Kirillov-Reshetikhin Crystals and Cacti

Balázs Elek

Cornell University,
Department of Mathematics

8/1/2018

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of trace 0 matrices and $V = \mathbb{C}^n$. The standard basis vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ form a basis for V that has several favorable properties:

1. Each basis vector is an eigenvector for the action of the subalgebra of diagonal matrices $\mathfrak{h}$, i.e. $\mathrm{diag}(t_1, \ldots t_n) \cdot \mathbf{v_k} = t_k \mathbf{v_k}$
2. The matrices $E_{ij} = (e_{mn})$ s.t. $e_{mn} = \begin{cases} 1 & \text{if } (m,n) = (i,j) \\ 0 & \text{else} \end{cases}$ for $i \neq j$ "almost permute" these vectors, i.e. $E_{ij} \cdot \mathbf{v_j} = \mathbf{v_i}$ and $E_{ij} \cdot \mathbf{v_k} = \mathbf{0}$ for $k \neq j$.
3. In fact we only need to use the matrices $F_i = E_{i+1\ i}$ to reach any basis vector from $\mathbf{v_1}$.

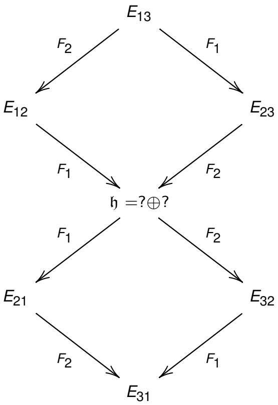
We say that this is a **good basis**.

Thus we can encode the representation as a colored directed graph, for example, $\mathfrak{sl}_3$ acting on $\mathbb{C}^3$ could be represented like this:

$$\mathbf{v_1} \overset{F_1}{\to} \mathbf{v_2} \overset{F_2}{\to} \mathbf{v_3}$$

Our aim is to generalize this idea and we'd hope that the good basis is compatible with tensor product decompositions and branching.

This works splendidly only as long as each weight space is one-dimensional. We already run into trouble with the adjoint representation of $\mathfrak{sl}_3$, as $\ker F_1, \ker F_2, \operatorname{im} F_1, \operatorname{im} F_2$ are all different subspaces of $\mathfrak{h}$.

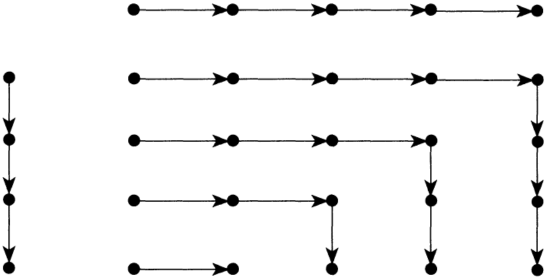


1. Consider the quantized enveloping algebra

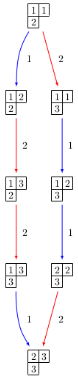
$$\mathfrak{g} \rightsquigarrow U_q(\mathfrak{g})$$
$$V \rightsquigarrow V_q$$

2. For $\mathfrak{g} = \mathfrak{sl}_n$, Date, Jimbo and Miwa [2] considered $V = \mathbb{C}^n$ with $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$, and computed tensor product decompositions of $(V_q)^{\otimes n}$.
3. Kashiwara later developed crystals by going to the $q \to 0$ limit in the general case.

What is the benefit of crystals? Combinatorics. For $\mathfrak{g} = \mathfrak{sl}_2$-crystals, tensor product decompositions are given by:



We know that for an irreducible $\mathfrak{sl}_n$-representation V_λ of highest weight λ, $\dim(V_\lambda) = \#SSYT(\lambda)$ with entries up to n. The crystal of V_λ has a realization as tableaux, for example, here is the crystal of the adjoint representation of $\mathfrak{sl}_3$



The lowering operators f_i work as follows:

1. Read the entries of the tableau from bottom to top, left to right, ignoring all numbers except i and $i+1$.
2. Replace $i+1 \to$ (and $i \to$),
3. Turn the rightmost unmatched (into an $i+1$

Example: applying f_2

1	2	2	4
2	3	4	
4	4		

$\to$ 442341224 $\to$)()) $\to$ 2323 $\to$

1	2	3	4
2	3	4	
4	4		

Now consider the affine Lie algebra $\tilde{\mathfrak{g}}$. This is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, we may consider $U(\tilde{\mathfrak{g}})$ and $U_q(\tilde{\mathfrak{g}})$.

