Pizzas and Richardson varieties

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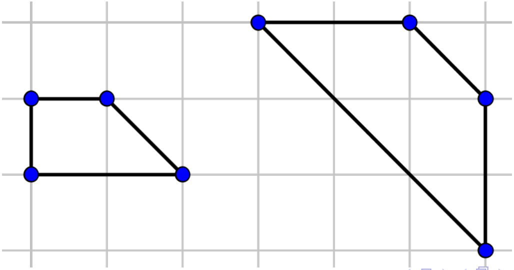
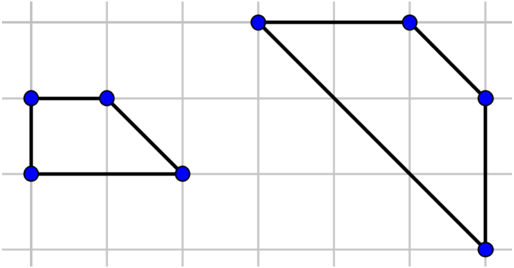
The following equivalence relation describes when two polygons define isomorphic toric surfaces:

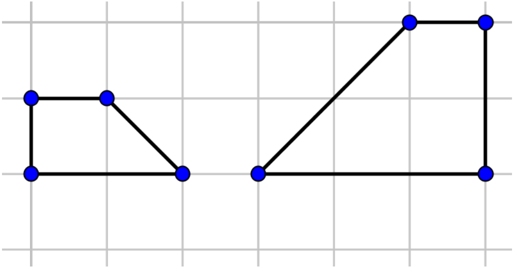
Definition 1

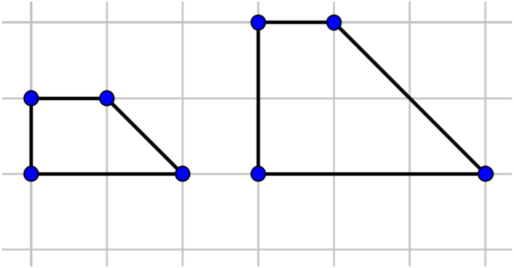
Two lattice polygons in the plane are **equivalent** if there is a continuous bijection between their edges and vertices such that, up to $GL(2, \mathbb{Z})$-transformations, the angles between the corresponding edges match simultaneously.

Question: Are the following two polygons equivalent?

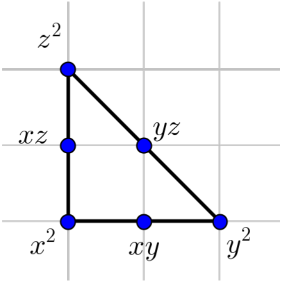






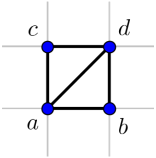


Consider the triangle with vertices $(0,0),(0,2),(2,0)$ in $\mathbb{R}^2$.



This gives us a map $(\mathbb{C}^\times)^2 \to \mathbb{P}^5$ that extends to the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. Note that if we choose the smaller triangle with vertices $(0,0),(0,1),(1,0)$ then we would get the identity map on $\mathbb{P}^2$ (so the underlying toric varieties are isomorphic).

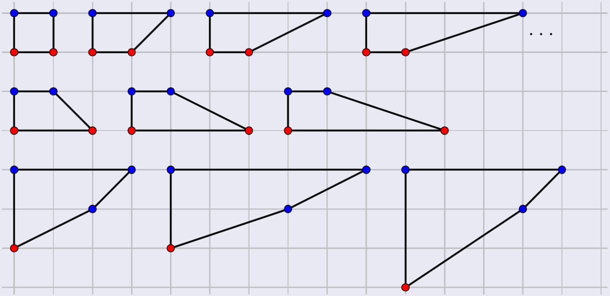
We are interested in when a toric variety degenerates into a union of Richardson varieties. Degenerations of toric varieties have a nice description in terms of subdivisions of the polytope. Consider the following example:



This is a picture of the image of the Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ degenerating to a union of two $\mathbb{CP}^2$-s meeting along a $\mathbb{CP}^1$.

Definition 2

A **pizza slice** is a quadrilateral equivalent to one of the quadrilaterals in the following figure:

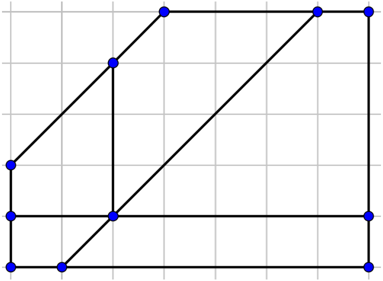


Alternatively, a pizza slice is the moment polytope of a Richardson surface.

