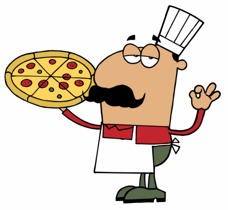
Pizzas and toric surfaces with Kazhdan-Lusztig atlases



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Let $Gr_k(\mathbb{C}^n)$ be the Grassmannian of k-subspaces of $\mathbb{C}^n$. Define the family

$$F = \left\{ (V_1, \dots, V_n, s) : V_i \in Gr_k(\mathbb{C}^n), \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ s & 0 & 0 & \dots & 0 \end{pmatrix} V_i \subseteq V_{i+1 \pmod n} \right\}.$$

For $s \neq 0$, if we know V_1, then the rest of the V_i's are uniquely determined, so

$$F_s \cong Gr_k(\mathbb{C}^n),$$

but the fiber F_0 is something new.

The Grassmannian, and hence F_s has an action of $T = (\mathbb{C}^\times)^n$, and for $s \neq 0$ the fixed points of which are identified with $F_s^T = \binom{[n]}{k}$ (where $[n] = \{1, 2, \ldots, n\}$). For the special fiber,

$$F_0^T = \left\{ (\lambda_1, \ldots, \lambda_n) \in \binom{[n]}{k}^n : \text{shift}_{-1}(\lambda_i) \subseteq \lambda_{i+1} \right\}$$

where $\text{shift}_{-1}(\lambda_i) = \left(\{\lambda_i^1 - 1, \ldots \lambda_i^k - 1\} \cap [n]\right)$.

Question

What are the objects that naturally index F_0^T?

Let $\widehat{W} = \{f : \mathbb{Z} \to \mathbb{Z} : f(i+n) = f(i) + n\}$ be the Weyl group of $\widehat{GL_n(\mathbb{C})}$. It contains the so-called bounded juggling patterns

$$\mathrm{Bound}(k,n) := \left\{ f \in \widehat{W} : f(i) - i \in [0,n], \left(\sum_{i=1}^{n} f(i) - i \right) \Big/ n = k \right\}.$$

Define a map $w : \mathrm{Bound}(k,n) \to F_0^T$ by $f \mapsto (\lambda_1, \ldots, \lambda_n)$, where

$$\lambda_i = \left((f(\leq i) - i) \setminus (-\mathbb{N}) \in \binom{[n]}{k} \right)$$

Then w is a bijection, but more is true:

Theorem 1

(Knutson, Lam, Speyer, [2]) The map w is an order-reversing map from the poset of positroid strata of $Gr_k(\mathbb{C}^n)$ to $\widehat{W}$.

The geometry agrees with the combinatorics, in the sense that

Theorem 2

(Snider, [5]) There is a stratified isomorphism between the standard open sets U_f of $Gr_k(\mathbb{C}^n)$ and $X_o^{w(f)} \subseteq \widehat{GL_n(\mathbb{C})}/B$.

and the T-equivariant degeneration of $Gr_k(\mathbb{C}^n)$ (via F) sits inside $\widehat{GL_n(\mathbb{C})}/B$ as a union of Schubert varieties. We would like to axiomatize this phenomenon, so for a stratified T_M-manifold $(M, \mathcal{Y})$, we want a Kac-Moody group H, and we want the stratifications to "match up" appropriately. A **Bruhat atlas** is a way of locally modeling the stratification of a variety on the stratification of Schubert cells by opposite Schubert varieties. More precisely,

Definition 3

(He, Knutson, Lu, [1]) An **Bruhat atlas** on a stratified T_M-manifold $(M, \mathcal{Y})$ is the following data:

1. A Kac-Moody group H with $T_M \hookrightarrow T_H$,
2. An atlas for M consisting of affine spaces U_f around the minimal strata, so $M = \bigcup_{f \in \mathcal{Y}_{\min}} U_f$,
3. A ranked poset injection $w : \mathcal{Y}^{\mathrm{opp}} \hookrightarrow W_H$ whose image is a union of Bruhat intervals $\bigcup_{f \in \mathcal{Y}_{\min}} [e, w(f)]$,
4. For $f \in \mathcal{Y}_{\min}$, a stratified T_M-equivariant isomorphism $c_f : U_f \xrightarrow{\sim} X_o^{w(f)} \subset H/B_H$,
5. A T_M-equivariant degeneration $M \rightsquigarrow M' := \bigcup_{f \in \mathcal{Y}_{\min}} X^{w(f)}$ of M into a union of Schubert varieties, carrying the anticanonical line bundle on M to the $\mathcal{O}(\rho)$ line bundle restricted from H/B_H.

