2015 Fall Olivetti: The Springer Resolution and Symplectic Singularities

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1 The cotangent bundle of the projective line

Everything is always over $\mathbb{C}$. Recall $\mathbb{P}^1 = \{[z_0 : z_1] | z_0, z_1 \in \mathbb{C}\}$. Let's find a way to describe $T^*\mathbb{P}^1$ explicitly. We have

$$T_{[z_0:z_1]}\mathbb{P}^1 = \left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathbb{C}^2 : \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \cdot \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = 0 \right\}$$

(by the exact sequence $0 \to \mathcal{L} \to \mathbb{P}^1 \times \mathbb{C}^2 \to T\mathbb{P}^1 \to 0$). Then

$$T^*_{[z_0:z_1]}\mathbb{P}^1 = \left\{ f : T_{[z_0:z_1]}\mathbb{P}^1 \to \mathbb{C} \text{ linear } \right\}.$$

Since $\left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \cdot \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \right\}$ is a basis of $\mathbb{C}^2$, we can cook up a map $A : \mathbb{C}^2 \to \mathbb{C}^2$ by defining

$$A \left(\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right) = f \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right) \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}.$$

Then the matrix A is nilpotent, so $\mathrm{tr}(A) = 0$. So we can describe the cotangent bundle of $\mathbb{P}^1$ as

$$T^*\mathbb{P}^1 = \left\{ (L, A) : L \in \mathbb{P}^1, A \in M_{2\times 2}(\mathbb{C}), A(\mathbb{C}^2) \subseteq L, A \text{ nilpotent} \right\}.$$

Thinking about $T^*\mathbb{P}^1$ this way gives us two natural maps

$$\begin{array}{ccc} T^*\mathbb{P}^1 & \xrightarrow{\mu} & \mathcal{N} \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & & \end{array}$$

where $\mu(L, A) = A, \pi(L, A) = L$, and $\mathcal{N}$ is the set of nilpotent matrices. The map π is just bundle projection, to understand μ better, we need to do some

2 Symplectic geometry of cotangent bundles

Let M be a manifold and ω a 2-form on M (e.g. $\mathbb{C}^2$ with $\langle z_1, z_2 \rangle = 1$). We say (M, ω) is a symplectic manifold if

1. ω is closed, i.e. $d\omega = 0$.

2. ω is non-degenerate, i.e. the map $\omega_m : T_m M \to T_m^* M$ given by $X \mapsto \omega(X, -)$ is an isomorphism for all $m \in M$.

Proposition 2.1. *Let M be a manifold. Then T^*M has a canonical symplectic form.*

Proof. Let $\pi : T^*M \to M$ be the bundle projection map and $\pi_* : T(T^*M) \to TM$ its differential. Then at any point $(m, \alpha) \in T^*M$,

$$\pi_{*(m,\alpha)} : T_{(m,\alpha)}(T^*M) \to T_m M$$

is a linear map. Define $\lambda \in \Omega^1(T^*M)$ by (go through typecheck in the formula)

$$\lambda_{(m,\alpha)}(X) = \alpha(\pi_{*(m,\alpha)} X)).$$

Let $\omega = d\lambda$. Exercise: check that ω is non-degenerate.

Q.E.D.

Example: $\mathbb{C}^n$ with basis $q_1, \ldots q_n$, $T^*\mathbb{C}^n$ with basis $q_1, \ldots, q_n, p_1, \ldots, p_n$, then

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Sometimes when a Lie group G acts on M preserving a symplectic form, there is a moment map

$$\mu : M \to \mathbb{C}^{\dim G}.$$

μ is quite magical, for instance, if G is a torus $T = (\mathbb{C}^\times)^k$, then $\mu(M)$ is the convex hull of M^T (Atiyah, Guillemin-Sternberg).