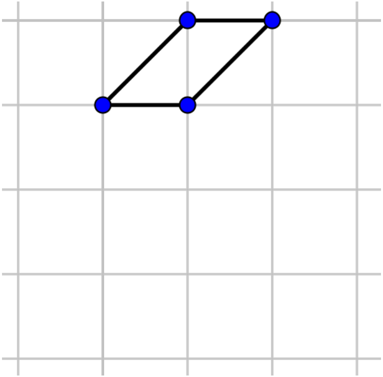
Definition 3

A **pizza** is a polygon subdivided into pizza slices in such a way that each pizza slice attaches to the center of the pizza at one of its red vertices, and each slice has exactly one vertex matching with a vertex of the polygon (its vertex opposite to the central one).

Alternatively, a pizza is a toric surface degenerating into a union of Richardson varieties.



How does one go about baking a pizza? We could just start putting slices together:



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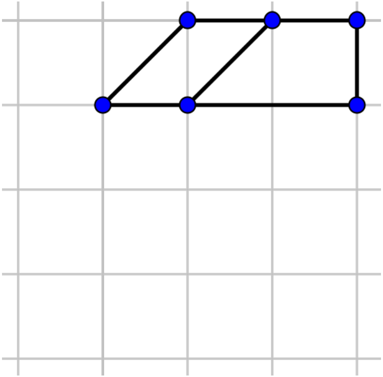

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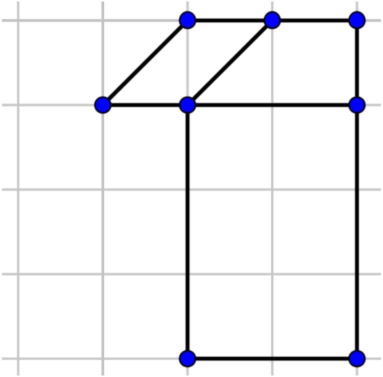

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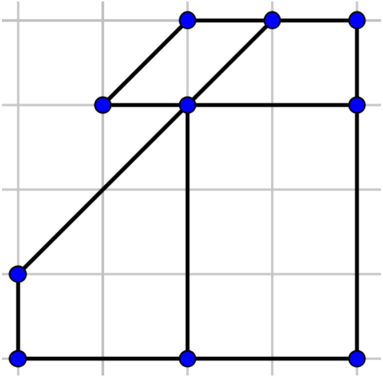

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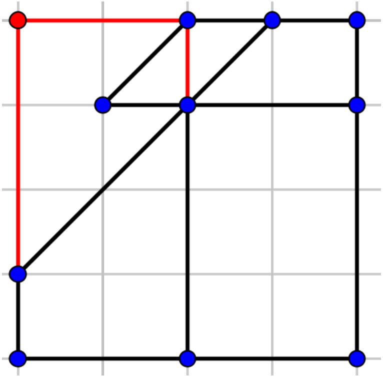
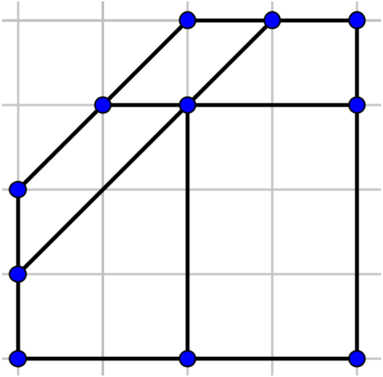
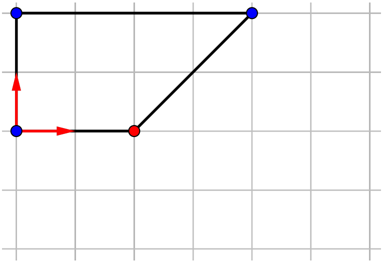


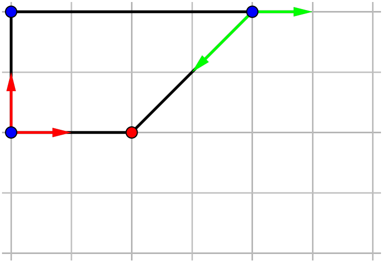
How does one go about baking a pizza? We could just start putting slices together:



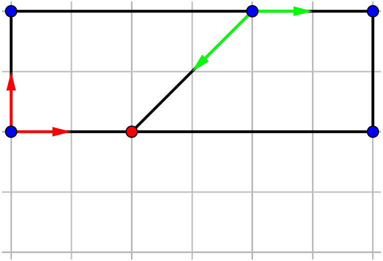
To do it more systematically, start with a single pizza slice sheared in a way that the bottom left basis of $\mathbb{Z}^2$ is the standard basis:



