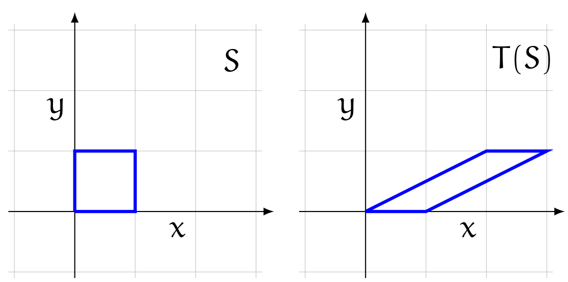
Determinants

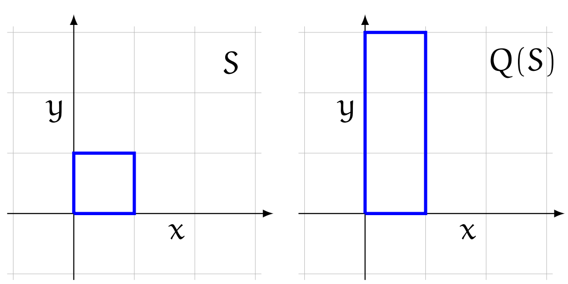
October 22, 2024

1 Motivation

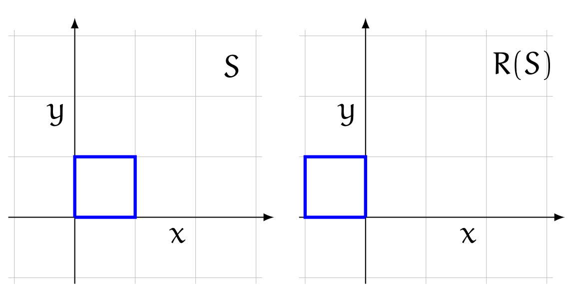
Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. We want to find a way to measure how the transformation affects areas. For example, the transformation $T(x, y) = (x + 2y, y)$ leaves the area of the unit square S unchanged:



Whereas the transformation $Q(x, y) = (x, 3y)$ scales its area by a factor of 3:



A rotation $R(x, y) = (-y, x)$ leaves the area invariant:



So we would like to assign the number 1 to the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, the number 3 to $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and the number 1 to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. You might notice that these matrices (almost) correspond to the different types of row operations.

2 Definitions

We claim there is a unique function $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ satisfying the following properties ($M \in \mathbb{F}^{n\times n}$ is an $n \times n$ matrix):

- $\det I_n = 1$, where I_n is the $n \times n$ identity matrix,
- Scaling a row of M by λ multiplies det by λ,
- Adding a multiple of a row to another row leaves det unchanged,
- swapping two rows multiplies det by -1.

Example 2.1. *Let's say we want to compute* $\det \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$. *We can argue as follows:*

$$\begin{aligned} \det \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} &= -\det \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \\ &= -\det \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix} \\ &= 7 \det \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ &= 7 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= 7. \end{aligned}$$

Note: we have not actually proved that this is well-defined (we will do this much later in the class)!

Example 2.2. *Let's compute* $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If $a \neq 0$, then*

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} \\ &= a \det \begin{pmatrix} 1 & b/a \\ 0 & d - (bc)/a \end{pmatrix} \\ &= (ad - bc) \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= ad - bc. \end{aligned}$$

and if $a = 0$, *then*

$$\begin{aligned}
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
&= -\det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \\
&= -c\det\begin{pmatrix} 1 & d/c \\ 0 & b \end{pmatrix} \\
&= -bc\det\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \\
&= -bc\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= -bc \\
&= ad - bc.
\end{aligned}$$

3 Properties

Definition 3.1. *A matrix* $M \in \mathbb{F}^{n\times n}$ *is called* ***upper triangular*** *if* $M_{i,j} = 0$ *whenever* $i > j$ *and* ***lower triangular*** *if* $M_{i,j} = 0$ *whenever* $j > i$.

Proposition 3.2. *If* M *is a triangular matrix, then* $\det M$ *is the product of the diagonal entries.*

Proof. If none of the diagonal entries are zero, then $\mathsf{RREF}(M) = I_n$, the identity matrix, and the scaling row operations that we need are precisely the product of the diagonal entries.

If some diagonal entries are zero, then $\mathsf{RREF}(M)$ has a zero row, since not every row has a pivot. Then $\det(M) = -\det(M)$, since we can multiply a zero row by -1 without chaning the matrix, and this implies that $\det(M) = 0$, which is also the product of the diagonal entries. □

Theorem 3.3. *Magical properties of* det*:*

1. *A matrix* $M \in \mathbb{F}^{n\times n}$ *is invertible if and only if* $\det M \neq 0$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof.

1. You will prove in HW5 that M is invertible if and only if $\mathsf{RREF}(M) = I_n$. The row operations don't change if the determinant is zero or not, and the determinant of a matrix whose RREF is not I_n is zero by the proposition above.
2. You can prove this with what we prove in HW5 and a little bit of extra work.
3. This follows from the above two.

□