MATH223 Homework 3
(due Monday, September 11, 30:59pm)

1. In this question we will try to understand this week's new definitions by putting them into a more concrete context of matrices.

   (a) (2 marks) Show that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, with $\vec{v}_i \in \mathbb{F}^n$ is linearly independent if and only if the RREF of the matrix whose columns are the vectors

   $$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$$

   has m pivots.

   (b) (2 marks) Show that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, with $\vec{v}_i \in \mathbb{F}^n$ spans $\mathbb{F}^n$ if and only if the RREF of the matrix

   $$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$$

   has n pivots.

   (c) (1 mark) Based on the previous parts, when does a list $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, with $\vec{v}_i \in \mathbb{F}^n$ form a basis for $\mathbb{F}^n$? Give your answer in terms of the RREF of the matrix

   $$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$$

2. (2 marks, this is Exercise 13 in Axler 2A) Suppose $\vec{v}_1, \ldots \vec{v}_m$ is linearly independent in V and $w \in V$. Show that

   $$v_1, \ldots, v_m, w \text{ is linearly independent} \Leftrightarrow w \notin \text{Span}(v_1, \ldots, v_m).$$

3. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (the rational numbers). You do not need to verify that it is a vector space.

   (a) (1 mark) Prove that the list $(1, \sqrt{2})$ is linearly independent.

   (b) (1 mark) Prove that the list $(1, \sqrt{2}, \sqrt[3]{2})$ is linearly independent.

   (c) (1 mark) Assuming that the list $(1, \sqrt{2}, \sqrt[3]{2}, \ldots, \sqrt[n]{2})$ is linearly independent (for any n), prove that $\mathbb{R}$ is an infinite-dimensional vector space over $\mathbb{Q}$.

4. Recall that a binomial coefficient $\binom{n}{k}$ is defined as

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)\cdots(2)(1)}.$$

   Similarly, we define, for a variable x the following *polynomial*:

   $$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

(a) (1 mark) Could we have instead defined this polynomial as

$$\binom{x}{k} = \frac{x!}{k!(x-k)!}?$$

Explain your answer carefully.

(b) (2 marks) Prove that the list

$$\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{m}$$

is a basis for $\mathcal{P}_m(\mathbb{F})$.[1]

5. (3 marks this is Exercise 15 in Axler 2C) Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

6. In this question we address a very commonly held false belief about dimensions.

(a) (2 marks) Let S_1, S_2, S_3 be finite sets. Prove that

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|,$$

where $|S|$ denotes the cardinality (number of elements) of a set. (**Hint:** this part has nothing to do with linear algebra).

(b) (2 marks) this is Exercise 19 in Axler 2C) Let U_1, U_2, U_3 be subspaces. We have seen (or will see) in class that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

You might guess (following part 6a) that this generalizes to

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) \\ - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$

but this would be wrong. Find an example of three subspaces U_1, U_2, U_3 that do not satisfy the above equality.

[1] We will see later that this basis is better for some purposes than $1, x, x^2, \ldots, x^m$. Here is something that you might find nice: what is $\frac{d}{dx}\binom{x}{k}$?