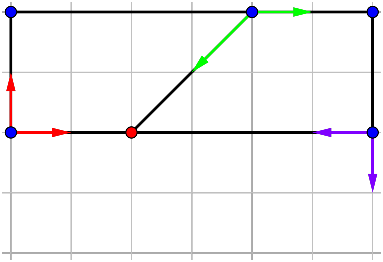
We know that the (clockwise) next slice will have to attach to the green basis



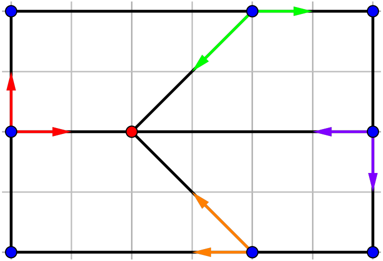
For instance,



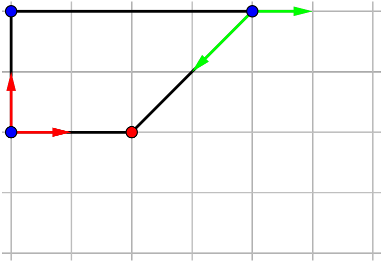
And the next slice will have to attach to the purple basis:



And if a pizza is formed, we must get back to the standard basis after some number of pizza slices

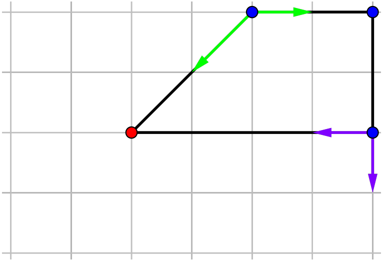


So we assign a matrix (in $SL_2(\mathbb{Z})$) for each pizza slice that records how it transforms the standard basis, for example



is assigned the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

And the second pizza slice



is assigned the matrix $\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$.

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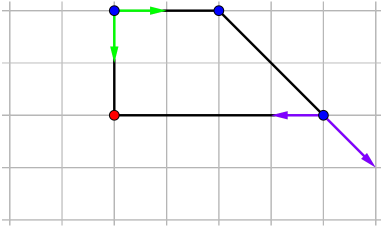

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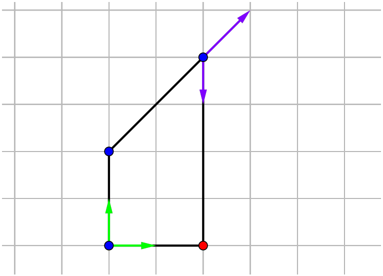


is assigned the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

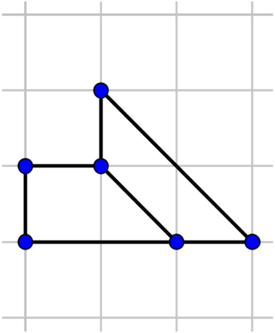
So if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to

So if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to $(MNM^{-1})M = MN$.

Theorem 4

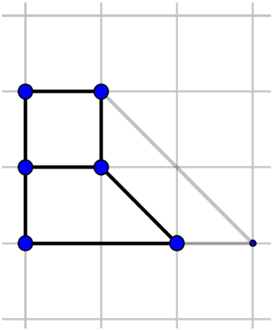
Let $M_1, M_2, \ldots, M_l$ be the matrices associated to a given list of pizza slices. If they form a pizza, then $\prod_{i=1}^{l} M_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

What is wrong with the following pizza?



The current matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

What is wrong with the following pizza?



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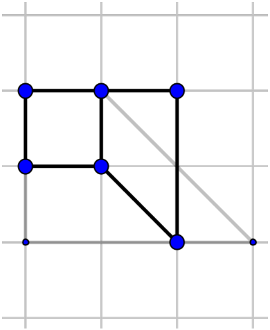

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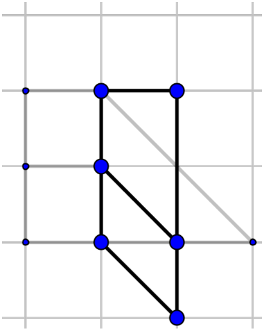

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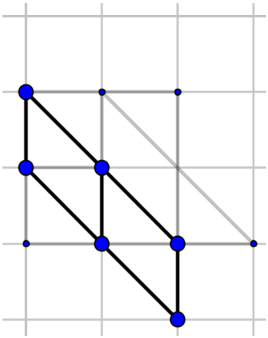