Some remarkable families of stratified varieties possess Bruhat atlases:

Theorem 4

(He, Knutson, Lu, [1]) Let G be a semisimple linear algebraic group. There are Bruhat atlases on every G/P, and on the wonderful compactification $\overline{G}$ of G.

A rather interesting fact about the Bruhat atlases on the above spaces related to G is that the Kac-Moody group H is essentially never finite, or even affine type, although H's Dynkin diagram is constructed from G's.

Bruhat atlases put the families G/P and $\overline{G}$ in the same basket, so one naturally wonders what other spaces could have this structure. Let $(H, \{c_f\}_{f \in \mathcal{Y}_{\min}}, w)$ be an equivariant Bruhat atlas on $(M, \mathcal{Y})$. We would like to understand what sort of structure a stratum $Z \in \mathcal{Y}$ inherits from the atlas. Each Z has a stratification,

$$Z := \bigcup_{f \in \mathcal{Y}_{\min}} U_f \cap Z, \quad \text{with} \quad U_f \cap Z \cong X_o^{w(f)} \cap X_{w(Z)}$$

since by (3), the isomorphism $U_f \cong X_o^{w(f)}$ is stratified. Therefore Z has an "atlas" composed of Kazhdan-Lusztig varieties.

Definition 5

A **Kazhdan-Lusztig atlas** on a stratified T_V-variety $(V, \mathcal{Y})$ is:

1. A Kac-Moody group H with $T_V \hookrightarrow T_H$,
2. A ranked poset injection $w_M : \mathcal{Y}^{\text{opp}} \to W_H$ whose image is
$$\bigcup_{f \in \mathcal{Y}_{\min}} [w(V), w(f)],$$
3. An open cover for V consisting of affine varieties around each $f \in \mathcal{Y}_{\min}$ and choices of a T_V-equivariant stratified isomorphisms
$$V = \bigcup_{f \in \mathcal{Y}_{\min}} U_f, \quad U_f \cong X_o^{w(f)} \cap X_{w(V)},$$
4. A T_V-equivariant degeneration $V \rightsquigarrow V' = \bigcup_{f \in \mathcal{Y}_{\min}} X^{w(f)} \cap X_{w(V)}$ carrying some ample line bundle on V to $\mathcal{O}(\rho)$.

Having gone through all this confusing nonsense, we'll bake some pizzas.

We start with the case that is often the easiest, namely, toric varieties. A **toric variety** is an algebraic variety containing an algebraic torus $(\mathbb{C}^\times)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

Example 6

$\mathbb{P}^n$ is a toric variety. The points with homogeneous coordinates $[1, t_1, t_2, \ldots, t_n]$ with $t_i \neq 0$ form an n-torus, and its action on itself clearly extends to all of $\mathbb{P}^n$.

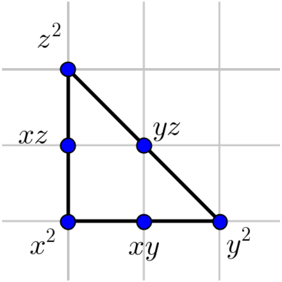
In this talk, we will focus on projective toric varieties.

Let $T \cong (\mathbb{C}^\times)^n$ be a torus. Then $M = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^\times)$ is T's character lattice. Let P be a lattice polytope in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with lattice points $p_0, \ldots p_k$. Consider the map

$$\begin{aligned} \Phi_P : T &\to \mathbb{P}^{|P|-1} \\ t &\mapsto [\chi^{p_0}(t), \chi^{p_1}(t), \ldots, \chi^{p_k}(t)]. \end{aligned}$$

We call $V_P = \overline{\Phi_P(T)} \subseteq \mathbb{P}^k$ the toric variety associated to P.

Consider the triangle with vertices $(0,0), (0,2), (2,0)$ in $\mathbb{R}^2$.



This gives us a map $(\mathbb{C}^\times)^2 \to \mathbb{P}^5$ that extends to the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. Note that if we choose the smaller triangle with vertices $(0,0), (0,1), (1,0)$ then we would get the identity map on $\mathbb{P}^2$ (so the underlying toric varieties are isomorphic).

Question

Which toric varieties admit Bruhat/K-L atlases?

