The Borel fixed point Theorem and some applications

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We'll prove the following theorem

Theorem 1 (Borel fixed point Theorem)

Let B be a connected solvable affine algebraic group over $\mathbb{C}$. Let X be a proper variety with a B-action. Then the set of fixed points X^B is nonempty.

Definition 2

A group B is **solvable** if the derived series

$$B \supset [B, B] \supset [[B, B], [B, B]] \supset \dots$$

terminates in $\{e\}$.

We only need a few facts about groups:

- The derived subgroup $[B, B]$ of a solvable group B is solvable and of strictly lower dimension.
- If $H \subseteq B$ is a subgroup such that $[B, B] \subseteq H$, then H is normal (H contains all commutators).
- If G is an affine algebraic group and $N \subseteq G$ is a closed normal subgroup, then G/N is an affine variety.

Our schemes are over $\mathbb{C}$.

Definition 3

A variety X is **proper** if for every scheme Z, the map

$$pr_2 : X \times Z \to Z$$

is closed.

This is equivalent to the statement that $X(\mathbb{C})$, with the classical topology is compact and Hausdorff.
We only need a few facts about proper varieties:

- A proper affine variety is a point.
- A closed subvariety of a proper variety is proper.
- If V is a vector space, then the space of lines $\mathbb{P}(V)$ in V is proper.

- Let B be a connected solvable group and X a proper variety.
- We will proceed by induction on $\dim B$, the base case being $\dim B = 0$, in which case $B = \{e\}$ and every point is a fixed point.
- The subgroup $D = [B, B]$ is connected, solvable and its dimension is strictly less than $\dim B$, therefore by induction $Y = X^D$ is nonempty.
- The set of fixed points Y is closed in X, so Y is proper.
- Since D is normal in B, for $b \in B, d \in D, y \in Y$, we have

$$d(b \cdot y) = bd'y = by,$$

  for some $d' \in D$, so B stabilizes Y.
- We can therefore assume that D fixes X pointwise.

- Since $X^D = X$, we have $D = [B, B] \subseteq \mathrm{Stab}_B(x)$ for all $x \in X$.
- In particular, all isotropy groups are normal in B, so for any $x \in X$, the quotient $B/\mathrm{Stab}_B(x)$ is an affine variety.
- Pick $x \in X$ such that the orbit $B \cdot x$ is closed (these always exist), then $B \cdot x$ is a proper variety.
- Since $B \cdot x \cong B/\mathrm{Stab}_B(x)$, the orbit $B \cdot x$ is a proper affine variety, hence it must be a point, so x is our fixed point.

- Let G be an affine algebraic group and V a finite-dimensional G-representation (i.e. a homomorphism $G \to GL(V)$).
- The variety $\mathbb{P}(V)$ is proper, and since the G-action is linear, it has a G-action.
- Let B be a Borel (maximal solvable) subgroup of G. Then B is a solvable group that acts on the proper variety $\mathbb{P}(V)$, and hence has a fixed point $[v]$ for some $v \in (V \setminus 0)$.
- So B stabilizes the line $\mathrm{Span}_{\mathbb{C}}(v)$, and therefore acts on it through a character

$$\chi_V : B \to \mathbb{C}^{\times}$$

- So v is a B-weight vector, i.e. a high weight vector.

- Let B be a Borel subgroup of maximal possible dimension.
- Find a representation V such that the stabilizer of the high weight vector above is exactly B.
- Repeat the previous argument on $V/\operatorname{Span}_{\mathbb{C}}(v)$ and use a bit of induction to obtain the Lie-Kolchin Theorem, i.e. for a suitable choice of basis, the image of B in $GL(V)$ consists of upper triangular matrices.
- Therefore we have a complete flag $F_\bullet \in Fl(V)$ (a proper variety), and $\mathrm{Stab}_G(F_\bullet) = B$.
- So the map $G/B \to Fl(V)$ given by $g \mapsto g \cdot F_\bullet$ is injective. The stabilizer of any flag is solvable, so it has dimension $\leq \dim B$.
- Therefore the orbit $G \cdot F_\bullet$ has smallest possible dimension, and is therefore closed, hence proper, so G/B is proper.

- Let B' be any Borel subgroup of G.
- The proper variety G/B has a left G-action, hence a B'-action.
- The action has a fixed point, i.e.

$$B'gB/B = gB/B$$

or, in other words,

$$g^{-1}B'g \subseteq B.$$

- If the above containment is not an equality, then $gBg^{-1} \subseteq G$ is a solvable group strictlylarger than B', contradicting the maximality of B'. Therefore

$$g^{-1}B'g = B$$

so B' is conjugate to B.