Heaps, Crystals and Preprojective algebra modules

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Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of trace 0 matrices and $V = \mathbb{C}^n$. The standard basis vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ form a basis for V that has several favorable properties:

1. Each basis vector is an eigenvector for the action of the subalgebra $\mathfrak{h}$ of diagonal matrices, i.e.

$$\mathrm{diag}(t_1, \ldots, t_n) \cdot \mathbf{v_k} = t_k \mathbf{v_k}$$

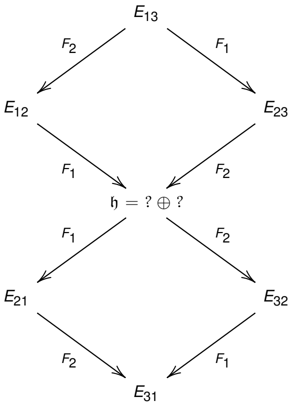
2. The matrices $E_{i,j} = (e_{mn})$ s.t. $e_{mn} = \begin{cases} 1 & \text{if } (m,n) = (i,j) \\ 0 & \text{else} \end{cases}$ for $i \neq j$ "almost permute" these vectors, i.e. $E_{i,j} \cdot \mathbf{v_j} = \mathbf{v_i}$ and $E_{i,j} \cdot \mathbf{v_k} = \mathbf{0}$ for $k \neq j$.
3. We only need to use the matrices $F_i = E_{i+1,i}$ to reach any basis vector from $\mathbf{v_1}$.

Thus we can encode the representation as a colored directed graph, for example, $\mathfrak{sl}_3$ acting on $\mathbb{C}^3$ could be represented like this:

$$\mathbf{v_1} \xrightarrow{F_1} \mathbf{v_2} \xrightarrow{F_2} \mathbf{v_3}$$

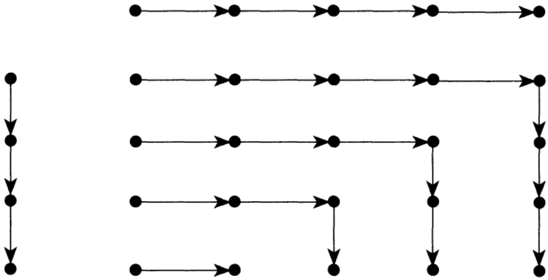
Our aim is to generalize this idea and we'd hope that the nice basis we found is compatible with things we want to do with $\mathfrak{g}$-representations, like tensor product decompositions and branching.

This works only as long as each weight space is one-dimensional. We already run into trouble with the adjoint representation of $\mathfrak{sl}_3$, as $\ker F_1$, $\ker F_2$, $\operatorname{im} F_1$, $\operatorname{im} F_2$ are all different subspaces of $\mathfrak{h}$.



Fortunately, thanks to Kashiwara [Kas91], there is a way of fixing this problem: by going first to the quantized universal enveloping algebra $U_q(\mathfrak{g})$ and then taking a limit as $q \to 0$ in a suitable sense. It turns out that in this setting, choosing a good basis is always possible. This object, the directed graph with vertices the basis elements and edges labeled by the action of the lowering operators is called a **crystal**. Since the representation theory of $U_q(\mathfrak{g})$ is very similar to that of $U(\mathfrak{g})$, we can use this combinatorial gadget to study representations.

Why do we like crystals? Because the rules for tensoring and branching are purely combinatorial. For $\mathfrak{g} = \mathfrak{sl}_2$-crystals, tensor product decompositions are given by:



We know that for an irreducible $\mathfrak{sl}_n$-representation V_λ of highest weight λ, $\dim(V_\lambda) = \#SSYT(\lambda)$ with entries up to n. The crystal of the adjoint representation of $\mathfrak{sl}_3$ is

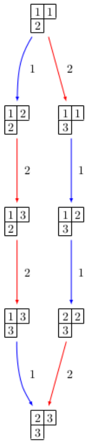


Figure: The crystal $B(\omega_1 + \omega_2)$ for A_2

Definition 1 (Stembridge [Ste96])

Let W be a Coxeter group. An element w is **fully commutative** if any reduced word for w can be obtained from any other by using only the Coxeter relations that involve commuting generators.

Example 2

If $W = S_n$, then w is fully commutative if and only if it is 321-avoiding.

We will mainly be interested in fully commutative elements associated to minuscule representations. Recall that a fundamental weight ω_p is minuscule if W acts transitively on the set of weights appearing in the representation $V(\omega_p)$. Let P_p be the maximal parabolic subgroup associated to ω_p, then the (unique) minimal length representative w_0^P for $w_0 W_{P_p}$ in W/W_{P_p} is fully commutative.

