MATH223 Homework 7
(due Friday, Nov/29, 11:59pm)

1. (This is Exercise 11 in Axler 5B) Suppose V is a two-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) (1 mark) Show that $T^2 - (a+d)T + (ad - bc)I = 0$.

   (b) (1 mark) Show that the minimal polynomial of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

2. (2 marks, this is Exercise 16 in Axler 5B) Suppose $a_0, \ldots, a_{n-1} \in \mathbb{F}$. Let T be the operator on $\mathbb{F}^n$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}.$$

   Here all the entries of the matrix are 0 except for all 1's on the line under the diagonal and the entries in the last column (some of which might also be 0). Show that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n.$$

3. (2 marks, this is Exercise 22 in Axler 5B) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \ldots, T^{\dim V})$.

4. A **Pingala sequence** is a sequence $P_0, P_1, P_2, \ldots$ of real numbers satisfying

$$P_n = P_{n-2} + P_{n-1}$$

   for $n \geq 2$.[1]

   (a) (1 mark) Show that the set of Pingala sequences is a subspace of $\mathbb{R}^{\mathbb{N}}$, the vector space of sequences of real numbers.

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[1] You might notice that if $P_0 = 0, P_1 = 1$, then you get the famous Fibonacci sequence (F_n). The Indian mathematician Pingala described this sequence about a thousand years before Fibonacci.

(b) (2 marks) Find a basis for the subspace of Pingala sequences and hence determine its dimension. Justify your answer.

(c) Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

(i) (1 mark) Show that if $P_0, P_1, \ldots$ is a Pingala sequence, then $T^n(P_0, P_1) = (P_n, P_{n+1})$ for $n \geq 0$.

(ii) (1 mark) Find the eigenvalues of T.

(iii) (2 mark) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(iv) (2 mark) Use the solution to the previous part to compute $T^n(0, 1)$. Conclude that

$$P_n = F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each nonnegative integer n (Not part of the question, just something to ponder: *isn't this a funny way to write a number that is clearly an integer?*).

(v) (1 mark) Use the previous part to conclude that if n is a nonnegative integer, then the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

5. (This is Exercise 32 in Axler 3F) The **double dual space** of V, denoted V'', is defined to be the dual space of V', in other words, $V'' = (V')'$. Define $\wedge : V \to V''$ by

$$(\wedge(\vec{v}))(\varphi) = \varphi(\vec{v})$$

for each $v \in V$ and $\varphi \in V'$.

(a) (1 mark) Show that $\wedge$ is a linear map from V to V''.

(b) (2 marks) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \wedge = \wedge \circ T$, where $T'' = (T')'$.

(c) (1 mark) Show that if V is finite-dimensional, then $\wedge$ is an isomorphism from V onto V''.