MATH223 Homework 5
(due Sunday, Oct/27, 11:59pm)

1. (3 marks) Recall the three types of row operations that we have:
   - Multiplying row i by a nonzero scalar λ.
   - Adding λ times row j to row i (with $i \neq j$).
   - Switching rows i and j.

   Suppose we are performing row operations on an $m \times n$ matrix A. For each of the row operations above, find an $m \times m$ matrix E such that the matrix product EA is the matrix A with the corresponding row operation performed. Justify your answer.

2. (2 marks) Instead of row operations, we may consider the following column operations:
   - Multiplying column i by a nonzero scalar λ.
   - Adding λ times column j to column i (with $i \neq j$).
   - Switching columns i and j.

   Suppose we are performing column operations on an $m \times n$ matrix A. For each of the column operations above, find an $n \times n$ matrix E such that the matrix product AE is the matrix A with the corresponding column operation performed. Justify your answer.

3. (2 marks, this is Exercise 2 in Axler 3D) Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

4. In this exercise we will connect invertibility to the RREF of a matrix. A square matrix A is said to be invertible if there exists another matrix B of the same size such that $AB = BA = I_n$ is the $n \times n$ identity matrix. If $A = \mathcal{M}(T)$ is the matrix of a linear transformation (with respect to some bases) then A is invertible (as a matrix) if and only if T is invertible (as a linear transformation).
   (a) (2 marks) Let $A = \mathcal{M}(T)$ be matrix of size $n \times n$. Prove that A is invertible if and only if the RREF of A is the $n \times n$ identity matrix. (**Hint:** Q1 on Homework 3 may be useful)
   (b) (4 marks) Suppose A is invertible. Consider the augmented matrix
   $$(A \mid I_n)$$
   (where I_n denotes the $n \times n$ identity matrix) of size $n \times 2n$. Show that the RREF of this matrix is
   $$\left(I_n \;\middle|\; A^{-1} \right).$$[1]

[1] This gives us a practical way to compute the inverse of a matrix.

(**Hint:** You might want to use Q1, Q3, and Q4 (a) for this)

5. (3 marks, this is Exercise 9 in Axler 3D) Suppose V is finite-dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W. *(Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U with $T|_U$ defined by $T|_U(\vec{u}) = T(\vec{u})$ for every $\vec{u} \in U$.* (**Hint:** Q5 on Homework 4 may be useful)

6. Let V be finite-dimensional and consider the vector space $\mathcal{L}(V)$ of linear maps from V to itself.

   (a) (2 marks) Is the set of invertible linear transformations a subspace of $\mathcal{L}(V)$? Justify your answer.

   (b) (2 marks) Is the set of non-invertible linear transformations a subspace of $\mathcal{L}(V)$? Justify your answer.

   (**Hint:** be careful with this one, your answer should depend on $\dim V$!)