Example 3

Let $\mathfrak{g} = \mathfrak{sl}_4$. All fundamental weights are minuscule, and $V(\omega_2) \cong \wedge^2\mathbb{C}^4$. Then $w_0^P = s_2 s_1 s_3 s_2$, which is indeed fully commutative.

Following Stembridge [Ste96], given a word $\mathbf{w} = r_1 r_2 \cdots r_k$ in W, we define the **heap** $H(\mathbf{w})$ of $\mathbf{w}$ to be the pair consisting of:

1. The poset on $\{1, \ldots, k\}$, where we declare $i \preceq j$ if $i > j$ and the corresponding entry of the Cartan matrix $a_{ij} \neq 0$ and we take transitive closure of this relation.
2. The labeling function π that sends i to s_i.

One can visualize a heap as a configuration of beads on runners arranged according to the Dynkin diagram as in Figure 2, where are dropping the beads one by one, and bead i is dropped on runner r_{k-i+1}.

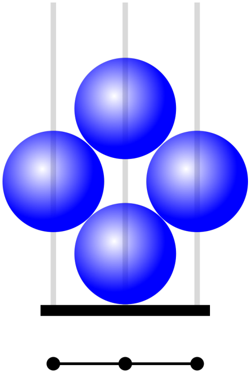


Figure: The heap of the element $s_2 s_1 s_3 s_2$ in type A_3

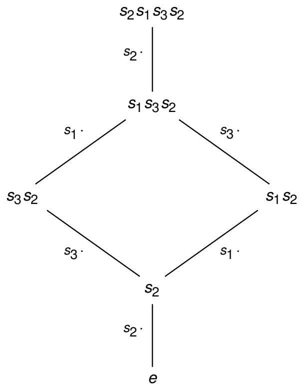
If w is a fully commutative element and $\mathbf{w}$ is a reduced word for w, then the heap $H(\mathbf{w})$ is independent of $\mathbf{w}$, so we'll refer to it as the heap $H(w)$ of w.

Let ω_p be a minuscule fundamental weight, then the weights occurring in $V(\omega_p)$ are in bijection with W/W_{P_p}. In this case, all minimal length representatives for elements of W/W_{P_p} are fully commutative, moreover, they are all elements v of W such that

$$v \leq_l w_0^{P_p}$$

where $\leq_l$ denotes the left weak order, i.e. $v \leq_l w$ if some terminal substring of a reduced word for w is a reduced word for v.

As an example, consider $V(\omega_2)$ for A_3. Then $w_0^{P_2} = s_2 s_1 s_3 s_2$, and the poset W/W_{P_2} is as follows:



Note that the heaps of these elements correspond to **order ideals** in $H(w_0^{P_2})$.

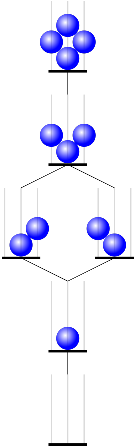


Figure: The heaps corresponding to elements of W/W_{P_2}

We can use these observations to describe a model for crystals of minuscule representations $B(\omega_p)$, where the underlying set is the order ideals (which are heaps themselves) $J(H(w_0^{P_p}))$ of $H(w_0^{P_p})$ and the lowering operators have an easy description: to apply f_j to a heap ϕ, try to remove a bead from runner j. If this is not possible because another bead on a neighboring runner is blocking it, then $f_j(\phi) = 0$, otherwise, $f_j(\phi)$ is ϕ with the highest bead on runner j removed.

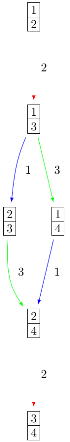


Figure: The crystal $B(\omega_2)$ using Young tableaux

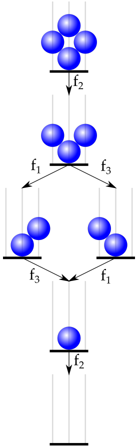


Figure: The crystal $B(\omega_2)$ using $J(H(s_2 s_1 s_3 s_2))$

So far we only considered minuscule representations, but we can use the language of heaps to construct models of more general crystals in a type-independent way.

Theorem 4

Let ω_p be a minuscule fundamental weight. Consider the set of k-fold tensor products of order ideals of $H(w_0^{P_p})$. The subset

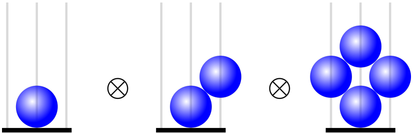
$$H(w_0^{P_p})_{\leq}^{\otimes k} = \left\{ \phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_k \;\middle|\; \phi_j \in J(H(w_0^{P_p})), \phi_i \subseteq \phi_{i+1} \right\},$$

with lowering operators defined using the tensor product rule for crystals, is a model for the crystal $B(k\omega_p)$.