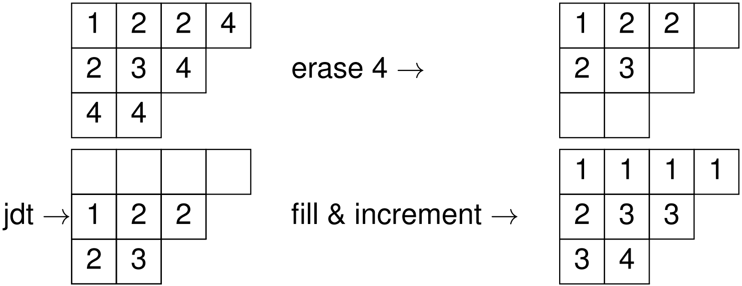
Conjecture 1 (Hatayama et al. [4])

Certain finite-dimensional $U(\tilde{\mathfrak{g}})$-modules, called ***Kirillov-Reshetikhin modules*** *would have crystal bases. A KR-module $W_s^{(r)}$ is determined by a choice of a non-affine node (r) of the Dynkin diagram and a positive integer s.*

As before, the situation is simplest for $\widetilde{\mathfrak{sl}_n}$. It turns out that in this case, each KR-module stays irreducible when restricted to $\mathfrak{sl}_n$, for example, for $U_q(\mathfrak{sl}_3)$, W_1^1 is the standard representation, and the KR crystal is

Question: can we extend the tableau model to the KR setting?
We want to give a combinatorial description of the f_0 operator.
Shimozono's [6] method for $\mathfrak{sl}_n$: use Schützenberger's **promotion** operator on tableaux.
If T is an SSYT, then $pr(T)$ is obtained by the following procedure:

1. Erase all entries n from the tableau.
2. Jeu-de-taquin the other boxes to the Southeast.
3. Fill the empty boxes in the Northwest with zeros.
4. Add one to each entry.


1 2 2 4
2 3 4
4 4
erase 4 →
1 2 2
2 3
jdt →
1 2 2
2 3
fill & increment →
1 1 1 1
2 3 3
3 4

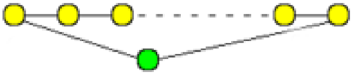
The promotion operator shifts the content of the tableau cyclically, and it is in fact cyclic of order n for rectangular tableaux, but not in general. Notice how pr interacts with the lowering operators for rectangular tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{pr} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{f_2} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \xrightarrow{pr^{-1}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{f_1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

So the promotion operator realizes the cyclic symmetry of the affine type A Dynkin diagram, at least for rectangular tableaux.



Theorem 2 (Shimozono)

$$f_0 = pr^{-1} \circ f_1 \circ pr$$

One would think that this is a phenomenon specific to type A because of the cyclic symmetry, but there exist tableaux models for other types, where the KR crystals were shown to exist on a case-by-case basis by finding a suitable analog of the promotion operator [3].

The **cactus group** $J_{\mathfrak{g}}$ was introduced by Henriques and Kamnitzer [8] in the context of coboundary categories. It has

1. Generators: s_J for J a connected subdiagram of $\mathfrak{g}$'s Dynkin diagram.
2. Relations:
   1. $s_J^2 = 1 \; \forall J$.
   2. $s_J s_{J'} = s_{J'} s_J$ if $J \cup J'$ is not connected.
   3. $s_J s_{J'} = s_{\theta_J(J')} s_J$ for $J' \subset J$.

   where θ_J is the Dynkin diagram automorphism $-w_0^J$ of J.

It surjects onto $W_{\mathfrak{g}}$ by $s_J \mapsto w_0^J$.

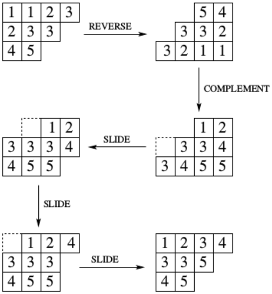
Halacheva [7] showed that there is an action of $J_{\mathfrak{g}}$ on any $\mathfrak{g}$-crystal, which we now describe.
The **Schützenberger involution** on a crystal B_λ of an irrep V_λ is the unique map $\xi_\lambda : B_\lambda \to B_\lambda$ on the vertices satisfying

1. $e_i(\xi_\lambda(b)) = \xi_\lambda(f_{\theta(i)}(b))$
2. $f_i(\xi_\lambda(b)) = \xi_\lambda(e_{\theta(i)}(b))$
3. $wt(\xi_\lambda(b)) = w_0 \cdot (wt(b))$

So, in effect, it flips the crystal upside down.

For $\mathfrak{sl}_n$-crystals, the operation is given by **evacuation** on tableaux, which is the following procedure:

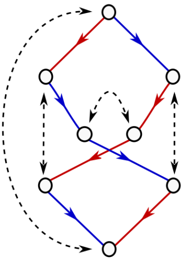
1. Rotate the tableau 180°.
2. Complement the entries $i \to n + 1 - i$.
3. Jeu-de-taquin the tableau back to the original shape.