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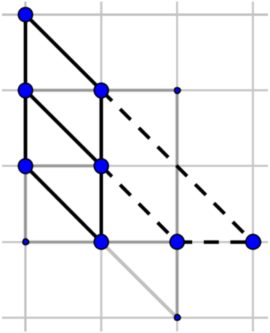
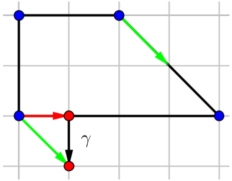


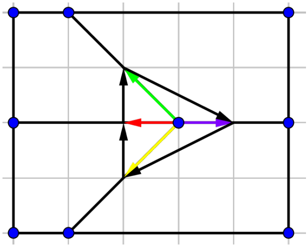
The current matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

To make sure our pizza is single-layered, we want to think of pizza slices not living in $SL(2,\mathbb{R})$ but in its universal cover $\widetilde{SL_2(\mathbb{R})}$. We will represent this by assigning the slice its matrix and the homotopy class of the straight line path connecting $\begin{pmatrix}1\\0\end{pmatrix}$ to $M\begin{pmatrix}1\\0\end{pmatrix}$, i.e.



and we think of multiplication in $\widetilde{SL_2(\mathbb{R})}$ as multiplication of the matrices and appropriate concatenatenation of paths.

Then for a pizza, we will have a closed loop around the origin. Also, as this path is equivalent to the path consisting of following the primitive vectors of the spokes of the pizza, its winding number will coincide with the number of layers of our pizza, as demonstrated by the following picture:

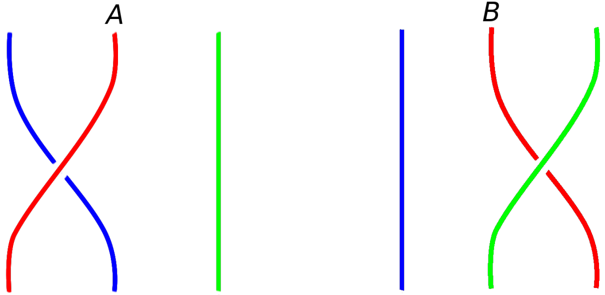


A fun fact about this lifting of pizza slices to $\widetilde{SL_2(\mathbb{R})}$:

Theorem 5 (Wikipedia)

The preimage of $SL_2(\mathbb{Z})$ inside $\widetilde{SL_2(\mathbb{R})}$ is Br_3, the braid group on 3 strands.

The braid group Br_3 is generated by the braids A and B (and their inverses):



with (vertical) concatenatenation as multiplication, satisfying the braid relation $ABA = BAB$.

The homomorphism $Br_3 \to SL(2, \mathbb{Z})$ is given by:

$$A \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$B \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Exercise: Check what the braid relation corresponds to via this mapping.

There is a very special element of Br_3, the “full twist” braid $(AB)^3$, that is mapped to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In fact, the kernel of the homomorphism is generated by $(AB)^6$.

Braids are really cool, but for computational reasons we would prefer to work with matrices:

Lemma 6

The map $Br_3 \to SL_2(\mathbb{Z}) \times \mathbb{Z}$, with second factor ab given by abelianization, is injective.

So for each pizza slice, we want to specify an integer.

This integer should be compatible with the abelianization maps:

Lemma 7 ([Kon])

The abelianization of $SL_2(\mathbb{Z})$ *is* $\mathbb{Z}/12\mathbb{Z}$*. Moreover, for*

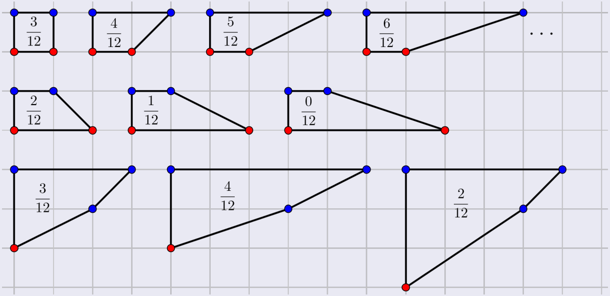
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

the image in $\mathbb{Z}/12\mathbb{Z}$ *can be computed by taking*

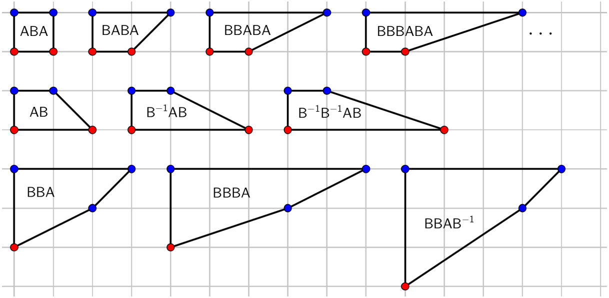
$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((1-c^2)(bd+3(c-1)d+c+3)+c(a+d-3))/12\mathbb{Z}.$$

