MATH223 Homework 8
(due Friday, Dec/6, 11:59pm)

1. (2 marks) Let $A \in \mathbb{R}^{m,n}$ be an $m \times n$ matrix with entries in $\mathbb{R}$. We define the following:

   - The **column space** $\mathrm{Col}(A)$ of A is the span of the columns of A (it is a subspace of $\mathbb{R}^{m,1}$).
   - The **row space** $\mathrm{Row}(A)$ of A is the span of the rows of A (it is a subspace of $\mathbb{R}^{1,n}$).
   - The **null space** $\mathrm{null}(A)$ of A is $\{X \in \mathbb{R}^{n,1} \mid AX = 0_{\mathbb{R}^{m,1}}\}$.

   Show that
   $$\left(\mathrm{Row}(A)^{\perp}\right)^{t} = \mathrm{null}(A)$$
   and
   $$\mathrm{Col}(A)^{\perp} = \mathrm{null}(A^{t})$$
   where $^{\perp}$ is considered with respect to the dot product and M^{t} denotes the transpose of the matrix M.

2. (3 marks) This exercise describes a practical way to compute the matrix of a projection onto a subspace. Let $V = \mathbb{R}^{n,1}$ be the vector space of column vectors with n components equipped with the standard dot product, and let $U \subseteq V$ be a subspace. Choose a basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ of U and let
   $$A = (u_1 u_2 \ldots u_m)$$
   be the $n \times m$ matrix whose columns are the u_j-s. Let $\vec{y} \in V$. Prove that $\exists \vec{x} \in \mathbb{R}^{m,1}$ such that
   $$A^{t}A\vec{x} = A^{t}\vec{y}$$
   and moreover, for any such $\vec{x}$, we have
   $$A\vec{x} = P_{U}\vec{y},$$
   where P_U is the orthogonal projection onto U.

3. (2 marks) Let $V = C[-1, 1]$ be the vector space of continuous real-valued functions defined on the interval $[-1, 1]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\ dt$. Let W_e and W_o denote the subspaces of even and odd functions respectively. Prove that
   $$W_e = W_o^{\perp}.$$
   (**Hint:** We have proved in class that if V is finite-dimensional inner product space, and U is a subspace then $V = U \oplus U^{\perp}$. Since $V = C[-1, 1]$ is infinite-dimensional, we could not use this theorem. However, it is true that $C[-1, 1] = W_e \oplus W_e^{\perp}$ and $C[-1, 1] = W_o \oplus W_o^{\perp}$. This should help you in answering this question. Before you jump to conclusions, note that if $V = U \oplus W$ and $V = U \oplus \widetilde{W}$, we can not in general conclude that $W = \widetilde{W}$.)

4. (2 marks, this is Exercise 6 in Axler 6C) Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.

$$T = TP_{(\mathrm{null}\, T)^{\perp}} = P_{\mathrm{range}\, T} T.$$

5. (2 marks) Let $M \in \mathbb{C}^{n,n}$ be an $n \times n$ matrix with complex entries. Let M^* denote the **conjugate transpose** of M, that is M^* is an $n \times n$ complex matrix whose entries are

$$M^*_{(i,j)} = \overline{M_{(j,i)}}.$$

Prove that

$$MM^* = I_{\mathbb{C}^n}$$

if and only if the rows of M form an orthonormal basis for $\mathbb{C}^n$.

6. (2 marks, this is Exercise 17 in Axler 8A) Suppose $T \in \mathcal{L}(V)$ is nilpotent and m is a positive integer such that $T^m = 0$.

   (a) Prove that $I - T$ is invertible and that $(I - T)^{-1} = I + T + \ldots + T^{m-1}$.

   (b) Explain how you would guess the formula above.

7. (2 marks, this is Exercise 3 in Axler 8B) Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

8. (2 marks, this is Exercise 9 in Axler 8B) Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that there exist $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.

9. (3 marks, this is Exercise 14 in Axler 8C) Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that there does not exist a direct sum decomposition of V into two nonzero subspaces invariant under T if and only if the minimal polynomial of T is of the form $(z - \lambda)^{\dim V}$ for some $\lambda \in \mathbb{C}$.