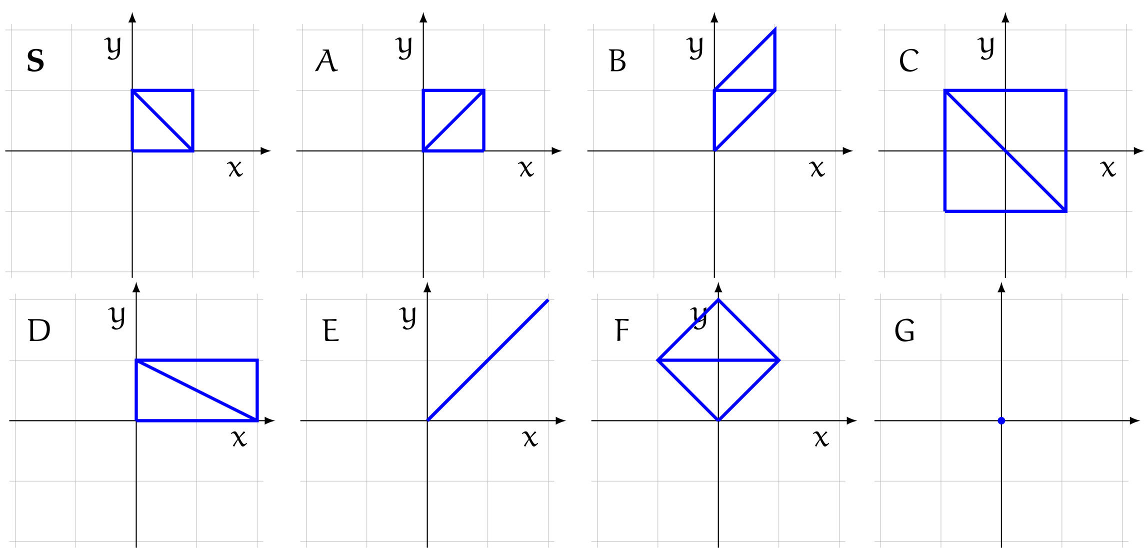
MATH223 Homework 4
(due Friday, Oct/18, 11:59pm)

1. (3 marks) If $S \subset V$ is an arbitrary subset (not necessarily a subspace), and $T : V \to V$ is a transformation (any function, not necessarily linear), we define the image of S under T to be
$$T(S) = \{T(\vec{v}) \mid \vec{v} \in S\}.$$
Consider the following figure:



On it you see an original subset of $\mathbb{R}^2$ (in blue, named **S**) and the images of **S** under some transformation (a different one for each of A, B, ...,G). For each of the images:

- Determine if the corresponding transformation can be linear or not.
- For the transformations that can be linear, find a matrix that could be the matrix of the transformation with respect to the standard basis of $\mathbb{R}^2$.
- For the transformations that can not be linear, explain why they can not be.

2. Let $\frac{d}{dx}$ denote differentiation, mult_x denote multiplication by x. We have seen that we can consider these as linear operators on $\mathcal{P}(\mathbb{R})$.[1] To clarify: a linear operator is a linear map from a vector space to itself.

   (a) (1 mark) If we restrict the domain of the linear map mult_x to the subspace $\mathcal{P}_m(\mathbb{R})$, what is
$$\text{range}\,(\text{mult}_x)\,?$$

[1] These two operators are the generators for the Weyl algebra, introduced by Herman Weyl to study the Heisenberg uncertainty principle in quantum mechanics.

(b) (1 mark) Consider both mult_x and $\frac{d}{dx}$ as linear maps from $\mathcal{P}_m(\mathbb{R})$ to $\mathcal{P}(\mathbb{R})$. Prove that

$$\text{range}\left(\frac{d}{dx}\text{mult}_x\right) \subseteq \mathcal{P}_m(\mathbb{R}),$$

so $\frac{d}{dx}\text{mult}_x$ can be considered as an operator on $\mathcal{P}_m(\mathbb{R})$.

(c) (2 marks) Find a simpler expression for the linear operator

$$\frac{d}{dx}\text{mult}_x - \text{mult}_x\frac{d}{dx}.$$

(**Hint:** If you consider this as an operator on $\mathcal{P}_m(\mathbb{R})$ then you may be able to use a Lemma)

(d) (1 mark) Use your result above to simplify the expression

$$\left(\frac{d}{dx}\right)^3\text{mult}_x - \text{mult}_x\left(\frac{d}{dx}\right)^3 - 2\left(\frac{d}{dx}\right)^2.$$

(e) (1 mark) Find the matrix of $\frac{d}{dx}\text{mult}_x$ (as an operator on $\mathcal{P}_m(\mathbb{R})$) with respect to the basis $1, x, x^2, \ldots, x^m$.

(f) (1 mark) Use parts (2c) and (2e) to find the matrix of $\text{mult}_x\frac{d}{dx}$ (with respect to the basis $1, x, x^2, \ldots, x^m$) without doing any more direct computation or matrix entries.

3. (2 marks, this is Exercise 11 in Axler 3A) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.[2]

4. Recall the notion of direct sums of vector spaces (definition 1.41 in Axler 1C). Let U, V be subspaces of some vector space, and let W be finite-dimensional vector space.

(a) (1 mark) Prove that if $\vec{u}_1 \ldots \vec{u}_m$ is a basis for U and $\vec{v}_1, \ldots \vec{v}_n$ is a basis for V, then

$$\vec{u}_1, \ldots, \vec{u}_m, \vec{v}_1, \ldots, \vec{v}_n$$

is a basis for $U \oplus V$.

(b) (1 mark) Given linear maps $S : U \to W, T : V \to W$, we define the direct sum of the maps $(S \oplus T) : (U \oplus V) \to W$ as follows:

$$(S \oplus T)(\vec{u} + \vec{v}) = S(\vec{u}) + T(\vec{v}).$$

Explain why the above rule uniquely determines the linear transformation (**Hint:** your explanation should involve clarifying the roles of $\vec{u}$ and $\vec{v}$).

(c) (2 marks) Let $\vec{w}_1, \ldots, \vec{w}_d$ be a basis for W. Find the matrix of $(S \oplus T)$ with respect to the bases $\vec{u}_1, \ldots, \vec{u}_m, \vec{v}_1, \ldots, \vec{v}_n$ and $\vec{w}_1, \ldots, \vec{w}_d$. in terms of the matrices of S and T (with respect to the bases $\vec{u}_1, \ldots, \vec{u}_m$ and $\vec{v}_1, \ldots, \vec{v}_n$).

[2]This is a special case of Schur's Lemma, an important result in representation theory.

(d) (1 mark) Given linear maps $Q : W \to U$ and $R : W \to V$, we define the direct sum of the maps $\begin{pmatrix} Q \\ \oplus \\ R \end{pmatrix} : W \to (U \oplus V)$ as follows:

$$\begin{pmatrix} Q \\ \oplus \\ R \end{pmatrix} (\vec{w}) = Q(\vec{w}) + R(\vec{w}).$$

Find the matrix of $\begin{pmatrix} Q \\ \oplus \\ R \end{pmatrix}$ with respect to the bases $\vec{w}_1, \ldots, \vec{w}_d$ and $\vec{u}_1, \ldots, \vec{u}_m, \vec{v}_1, \ldots, \vec{v}_n$.2

(e) (1 mark) Find the null space of $\begin{pmatrix} Q \\ \oplus \\ R \end{pmatrix}$ in terms of $\operatorname{null}(Q)$ and $\operatorname{null}(R)$.

5. (2 marks, this is Exercise 5 in Axler 3C) Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.