As a first step toward answering the question above, we ask

Question

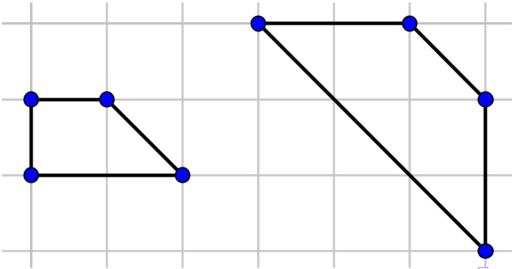
*Which **smooth** toric **surfaces** admit Bruhat/K-L atlases?*

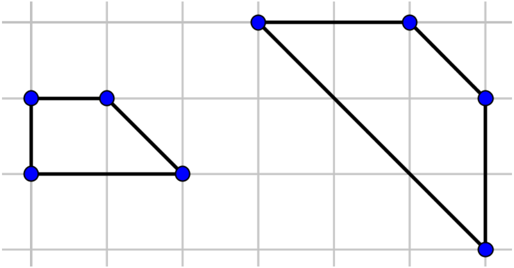
The following equivalence relation describes when two polygons define isomorphic toric surfaces:

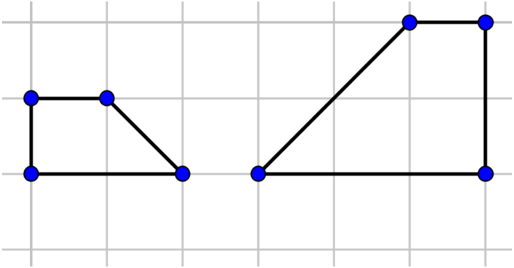
Definition 7

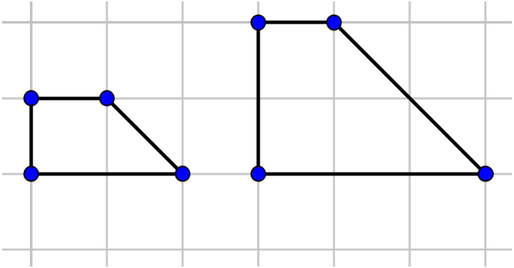
Two lattice polygons in the plane are **equivalent** if there is a continuous bijection between their edges and vertices such that, up to $GL(2,\mathbb{Z})$-transformations, the angles between the corresponding edges match simultaneously.

Question: Are the following two polygons equivalent?

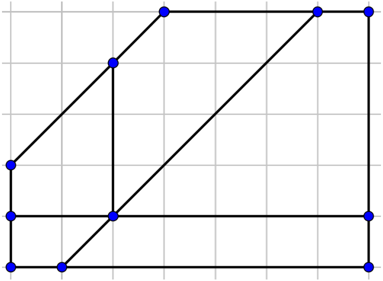








Let M be a smooth toric surface with an equivariant Kazhdan-Lusztig atlas. Part (4) of definition 5 gives us a decomposition of M's moment polygon into the moment polygons of the Richardson surfaces $X^{w(f)} \cap X_{w(V)}$, or, more pictorially, a slicing of the polytope into "pizza slices":

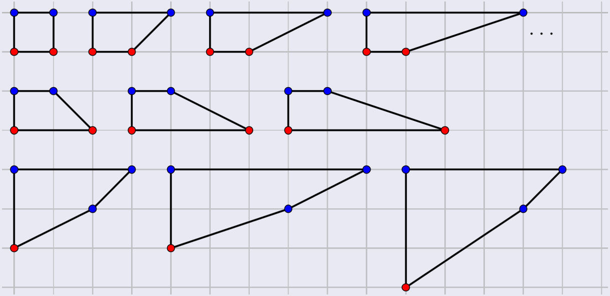


The definition of a Kazhdan-Lusztig atlas is a big package, so we summarize what we are after as a checklist. To put a Kazhdan-Lusztig atlas on a **smooth** toric surface M, we need:

- ☐ A subdivision of M's moment polygon into a “pizza”.
- ☐ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

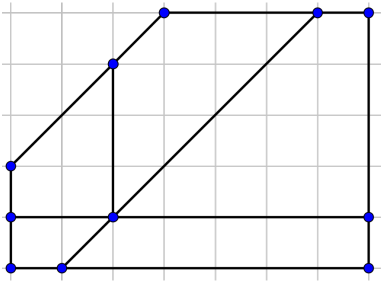
Proposition 8

*The moment polytope of a Richardson surface in any H appears in a rank 2 Kac-Moody group, and the following is a complete list of the ones who are smooth everywhere except possibly where they attach to the center of the pizza (possible center locations in red). We refer to these as **pizza slices**.*



Definition 9

A **pizza** is a polygon subdivided into pizza slices in such a way that each pizza slice attaches to the center of the pizza at one of its red vertices, and each slice has exactly one vertex matching with a vertex of the polygon (its vertex opposite to the central one).

