Flag varieties, Bott-Samelson varieties
GRT learning seminar

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1 Flag varieties

1.1 Notational conventions

Let G be a semisimple algebraic group over $\mathbb{C}$. Fix a Borel (maximal solvable) subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ denote the respecive Lie algebbras. These choices determine a weight lattice $P \subset \mathfrak{h}^*$ and a root system $\Delta \subset P^*$ with a set of positive roots Δ^+ and a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. They also determine a Weyl group $W = N_G(T)/T$ with a given set of generators s_i and length function $l : W \to \mathbb{N}$.

Example 1.1. *The main example to have in mind is $G = SL_n(\mathbb{C})$, with B upper triangular matrices, and T diagonal matrices. Here the weight lattice is $P = \{\vec{x} \in \mathbb{Z}^n | \vec{x} \cdot (1, 1, \ldots, 1)^T = 0\}$, the set of roots is $\{e_i - e_j | i \neq j\}$, where e_i is the i-th standard basis vector. The positive roots are $\Delta^+ = \{e_i - e_j | i < j\}$ and the simple roots are $\Pi = \{e_i - e_{i+1}\}$. The Weyl group W is the symmetric group S_n.*

1.2 Introduction

We are interested in the **flag variety** G/B of G. Since B is a closed subgroup, this is a smooth variety with a transitive G-action.

Example 1.2. *For $G = SL_n(\mathbb{C})$, G/B is the **variety of complete flags** in $\mathbb{C}^n$*

$$\{0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = \mathbb{C}^n | \dim F_i = i\}.$$

To see this, notice that G/B is isomorphic to $\mathcal{B}$, the variety of all Borel subgroups via

$$gB/B \mapsto gBg^{-1}$$

and the stabilizer of a complete flag is a Borel subgroup. Under this identification, the point B/B corresponds to the Borel subgroup B and to the base flag $\{0 \subset \mathrm{Span}\{e_1\} \subset \mathrm{Span}\{e_1, e_2\} \subset \ldots \subset \mathrm{Span}\{e_1, \ldots, e_{n-1}\} \subset \mathbb{C}^n\}$.

The flag variety has a T-action (since $T \subset G$).

Proposition 1.3. *The T-fixed points in G/B are in bijection with the Weyl group, more precisely, we have*

$$(G/B)^T = \{\dot{w}B/B\}_{w \in W},$$

where $\dot{w}$ denotes a representative of an element of $W = N_G(T)/T$ in G.

Example 1.4. *For $G = SL_n(\mathbb{C})$, the T-fixed flags are precisely the coordinate flags*

$$\left\{0 = F_0 \subset \mathrm{Span}\, e_{w(1)} \subset \mathrm{Span}\{e_{w(1)}, e_{w(2)}\} \subset \ldots \subset \mathrm{Span}\{e_{w(1)}, \ldots, e_{w(n-1)}\} \subset \mathbb{C}^n\right\}.$$

The flag variety is also a projective variety, which means that we can gain a lot of leverage on it by looking at its T-moment map image (see Figure 1) which is know to be given by the convex hull of the images of the T-fixed points.

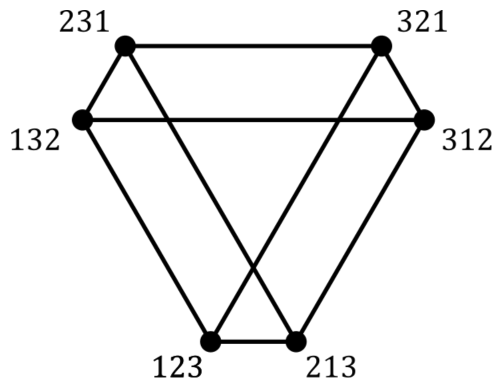


Figure 1: The moment map image of $SL_3(\mathbb{C})$'s flag manifold

1.3 The Bruhat decomposition

The sets $X^w_o = BwB/B$ are called **Bruhat cells**. They are cells in the sense of algebraic topology, i.e. $X^w_o \cong \mathbb{C}^{l(w)}$. Their closures $X^w = \overline{X^w_o}$ are called **Schubert varieties**.