3 The Springer resolution

Now we again focus on the map $\mu : T^*\mathbb{P}^1 \to \mathcal{N} \subset \mathbb{C}^3$. It was not an accident that we denoted it μ, as it is in fact the moment map for the $SL_2(\mathbb{C})$-action on $\mathbb{P}^1$. The set of nilpotent matrices $\mathcal{N}$ in this case is

$$\mathcal{N} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a^2 + bc = 0 \right\},$$

so it is a quadratic cone in $\mathbb{C}^3 \cong \mathfrak{sl}_2(\mathbb{C})$. **(Draw C-G picture somewhere where it doesn't have to be erased and explain it)** From the picture, we see that μ is *birational.* Note that $\mathcal{N}$ has a singular point at the origin, and as $T^*\mathbb{P}^1$ is smooth, μ is a *resolution of singularities.*

There is a completely analogous picture for any semisimple algebraic group G (e.g. $SL_n(\mathbb{C})$), $\mathbb{P}^1$ is replaced by the flag manifold G/B, and the moment map has its image in the nilcone. The map

$$\mu : T^*(G/B) \to \mathcal{N}$$

is called the Springer resolution.

Remark 3.1. *Just the sheer number of important and well-studied things that are involved with setting up μ, cotangent bundles, moment maps, resolutions of singularities make it awesome. And if that wasn't enough, applications of μ include a construction of all representations of finite reflection groups. The resolution μ is also* semismall*, so it is a place where perverse sheaves work well.*

4 Symplectic singularities

Motivated by the awesomeness of the Springer resolution, one might want to find more examples of this phenomenon.

Definition 4.1. *A symplectic variety is a variety X such that the smooth locus of X has a symplectic form ω (and technical conditions, X should be normal and ω should extend to any resolution as a holomorphic 2-form).*

Note that $\mathcal{N}$ is a symplectic variety, as its smooth locus is the orbit of a regular nilpotent element (draw in Jordan form) under G, so it is an example of a coadjoint orbit (Ben's talk later). If one wants to get even closer to the nilcone, one defines

Definition 4.2. *A conical symplectic variety is a symplectic variety X with a $\mathbb{C}^\times$-action contracting X to a point such that for $t \in \mathbb{C}^\times$, $t^*\omega = t^l \cdot \omega$.*

Maybe mention that there is also a lot of buzz about symplectic singularities recently, somewhat coming from Physics.

5 Complete intersections

And now for something (seemingly) completely different. A variety $Y \subseteq \mathbb{P}^n$ is a *complete intersection* if $I(Y)$ can be generated by $n - \dim(Y)$ elements. A fun example to play with is the twisted cubic curve

$$Y = \overline{[1 : t : t^2 : t^3]} \subset \mathbb{P}^3.$$

Even though Y is the projective closure of

$$Y^o = \{(t, t^2, t^3\} \subset \mathbb{C}^3,$$

and $I(Y^o) = \langle z_1^2 - z_2, z_1^3 - z_3$ is easily generated by 2 elements, if one naively tries to homogenize the generators and lets

$$I(X) = \langle z_1^2 - z_2 z_0, z_1^3 - z_3 z_0^2 \rangle,$$

then

$$X = Y \sqcup [0 : 0 : z_2 : z_3]$$

is a union of the twisted cubic curve and a line. Actually Y isn't a complete intersection, which one can prove by arguing that since $\deg(Y) = 3$ (whatever this means), by hyperplane Bézout theorem the two generators would need to have degrees 1 and 3, but direct checking shows that Y is too twisted to lie on a hyperplane. Note that *as a set*, Y is the intersection of

$$V(\langle z_1^2 - z_0 z_2 \rangle) \cap V(\langle z_2^2(z_1 z_3 - z_2^2) - z_3(z_0 z_3 - z_1 z_2) \rangle)$$

but this intersection is really the twisted cubic twice. Anyway, the point is that this is some very classical algebraic geometry property.

6 Namikawa's theorem

Now the punchline:

Theorem 6.1. *[1] Let X be a conical symplectic variety which is a complete intersection. Then X is the nilcone of some semisimple algebraic group with the symplectic structure Ben will talk about.*

The proof is by a lot of high powered algebraic geometry and contact structures.

Remark 6.2. *This theorem is completely crazy. We start with a variety X with the following properties:*

1. *It has a $\mathbb{C}^\times$-action contracting it to a point,*

2. *It has a symplectic structure ω on the smooth locus that is a weight vector for the $\mathbb{C}^\times$-action,*

3. *It is a complete intersection.*

And out of the blue, there is a G. This also leads us to Allen Knutson's favorite definiton of a Lie algebra

Definition 6.3. *A semisimple complex Lie algebra is a conical symplectic variety which is a complete intersection.*

References

[1] Namikawa, Yoshinori On the structure of homogeneous symplectic varieties of complete intersection. Invent. Math. 193 (2013), no. 1, 159185.