Definition 5

The set $RPP(H(w_0^{P_p}), k)$ of order-reversing maps from the poset $H(w_0^{P_p})$ to $\{0, \ldots k\}$ (with the standard ordering) is called a **reverse plane partition** of shape $H(w_0^{P_p})$.

There is a bijection between elements of $H(w_0^{P_p})^{\otimes k}_{\leq}$ and $RPP(H(w_0^{P_p}), k)$, for example



corresponds to the rpp

$$\begin{pmatrix} & 1 & \\ 1 & & 2 \\ & 3 & \end{pmatrix}$$

The lowering operator f_i acts on rpps by decreasing an entry on the i-th column. For example, for $RPP(H(s_2s_1s_3s_2), k)$ in type A_3, we have (as long as the resulting array is an rpp)

$$f_1\begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{pmatrix} & a & \\ b-1 & & c \\ & d & \end{pmatrix}$$

$$f_2\begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{cases} \begin{pmatrix} & a-1 & \\ b & & c \\ & d & \end{pmatrix} & \text{if } a+d \leq b+c \\ \begin{pmatrix} & a & \\ b & & c \\ & d-1 & \end{pmatrix} & \text{if } a+d > b+c \end{cases}$$

$$f_3\begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix} = \begin{pmatrix} & a & \\ b & & c-1 \\ & d & \end{pmatrix}$$

To summarize our discussion up to this point, we have constructed a model for crystals of the form $B(k\omega_p)$ where ω_p is a minuscule fundamental weight using either k-fold tensor products of heaps, or as reverse plane partitions of shape $H(w_0^{P_p})$. We found that (at least in type A_3) the lowering operators have a nice description in terms of the rpps.

Let Q be an orientation of $\mathfrak{g}$'s Dynkin diagram with vertex set I, and Q^* be the opposite orientation. Consider the doubled quiver $\overline{Q} = Q \cup Q^*$. Let $\mathbb{C}\overline{Q}$ be the path algebra of $\overline{Q}$. Consider the element

$$\rho = \sum_{e \in E(\overline{Q})} \varepsilon(e) e^* e$$

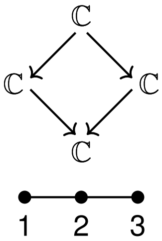
where $\varepsilon(e) = 1$ if $e \in E(Q)$ and -1 if $e \in E(Q^*)$. The algebra $\Lambda(Q) = \mathbb{C}\overline{Q}/(\rho)$ is called the **preprojective algebra of Q.**

Given a heap of the form $H(w_0^{P_p})$, we make the following construction:

1. Replace each element in the heap by $\mathbb{C}$.
2. Replace each covering relation in $H(w_0^{P_p})$ by the identity map $1 : \mathbb{C} \to \mathbb{C}$.
3. Define an I-graded vector space $L(\omega_p)$ by letting $L(\omega_p)_j$ be the direct sum of the 1-dimensional spaces which are labeled by $j \in I$.
4. Define a map from $L(\omega_p)_j$ to $L(\omega_p)_i$ by extending the above-defined maps linearly.

The resulting quiver representation $L(\omega_p)$ is a $\Lambda(Q)$-module, and it is the projective cover of the simple quiver representation $S(p)$ supported at vertex p.

As an example, consider the heap $H(s_2 s_1 s_3 s_2)$ for A_3. Following the construction we get



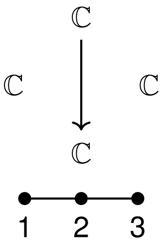
and we see that this is indeed a (projective) $\Lambda(Q)$-module.

We can repeat the above steps with Weyl group elements other than $w_0^{P_p}$. We need w to be **dominant minuscule**, meaning that there is a weight λ and a reduced word $s_{i_1} \cdots s_{i_k}$ for w such that

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_k}(\lambda) = \lambda - \alpha_{i_k} - \alpha_{i_{k-1}} - \ldots - \alpha_{i_j}$$

(this is stronger than fully commutative). In this case, the set $RPP(H(w), k)$ serves as the underlying set of the Demazure crystal $B_w(\lambda)$. The resulting quiver representation is still a $\Lambda(Q)$-module. For simplicity, we'll continue to assume that $w = w_0^{P_p}$.