Theorem 1.5 (Bruhat decomposition)**.** *The Bruhat cells form a cell decomposition of* G/B*, i.e.*

$$G/B = \bigsqcup_{w \in W} X^w_o.$$

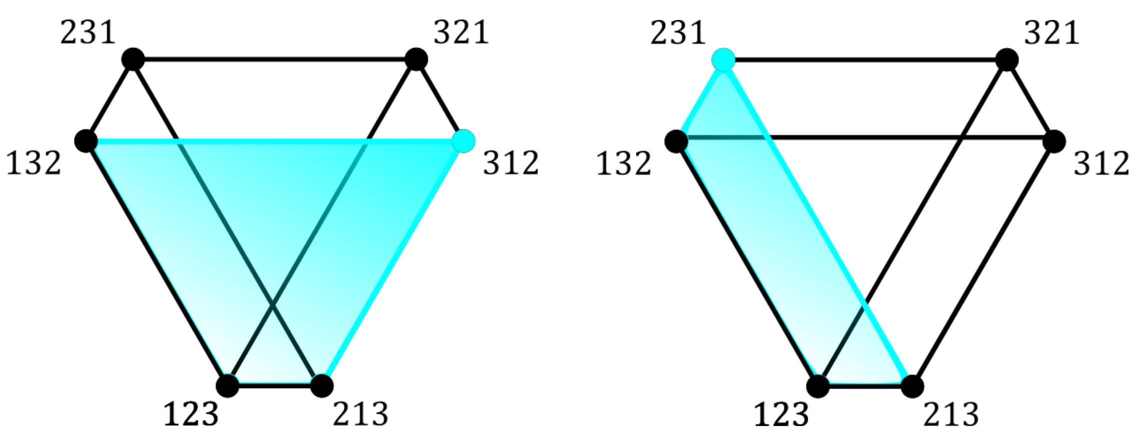
Moreover, any Schubert variety is a union of Bruhat cells, and the closure relations define a partial ordering on W*, called the **Bruhat order***

$$X^w = \bigsqcup_{v \leq w} X^v_o.$$

Example 1.6. *For* $G = SL_n(\mathbb{C})$*, the* B*-orbit of a standard basis vector* e_k *is*

$$\left\{ c_k e_k + \sum_{i=1}^{k-1} c_i e_i \;\middle|\; c_k \neq 0 \right\},$$

in particular, if we start at a coordinate flag wB/B*, and apply elements of* B*, we can get arbitrarily close to other coordinate flags where some of the inversions of the permutation* w *are eliminated, i.e. where instead of the standard basis* $e_{w(i)}$ *vector occuring at step* i *of the flag, any of the standard basis vectors* e_k *with* $k \leq w(i)$ *occurs instead (with* $e_{w(i)}$ *occuring later). See Figure 2 for an example in* $SL_3(\mathbb{C})$*.*



(a) X^{312} (b) X231

Figure 2: Two Bruhat cells in $SL_3(\mathbb{C})/B$.

If we take closures, these points are added, and we see that the Bruhat order then has the description that $v \leq w$ *if for all* $i = 1, \ldots, n$*,*

$$\mathrm{sort}(v(1), v(2), \ldots, v(i)) \leq \mathrm{sort}(w(1), w(2), \ldots, w(i)),$$

and the $\leq$ *stands for comparing sequences entry-wise.*

If $G = SL_n(\mathbb{C})$, given a flag F, we can decice which Schubert cell it belongs to by looking at the $(n-1) \times (n-1)$ **rank matrix** whose (i,j)-th entry is $\dim(F_i \cap \mathrm{Span}\{e_1, \dots, e_j\}$, and comparing it to the rank matrices of the coordinate flags.

Example 1.7. *The flag* $F = (0 \subset \mathrm{Span}\{e_1 + e_3\} \subset \mathrm{Span}(e_1 + e_3, e_1) \subset \mathbb{C}^3$ *has rank matrix*

	$\mathrm{Span}\{e_1\}$	$\mathrm{Span}\{e_1, e_2\}$
$\mathrm{Span}\{e_1 + e_3\}$	0	0
$\mathrm{Span}(e_1 + e_3, e_1)$	1	1

The coordinate flag corresponding to the permutation 312 *has the same rank matrix, so* $F \in X_o^{312}$.

2 Bott-Samelson varieties

2.1 Motivation: Desingularizations of Schubert varieties

Schubert varieties are in general singular.

Example 2.1. *For* $G = SL_n(\mathbb{C})$, *a Schubert variety* X^w *is singular if and only if the permutation does not contain any* 4×4 *permutation submatrix equal to the permutation* 3412 *or* 4231.

