety $\widetilde{X}$ with a birational map (roughly speaking, an isomorphism of open sets) $r : \widetilde{X} \to X$, then we say that $\widetilde{X} \to X$ is a resolution of singularities of X.[1] Bott-Samelson varieties provide natural resolutions of singularities to Schubert varieties and also have other applications. They were introduced independently by Demazure and Hansen, and Demazure called them Bott-Samelson varieties, hence the title of the section. They are defined (for $G = GL_n(\mathbb{C})$) for a word $Q = (s_{\alpha_1}, \ldots, s_{\alpha_n})$ in the simple reflections generating W. If $|Q| = k$, then BS^Q is a k-tuple of flags $(V^0, \ldots, V^k)$ satisfying certain incidence conditions. We define BS^Q inductively as follows:

$$V^0 = \left(\{0\} \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n\right),$$

and if

$$V^i = \left(\{0\} \subset V_1^i \subset \ldots \subset V_n^i = \mathbb{C}^n\right),$$

then V^{i+1} is obtained from V^i be replacing the α_i-dimensional subspace $V_{\alpha_i}^i$ (note: $1 \leq \alpha_i \leq n-1$) of V^i by a new one $V_{\alpha_i}^{i+1}$ contained in $V_{\alpha_i+1}^i$ and containing $V_{\alpha_i-1}^i$ (sic). The picture should clarify this.

3.2 The structure of BS^Q

Notice that we have a forgetful map $BS^Q \to BS^{Q\setminus\text{last}}$ that takes

$$\pi : (V^0, \ldots, V^{k-1}, V^k) \mapsto (V^0, \ldots, V^{k-1}).$$

Note that V^k only differs from V^{k-1} by a choice of a point in

$$\left(V_{\alpha_{k-1}+1}^{k-1} / V_{\alpha_{k-1}-1}^{k-1}\right) \cong \mathbb{CP}^1$$

So each fiber of π is a $\mathbb{CP}^1$, and by induction, BS^Q is an iterated $\mathbb{CP}^1$-bundle. Therefore it is connected, smooth, projective, and irreducible.

[1] The morphism should be proper, and the map should be an isomorphism away from the singular locus of X, the map $BS^Q \to X^w$ generally has a larger ramification locus than $\mathrm{Sing}(X^w)$.

Remark 3.1. *(Fun fact) This makes it really easy to compute the cohomology of BS^Q, by the Leray-Hirsch and binomial theorems:*

$$\dim_{\mathbb{R}}(H^{2i}(BS^Q, \mathbb{R})) = \binom{k}{i}.$$

3.3 Map to G/B

There is an easy map from BS^Q to G/B, namely,

$$\begin{aligned} m : & BS^Q \to G/B \\ & (V^0, \dots, V^k) \mapsto V^k. \end{aligned}$$

We will see a little later why m is B-equivariant. If Q is an arbitrary word, then

$$m : BS^Q \twoheadrightarrow X^{\mathrm{Dem}(Q)},$$

where Dem is the Demazure/nil Hecke product. For any word Q, $\mathrm{Dem}(Q)$ is the unique maximum (in Bruhat order) of

$$\prod_{i \in S} s_{\alpha_i}$$

for $S \subseteq Q$. It is probably of combinatorial interest that Dem exists, and makes W into a monoid, so there is one more reason why BS^Q's are really awesome.

Also, the image of m in G/B must be something B-invariant, irreducible and closed, so it must be some X^v, in particular, if Q is a reduced word for $w \in W$, i.e. $\prod_{i=1}^k s_{\alpha_i} = w$ and k is minimal, then

$$m : BS^Q \twoheadrightarrow X^w,$$

since $wB/B \in \mathrm{Im}(m)$, and the dimensions match. In this case, m is also generically one-to-one.

3.4 For $G \neq GL_n(\mathbb{C})$

Bott-Samelson varieties exist for other groups as well, let P_i denote the minimal parabolic associated to s_i, and let $Q = (s_{\alpha_1}, \dots, s_{\alpha_k})$. Then

$$BS^Q = P_{\alpha_1} \times P_{\alpha_2} \times \dots \times_B P_{\alpha_k}/B^k$$

where the action of B^n is defined as

$$(b_1, \dots b_k) \cdot (p_1, \dots, p_k) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, p_{k-1}^{-1} b_k p_k).$$

The map to the flag variety G/B is

$$m(p_1, \dots, p_k) = p_1 p_2 \cdots p_k.$$

and now we see that m is obiviously B-equivariant. Essentially everything is true in the general case that is true the $GL_n(\mathbb{C})$-case, the Bott-Samelsons still desingularize Schubert varieties, and they are still iterated $\mathbb{CP}^1$-bundles.