The construction of the module $L(\omega_p)$ from the heap $H(w_0^P)$ reveals an additional piece of structure. Consider the following map on the elements of $H(w_0^P)$: send each bead to the bead on the same runner just below it, or, if a bead is the lowest bead on the runner, send it to 0. Extend this to a nilpotent endomorphism T of $L(\omega_p)$.



Consider the space $L(k\omega_p) = \{M \subseteq L(\omega_p)^{\oplus k}\}$ of $\Lambda(Q)$-submodules. $L(k\omega_p)$ has connected components indexed by possible dimension vectors of M. We'll also denote the nilpotent endomorphism $T^{\oplus k}$ of $L(k\omega_p)$ by T.
We can obtain a finer decomposition of $L(k\omega_p)$ by considering the Jordan type of T restricted to each M_i. This gives us a partition over each vertex, but there are also conditions between the Jordan types over neighboring vertices, and the Jordan type of T restricted to M is given by an rpp of shape $H(w_0^{P_p})$ with the sum of the entries in column i adding up to $\dim(M_i)$.

Conjecture 6

The irreducible components of $L(k\omega_p)$ are indexed by reverse plane partitions.

For example, consider $L(2\omega_2)$ in type A_3. Let M be a submodule with dimension vector $(1, 2, 1)$. Write M_i for the subspace corresponding to the i-th node of the Dynkin diagram. To choose M_2, we have to choose a 2-dimensional subspace of $\mathbb{C}^2 \oplus \mathbb{C}^2$ stable under the linear map

$$T(x, y, z, w) = (0, 0, x, y).$$

1. If $M_2 = \ker T$, and we can choose M_1 and M_3 arbitrarily, this corresponds to the rpp

$$\begin{pmatrix} & 0 & \\ 1 & & 1 \\ & 2 & \end{pmatrix}$$

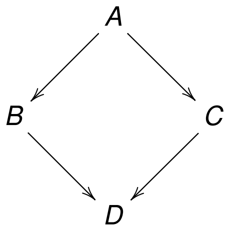
2. If $M_2 \neq \ker T$, then M_1 and M_3 are determined, this corresponds to the rpp

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

Conjecture 7

The lowering operator f_i on the rpps corresponds to taking a generic submodule with quotient the simple module $S(i)$.

Consider the case $L(k\omega_2)$ in type A_3. Let $\phi = \begin{pmatrix} & a & \\ b & & c \\ & d & \end{pmatrix}$ be an rpp, and let M be a module in the component indexed by ϕ. Note that in this case, this just means that $\dim(\ker T \cap M_2) = d$. For simplicity, we identify the subspaces $M_1 = B$ and $M_3 = C$ with their images in $M_2 = A + D$. Visually M looks like this:



We are looking for a submodule of M that fits into the SES

$$0 \to f_2(M) \to M \to S(2) \to 0$$

then we have to choose an $a + d - 1$-dimensional subspace of $M_2 = A + D$. To be a submodule of M, this subspace needs to contain B and C, and therefore $B + C$. Generically, this subspace will not contain all of D, unless $B + C = D$, in which case we are forced to contain all of D.

We claim that

$$\dim(B+C) = \min(b+c-a, d).$$

To get these upper bounds we use the rank-nullity theorem for the operators T^0 and T^1 restricted to M_2.

1. To see that $\dim(B+C) \leq b+c-a$, note that

$$\begin{aligned}\dim(B+C) &= \dim(B) + \dim(C) - \dim(B \cap C) \\ &\leq b+c-a\end{aligned}$$

2. To see that $\dim(B+C) \leq d$, note that

$$\begin{aligned}\dim(B+C) &= \dim(T(B+C)) + \dim((B+C) \cap (\ker T)) \\ &\leq 0+d\end{aligned}$$

(in general we get an upper bound from rank-nullity applied to each operator $T^0, T^1, \ldots T^{m-1}$ where m is the number of beads on the runner in the heap $H(w_0^{P_p})$).

Therefore $B + C = D$ if and only if $b + c - a \geq d$, or equivalently, if

$$a + d \leq b + c.$$

and we see that this is the same rule as the lowering operator on the rpps (16).
In general, to compute $f_i(M)$, we need to know which subspaces of the form $M_i \cap \ker T^j$ must be contained in a generic submodule with quotient $S(i)$. This coincides with the upper bound coming from the rank-nullity theorem applied to T^j being attained, and we want the largest j for which this happens. Then $f_i(M)$ will contain $M_i \cap \ker T^j$ but not $M_i \cap \ker T^{j+1}$, so we know the Jordan type of T restricted to $f_i(M)$.

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