# MATH223 Homework 1
# (due Friday, September 12, 11:59pm)

1. (2 marks) Solve the system

$$\begin{aligned}
2x_1 + 6x_2 + x_3 + 3x_4 - x_5 + 6x_6 &= 1\\
-x_1 - 3x_2 + x_3 \phantom{+ 3x_4 - x_5} - x_6 &= 0\\
-2x_1 - 6x_2 + x_3 - x_4 + 2x_5 - 5x_6 &= 3\\
x_1 + 3x_2 + x_3 + 2x_4 + x_5 + 2x_6 &= 4
\end{aligned}$$

   using row reduction. Show your steps until you get to reduced row echelon form. Give your answer in parametric form (see ILA Chapter 2.3 for the details).

   **Warning:** It is extremely tempting to put this into a computer and copy the result. Please do this yourself by hand. You are allowed to check your work using a computer afterwards, but you don't want to deny yourself the experience of doing this manually at least once in your life.

2. In this question assume that we are considering a system of linear equations.

   (a) (2 marks) If we have 4 equations in 6 variables, how many free variables may the system have? Justify your answer.

   (b) (2 marks) If we have 4 equations in $n$ variables, how many free variables may the system have? Justify your answer.

   (c) (2 marks) If we have $m$ equations in $n$ variables, how many free variables may the system have? Justify your answer.

3. (4 marks) Recall that in the row reduction (Gaussian elimination) algorithm we have three row operations:

   - Multiplying a row by a nonzero scalar.
   - Adding a multiple of a row to another row.
   - Swapping two rows.

   Show that one of the three operations is not really necessary by exhibiting a sequence of row operations of the other two types that has the same effect on an arbitrary matrix.

4. Recall that in the row reduction (Gaussian elimination) algorithm we have three row operations:

   - Multiplying a row by a nonzero scalar.
   - Adding a multiple of a row to another row.

- Swapping two rows.

It is always possible to put a matrix into **reduced row echelon** form (for the details, see ILA Chapter 2) with these operations.

In this question, we will study what happens when you don't have all of these operations at your disposal. Assume that you only have the following two row operations (we'll call this **restricted row operations**)[1]:

- Multiplying a row by a nonzero scalar.
- Adding a multiple of a row to a row **below** it.

(a) (2 marks) Consider the matrix

$$\begin{pmatrix} 3 & 6 & 3 \\ -1 & -2 & 0 \\ 1 & 3 & 0 \end{pmatrix}$$

Use the **restricted row operations** to get this matrix as close to row echelon form as you can (**Hint:** you should be able to get to a form where each row has a pivot that is equal to 1). Highlight the pivots in your answer.

Show that it is not possible to get this matrix all the way to reduced row echelon form using the restricted row operations.

(b) (3 marks) Now consider the general case of the above problem. That is, consider a matrix of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

with unknown entries. Assume that the reduced row echelon form of this matrix (obtainable using the regular row operations) is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(**Hint:** It may not be immediately obvious how you can use this assumption to answer the question. You might want to rephrase it differently.)

Describe all possible end results of row reduction of this matrix using the **restricted row operations**. (**Hint:** Your answer will have to involve several cases, but in each of them, all the entries of the matrix should be 1-s, 0-s or $*$-s).

(c) (3 marks) We will generalize to the $4 \times 4$ case, that is, start with a matrix of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

[1] This question deals with the baby version of the Bruhat decomposition. This is an important theorem in a branch of mathematics called Schubert Calculus. Schubert Calculus is concerned with questions like "What is the number of smooth conic plane curves tangent to five given general conics?", (it is 3264).

whose reduced row echelon form (obtainable using the regular row operations) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

. Describe the possible end results of row reduction using **restritcted row operations** (you do not need to list every single one, but you should say how many there are, and what they look like).

(d) (Challenge question, will not be marked): Can you generalize this to the $n \times n$ case? Try to find a set that is in bijection with the possible end results of row reduction using the **restricted row operations** on a matrix with unknown entries (still maintaining the assumption that the reduced row echelon form has 1-s in every row).