MATH223 Homework 6
(due Sunday, Nov/3, 11:59pm)

1. (this is Exercise 14 in Axler 4) Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Use linear algebra as outlined below in (a)-(c) to prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that

$$rp + sq = 1.$$

   (a) (2 marks) Define $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$ by

$$T(r, s) = rp + sq.$$

   Show that the linear map T is injective.

   (b) (1 mark) Show that the linear map T in (a) is surjective.

   (c) (1 mark) Use (b) to conclude that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.

2. As in question 1, let $p(x) = a_0 + a_1 x + \ldots + a_m x^m$ and $q(x) = b_0 + b_1 x + \ldots + b_n x^n$ be nonconstant polynomials (with $a_m \neq 0$, $b_n \neq 0$), but in this question do not assume that they do not share a root. Let $T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$ be the same linear map

$$T(r, s) = rp + sq.$$

   (a) (1 mark) In question 1 you showed that T is an isomorphism if p and q do not share a root. Show the converse, i.e. prove that if T is an isomorphism then p and q do not share a root.

   (b) (2 marks) Recall that the list

$$\Big((1, 0), (x, 0), \ldots, (x^{n-1}, 0), (0, 1), (0, x), \ldots, (0, x^{m-1})\Big)$$

   is a basis of $\mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C})$ and that

$$(1, x, x^2, \ldots, x^{m+n-1})$$

   is a basis of $\mathcal{P}_{m+n-1}(\mathbb{C})$. Compute the matrix of T with respect to these bases.

   (c) (2 marks) The expression $\det(\mathcal{M}(T))$ is called the **resultant** of the polynomials p and q. If $\det(\mathcal{M}(T)) = 0$, then what can you say about the roots of p and q? Justify your answer.

3. As in questions 1 and 2, let $p(x) = a_0 + a_1 x + \ldots + a_m x^m$ be a degree m polynomial. Let $q(x) = \frac{d}{dx}(p(x))$ be the derivative of p. We say that $p(x)$ **has a double root at** λ if $p(x) = (x-\lambda)^2 s(x)$ for some polynomial s of degree $m-2$.

   (a) (1 mark) Prove that $p(x)$ has a double root at λ if and only if $p(\lambda) = q(\lambda) = 0$ (**Hint:** This is a Calculus question).

   (b) (3 marks) Let $m = 2$ (i.e. p is a quadratic polynomial). Use question 2 to find a condition on the coefficients of p that is equivalent to p having a double root (**Hint:** remember, $a_2 \neq 0$). Do you recognize this expression? (**Hint:** It is generally best to leave the computation of determinants to a computer, especially if the matrix involved is large. There are numerous software that can compute determinants involving formal variables, sage is one of them that is relatively easy to use.)

   (c) (2 marks) Let $m = 3$ (i.e. p is a cubic polynomial). Find a condition on the coefficients of p that is equivalent to p having a double root.

4. In this question we'll explore another application of resultants. We want to find solutions of the system

$$\begin{aligned} y^2 \qquad\qquad\qquad\qquad\qquad\qquad &- (x^3 + 3x^2 + 2x + 1) = 0 \\ y^3 + y^2(3x+3) + y(3x^2+6x+2) &+ (x^3 + 3x^2 + 4x + 2) = 0 \end{aligned}$$

   of non-linear polynomial equations. That is, we want to find pairs (x, y) of (real) numbers that satisfy both of the above equations.

   (a) (3 marks) Define two polynomials

$$\begin{aligned} p(x,y) &= y^2 - (x^3 + 3x^2 + 2x + 1) \\ q(x,y) &= y^3 + y^2(3x+3) + y(3x^2+6x+2) + (x^3 + 3x^2 + 4x + 2). \end{aligned}$$

   Think of them as polynomials in the variable y (so, we treat functions of x as coefficients, notice that we already wrote them this way). Use question 2 to find a polynomial equation $s(x)$ in just the variable x that is equivalent to $p(x,y)$ and $q(x,y)$ sharing a root.[1]

   (b) (2 marks) Find a root of $s(x)$ and use this to find some solutions to the system[2] (you should be able to find three of them).

[1] This technique is one of the main tools of elimination theory. Notice how much harder polynomial elimination is than Gaussian (linear) elimination.

[2] This polynomial will be high degree so we don't have a formula for its roots. Computers can also help with factorization sometimes, here is a sage example.