MATH 223 — Final Exam — 150 minutes

18th December 2024

- The test consists of 17 pages and 14 questions worth a total of 0 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
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Cheat sheet

These are the most important definitions that we encountered:

- A list $\vec{v}_1, \ldots \vec{v}_m$ of vectors is **linearly independent** if
$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$
has only the trivial solution $a_1 = \ldots = a_m = 0$.
- The **span** of a list $\vec{v}_1, \ldots \vec{v}_m$ is the set
$$\{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m \in V \mid a_i \in \mathbb{F}\}$$
- A **basis** of a vector space is a linearly independent spanning list.
- The **dimension** of a vector space is the number of elements in a basis.
- For $T \in \mathcal{L}(V, W)$, the **null space** of T, denoted null(T) is
$$\text{null}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
- For $T \in \mathcal{L}(V, W)$, the **range** of T, denoted range(T) is
$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$
- A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.
- Suppose $T \in \mathcal{L}(V)$. A nonzero vector $\vec{v} \in V$ is called an **eigenvector** of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$ if
$$T(\vec{v}) = \lambda\vec{v}.$$

- Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. The **minimal polynomial** of T is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0_V$ is the zero operator.

- Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** in an inner product space if
$$\langle \vec{v}, \vec{w} \rangle = 0.$$

- A list of vectors is **orthonormal** if each vector $\vec{v}$ in the list has $\langle vecv, \vec{v} \rangle = 1$ and is orthogonal to all the other vectors in the list.

- Suppose $T \in \mathcal{L}(V)$. A basis of V is called a **Jordan basis** for T if with respect to this basis T has a block diagonal matrix
$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$
in which each A_k is an upper-triangular matrix of the form
$$A_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}.$$

- An m**-linear form** on V is a function $\beta : V^m \to \mathbb{F}$ that is linear in each slot when the other slots are held fixed. The set of m-linear forms on V is denoted by $V^{(m)}$.

- An m-linear form α on V is called **alternating** if $\alpha(\vec{v}_1, \ldots, vecv_m) = 0$ whenever $\vec{v}_1, \ldots, \vec{v}_m$ is a list of vectors in V with $\vec{v}_j = \vec{v}_k$ for some $j \neq k$. The set of alternating m-linear forms is denoted by $V_{\text{alt}}^{(m)}$.

- The **determinant** $\det T$ of an operator $T \in \mathcal{L}(V)$ is the unique scalar in $\mathbb{F}$ such that
$$\alpha(T\vec{v}_1, \ldots, T\vec{v}_m) = (\det T)\alpha(\vec{v}_1, \ldots, \vec{v}_m)$$
for all $\alpha \in V_{\text{alt}}^{(\dim V)}$.

1. Let

$$U = \left\{ p \in \mathcal{P}_3(\mathbb{R}) \;\middle|\; \int_0^1 p(x)\, dx = p(1) \right\}.$$

   (a) Prove that U is a subspace.

   (b) Find a basis for U and compute its dimension.

2. Let V, W be vector spaces and $T : V \to W$ a linear map. Prove that if

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$$

is linearly dependent, then

$$T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n$$

is linearly dependent.

3. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be given by reflection across the line $y = 2x$. Find all eigenvalues and eigenvectors of T.

4. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ be the linear transformation given by

$$T(p) = x^2 p.$$

Compute the matrix of T with respect to the bases $(1, x+1, x^2+x+1)$ of $\mathcal{P}_2(\mathbb{R})$ and $(1, x, x^2, x^3, x^4)$ of $\mathcal{P}_4(\mathbb{R})$.

5. For each of the following parts, give a concrete example. For this question only, you do not need to justify your answers.

   (a) A nilpotent operator $T \in \mathcal{L}(V)$ that has $\dim \operatorname{range}(T) = \dim V - 1$

   (b) Two linear maps S, T such that $ST = I$ but S is not invertible.

   (c) An operator on $V = \mathbb{C}^2$ whose minimal polynomial is $z - 1$.

   (d) An inner product on $\mathbb{R}^2$ different from the usual dot product.

   (e) A nonzero vector in $\mathcal{P}_2(\mathbb{R})$ orthogonal to $p(x) = x$ under the inner product $\langle p, q \rangle = \int_0^1 pq$.

6. Let $V = \mathcal{P}_1(\mathbb{R})$, and, for $p \in V$, define $\varphi_1, \varphi_2 \in V'$ by

$$\varphi_1(p) = \int_0^1 p(x)\, dx, \quad \text{and} \quad \varphi_2(p) = \int_0^2 p(x)\, dx.$$

Prove that (φ_1, φ_2) is a basis for V' and find a basis for V for which it is the dual basis.

7. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis for an inner product space V over $\mathbb{R}$ and that

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= 1 \\
\langle \vec{v}_2, \vec{v}_2 \rangle &= 2 \\
\langle \vec{v}_3, \vec{v}_3 \rangle &= 2 \\
\langle \vec{v}_1, \vec{v}_2 \rangle &= -1 \\
\langle \vec{v}_1, \vec{v}_3 \rangle &= -1 \\
\langle \vec{v}_2, \vec{v}_3 \rangle &= 1
\end{aligned}$$

Find an orthonormal basis for V. Justify your answer.

8. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null}(T) = U$ if and only if $\dim U \geq \dim V - \dim W$.

9. Suppose that V is finite-dimensional inner product space and U is a subspace of V. Show that

$$P_{U^\perp} = I_V - P_U,$$

where I_V is the identity map on V and P_W denotes the orthogonal projection map onto a subspace $W \subseteq V$.

10. Two linear operators $S, T \in \mathcal{L}(V)$ are said to be simultaneously diagonalizable if there exists a basis $(\vec{v}_1, \ldots, \vec{v}_n)$ of V such that the matrices $\mathcal{M}(S, (\vec{v}_1, \ldots, \vec{v}_n))$ and $\mathcal{M}(T, (\vec{v}_1, \ldots, \vec{v}_n))$ of S and T with respect to this basis are both diagonal. Prove that if S and T are simultaneously diagonalizable, then they commute, i.e.
$$ST = TS$$

11. Suppose that $T \in \mathcal{L}(\mathbb{C}^3)$ is an operator such that the minimal polynomial of T is

$$z^3 + 2z^2 + z.$$

Find all the possibilities for the matrix of $\mathcal{M}(T)$ with respect to a Jordan basis (up to reordering the basis).

12. Suppose $V = \mathbb{F}^n$ and $T \in \mathcal{L}(V)$ is given by

$$T(x_1, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n).$$

Find the minimal polynomial of T. Justify your answer.

13. Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I + T) = 1$.

14. Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.

(a) Prove that $\dim \operatorname{null}(\mathcal{A}) = (\dim V)(\dim \operatorname{null}(S))$.

(b) Prove that $\dim \operatorname{range}(\mathcal{A}) = (\dim V)(\dim \operatorname{range}(S))$.