Definition 8

The **nutritive value** ν of a pizza slice S is the rational number $\frac{\mathrm{ab}(S)}{12}$. They are given by



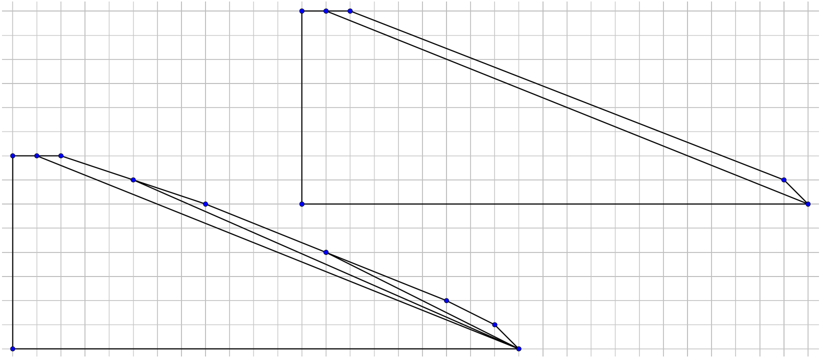
Assigning the nutritive value of pizza slices is equivalent to lifting their matrices to Br_3:



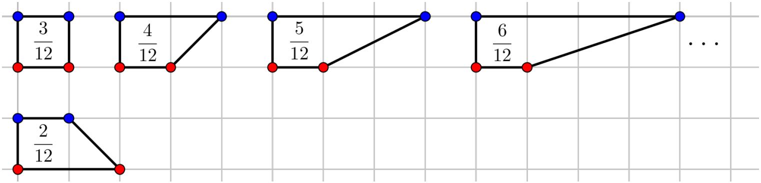
(Notice: $\nu(S)$ is equal to the number of *A*s and *B*s, minus the number of A^{-1}s and B^{-1}s)

Now we can make sure our pizza is bakeable in a conventional oven by requiring that the product of the matrices is the identity, and the sum of the nutritive values of the slices in the pizza is $\frac{12}{12}$. This almost reduces the classification to a finite problem. Rephrasing this in terms of braids, a pizza is a list of the words of the slices whose product is equal to the double full twist element $(AB)^6$.

There is still a problem, as the slice with nutritive value $\frac{0}{12}$, one fears it might appear arbitrarily many times in a pizza. In fact it can



One way to circumvent this issue is to chicken out and only consider pizzas made of the following pizza slices (a condition that we will refer to as "simply laced"):



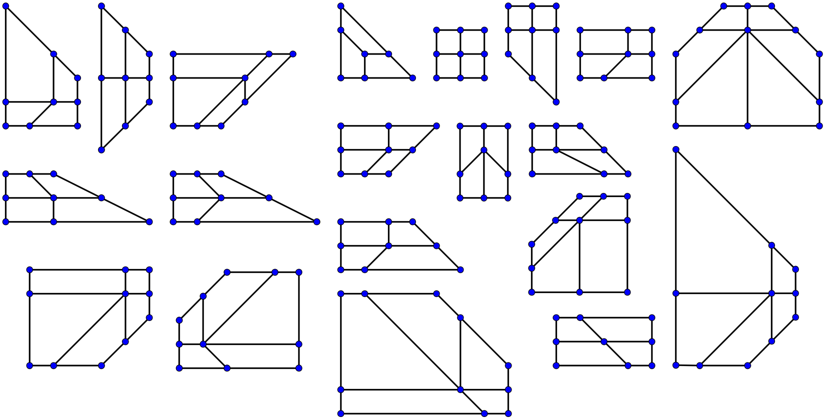
Our main results are the following:

Theorem 9

There are 20 *non-equivalent pizzas made of simply laced pizza slices.*

Theorem 10

Without the simply-laced assumption, there are at most 7543 *pizzas for which the degeneration can be carried out inside the flag manifold of a Kac-Moody group.*



[Kon] Keith Konrad.
$SL(2,\mathbb{Z})$, notes posted online:.
available online at http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/SL(2,Z).pdf.