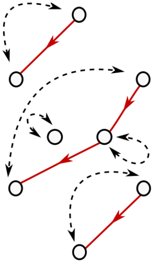
Halacheva [7] showed that $J_{\mathfrak{g}}$ acts o a $\mathfrak{g}$-crystal B by

$$s_J(b) = \xi_{B_J}(b)$$

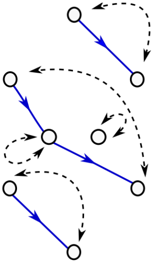
where ξ_{B_J} is the Schützenberger involution on the restricted crystal using only the lowering operators in J. For example, on the adjoint representation of $\mathfrak{sl}_3$,



The s_{12} action.



The s_1 action.



The s_2 action.

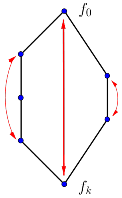
Our main result is the following:

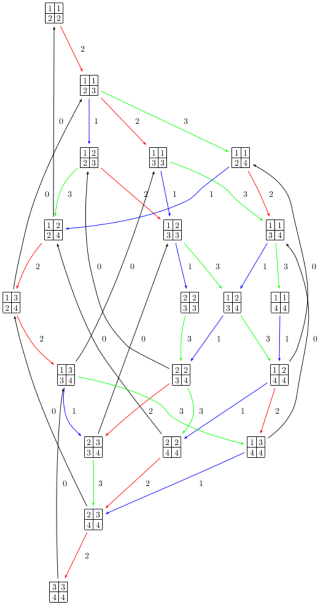
Theorem 3

For a KR-module corresponding to a cominuscule fundamental weight (k), let J be the subset of $\mathfrak{g}$'s Dynkin diagram complementary to $\{k\}$. Then

$$f_0^{-1} = e_0 = s_J f_k s_J.$$

In fact, s_J corresponds to





Stembridge [9]:

1. Check that $s_J e_k s_J$ is a lowering operator.
2. See how $s_J e_k s_J$ interacts with the other lowering operators (e.g. $f_1(s_J e_k s_J)^2 f_1(b) = (s_J e_k s_J) f_1^2 (s_J e_k s_J)(b)$ under certain local conditions).
3. By the uniqueness of crystals, we are done.

1. Even though our method may be less ad hoc than trying to extend *pr* to other types, it only works for $W_s^{(r)}$ for (r) cominuscule.
2. We are only using a single element of the cactus group, what do others do?
3. When are other representations of $\mathfrak{g}$ also representations of some Kac-Moody algebra containing $\mathfrak{g}$?
4. What does this say about promotion and evacuation on Young tableaux?

[1] Chari, V; Pressley, A., *Quantum affine algebras and their representations* Representations of groups, CMS Conf. Proc. 16, Amer. Math. Soc., Providence, RI (1995) 59-78.

[2] Date, E.; Jimbo, M.; Miwa, T., *R-matrix for cyclic representations of* $U_q(\widehat{\mathfrak{sl}}(3,\mathbb{C})$ *at* $q^3 = 1$. Phys. Lett. A 148 (1990), no. 1-2, 4549.

[3] Fourier, Ghislain; Okado, Masato; Schilling, Anne, *Kirillov-Reshetikhin crystals for nonexceptional types.* Adv. Math. 222 (2009), no. 3, 10801116. 17B37 (17B10 17B67)

[4] Hatayama, Goro; Kuniba, Atsuo; Okado, Masato; Takagi, Taichiro; Tsuboi, Zengo, *Paths, crystals and fermionic formulae.* MathPhys odyssey, 2001, 205272, Prog. Math. Phys., 23, Birkhuser Boston, Boston, MA, 2002. 17B37 (81R10 82B23)

[5] Littelmann, P., *Paths and root operators in representation theory* Ann. of Math. (2) 142 (1995), no. 3, 499-525

[6] Shimozono, Mark *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties.* J. Algebraic Combin. 15 (2002), no. 2, 151187. 17B10 (05E10)

[7] Halacheva, Iva *Alexander Type Invariants of Tangles, Skew Howe Duality for Crystals and The Cactus Group.* Thesis (Ph.D.)University of Toronto (Canada). 2016. 88 pp. ISBN: 978-1369-67293-0

[8] Henriques, André; Kamnitzer, Joel *Crystals and coboundary categories.* Duke Math. J. 132 (2006), no. 2, 191216. 17B37 (17B10 18D10 18D50)

[9] Stembridge, John R. *A local characterization of simply-laced crystals.* Trans. Amer. Math. Soc. 355 (2003), no. 12, 48074823. 17B37 (05E15)