If X and Y are varieties with a right action of B on X and a left action of B on Y, then we define the quotient

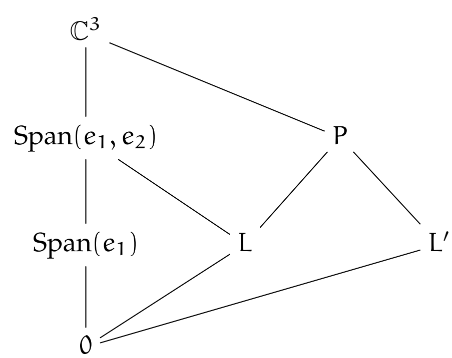
$$X \times^B Y = \{[x, y] \mid x \in X, y \in Y, [x, y] = [xb^{-1}, by]\}$$

Definition 2.2. *Let* $Q = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ *be a word in the simple reflections. The* ***Bott-Samelson*** *variety is*

$$BS^Q = P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_k}/B,$$

where P_j *denotes the minimal parabolic containing the root subgroup for* $-\alpha_j$.

Example 2.3. *For* $G = SL_n(\mathbb{C})$ *the Bott-Samelson variety* G^Q *can be interpreted as the* ***incidence variety***, *where start from the base flag and at every step of* Q, *we change only the subspace corresponding to the simple reflection. More concretely, for* $G = SL_3(\mathbb{C})$ *and* $Q = (s_1, s_2, s_1)$, *we have that* $BS^Q = \{(L, P, L') \mid L \subset \mathrm{Span}(e_1, e_2) \cap P, L' \subset P\}$, *or, more visually*



Theorem 2.4. *The Bott-Samelson variety* BS^Q *has a map to the flag variety*

$$\begin{aligned} m : BS^Q &\to G/B \\ [p_1, p_2, \dots, p_k] &\mapsto p_1 p_2 \cdots p_k B/B. \end{aligned}$$

Moreover, if Q *is a reduced word, then the image* $m(BS^Q)$ *is the Schubert variety* X^w *(where* $w = \prod Q$*), and this map is generically one-to-one.*

Example 2.5. *For* $G = SL_n(\mathbb{C})$, *the map is "take the rightmost flag in the incidence variety picture".*

Remark 2.6. *Note that the Bott-Samelson variety is not a resolution of singularities in the strictest sense, since it is not generically one-to-one to the smooth locus of the Schubert variety. For example,* G/B *is smooth, but* $m : BS^Q \to G/B$ *is not an isomorphism.*

2.2 Charts on Bott-Samelson varieties

The Bott-Samelson variety is an iterated $\mathbb{P}^1$-bundle because each quotient P_k/B is isomorphic to $\mathbb{P}^1$. Therefore it has many natural coordinate charts.

Proposition 2.7. *On $P_k/B \cong \mathbb{P}^1$, we have two charts $u_+, u_- : \mathbb{C} \to P_k/B$ given by*

$$u_+(z) = u_{\alpha_k}(z) \cdot s_k$$
$$u_-(w) = u_{-\alpha_k}(w)$$

where $u_\beta : SL_2(\mathbb{C}) \to G$ is the root subgroup corresponding to β. The change of coordinates between the two charts is $w = \frac{1}{z}$.

Example 2.8. *For $SL_3(\mathbb{C})$, and $Q = (s_1, s_2)$, the $+-$-chart is given by*

$$\left[\begin{pmatrix} z_1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & w_2 & 1 \end{pmatrix}\right].$$

Theorem 2.9. *For Q a reduced word for w, the $+^{|Q|}$-chart of BS^Q is an isomorphism from $\mathbb{C}^{l(w)}$ to X_o^w.*

Example 2.10. *For $Q = (s_1, s_2)$, the image of the $++$ chart in G/B is*

$$\begin{pmatrix} z_1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z_2 & -1 \\ 0 & 1 & 0 \end{pmatrix} /B = \begin{pmatrix} z_1 & -z_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} /B.$$

Notice that the origin $z_1 = z_2 = 0$ is mapped to the T-fixed flag 312, which is in X_o^{312}.

For us, the most important application of Bott-Samelson varieties is to give explicit coordinates to the big cell $X_o^{w_0}$.

3 Actions of vector fields

3.1 $SL_2(\mathbb{C})$

For $G = SL_2(\mathbb{C})$, the Bott-Samelson variety is isomorphic to the flag variety

$$BS^{(s)} = G/B.$$

The big cell is parametrized by

$$\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} /B.$$

Recall that we have a left $U(\mathfrak{g})$-action on G/B generated by the vector fields corresponding to the basis e, f, h of $\mathfrak{sl}_2(\mathbb{C})$. We compute these actions in this coordinate chart.

We have

$$\exp(-te) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \qquad \exp(-tf) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}, \qquad \exp(-th) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix},$$

Since

$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} /B = \begin{pmatrix} e^{-t}z & -e^{-t} \\ e^t & 0 \end{pmatrix} /B = \begin{pmatrix} e^{-2t}z & -1 \\ 1 & 0 \end{pmatrix} /B,$$

and we have

$$\frac{d}{dt}(e^{-2t}z) = -2e^{-2t}z$$
$$h \cdot z = \left.\frac{d}{dt}\right|_{t=0} (e^{-2t}z) = -2z$$
$$h \mapsto -2z\frac{d}{dz}.$$

Similarly, we compute the action of f:

Since

$$\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}/B = \begin{pmatrix} z & -1 \\ -tz+1 & t \end{pmatrix}/B = \begin{pmatrix} \frac{z}{-tz+1} & tz-1 \\ 1 & -t(tz-1) \end{pmatrix}/B = \begin{pmatrix} \frac{z}{-tz+1} & -1 \\ 1 & 0 \end{pmatrix}/B,$$

and we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{z}{-tz+1}\right) &= \frac{z^2}{(-tz+1)^2} \\ f \cdot z = \left.\frac{d}{dt}\right|_{t=0}\left(\frac{z}{-tz+1}\right) &= z^2 \\ f \mapsto z^2 \frac{d}{dz}. \end{aligned}$$

Exercise 3.1. *Using this coordinate chart, verify that* $e \mapsto -\frac{d}{dz}$.

4 $SL_3(\mathbb{C})$

Let $G = SL_3(\mathbb{C})$ and $Q = (s_1, s_2, s_1)$. Then $BS^Q \to G/B$ is generically one-to one. Let us compute the image of the $+++$ chart.

$$\begin{pmatrix} z_1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & z_2 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} z_3 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}/B = \begin{pmatrix} z_1z_3 - z_2 & -z_1 & 1 \\ z_3 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}/B$$

The z_2 coordinate can be recovered by taking the top left 2×2 minor (this is preserved under the right action of B, if the antidiagonal entries are scaled appropriately).

We have to compute the action of the vector fields $e_1, f_1, h_1, e_2, f_2, h_2$. We have

$$\exp(-te_1) = \begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(-tf_1) = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(-th_1) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp(-te_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(-tf_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{pmatrix}, \quad \exp(-th_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

Exercise 4.1. *Verify that or find a sign mistake in*

$$\begin{aligned} e_1 &\mapsto -\partial_{z_1} \\ f_1 &\mapsto z_1^2\partial_{z_1} - z_1z_2\partial_{z_2} + (z_2 - z_1z_3)\partial_{z_3} \\ h_1 &\mapsto -2z_1\partial_{z_1} + z_2\partial_{z_2} + z_3\partial_{z_3} \\ e_2 &\mapsto z_1\partial_{z_2} - \partial_{z_3} \\ f_2 &\mapsto z_2\partial_{z_1} + z_3^2\partial_{z_3} \\ h_2 &\mapsto z_1\partial_{z_1} - z_2\partial_{z_2} - 2z_3\partial_{z_3} \end{aligned}$$

Example 4.2. *Note that* $[e_1, f_1] = h_1$

Exercise 4.3. *Verify the remaining relations in* $\mathfrak{sl}_3(\mathbb{C})$ *or find a sign mistake in the formulas.*

4.1 The principal block of category $\mathcal{O}$

Similarly to the situation with $\mathbb{P}^1$ described by Dylan in the first lecture, we realize see the dual Verma module $M(0)^\vee$ as $\mathbb{C}[z_1, z_2, z_3]$. Notice that there is a highest weight vector of weight 0 (corresponding to the scalars) that is annihilated by all of the operators (this realizes the trivial representation as a submodule).

Recall that we have the BGG resolution

$$L(0) \to M(0)^\vee \to M(s_1.0)^\vee \oplus M(s_2.0)^\vee \to M(s_1s_2.0)^\vee \oplus M(s_2s_1.0)^\vee \to M(s_1s_2s_1.0)^\vee \to 0$$

Exercise 4.4. *Verify that in the above resolution the highest weight vectors of* $M(s_1.0)^\vee$ *and* $M(s_2.0)^\vee$ *are* z_1 *and* z_3*, respectively.*

Note that the maps in the BGG resolution are given by taking residues with respect to some of the variables. For example, the map $M(0)^\vee \to M(s_1.0)^\vee \oplus M(s_2.0)^\vee$ is $\mathrm{Res}_{z_1} \oplus \mathrm{Res}_{z_3}$. This corresponds to sending the coordinates z_1, z_3 to ∞, respectively.