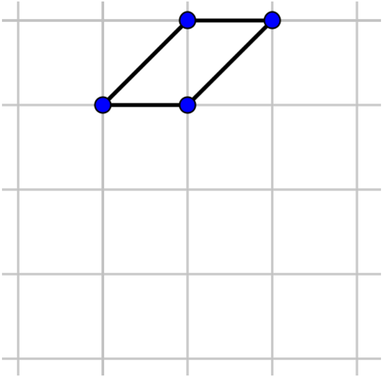


We will return to other aspects of the pizza after we are finished with the crust, but until then, we ask:

Question

Up to equivalence, how many pizzas are there?

How does one go about baking a pizza? We could just start putting pieces together:



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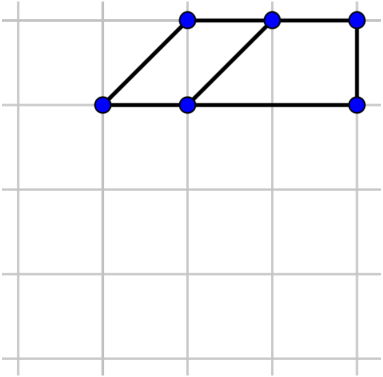

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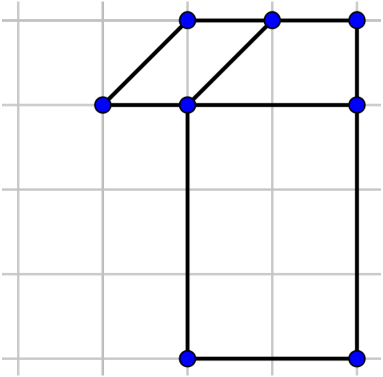

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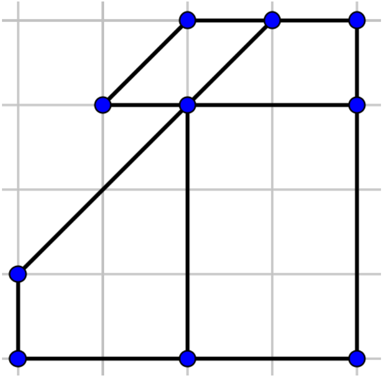

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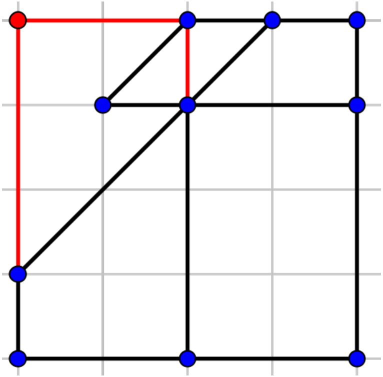
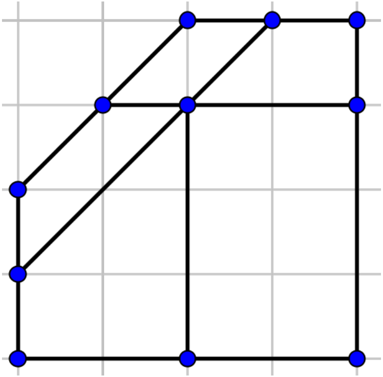
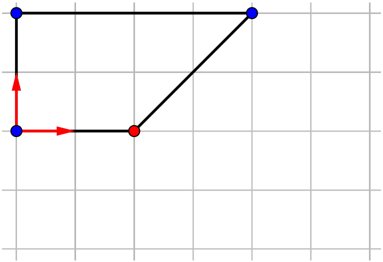


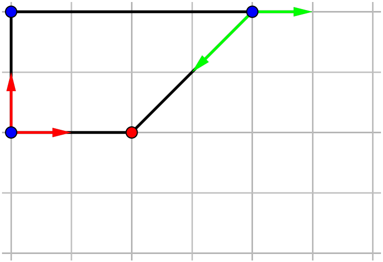
How does one go about baking a pizza? We could just start putting pieces together:



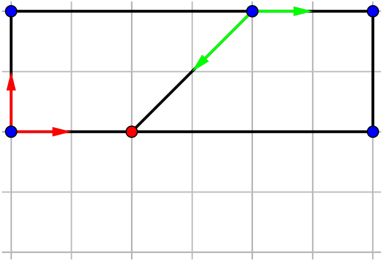
To do it more systematically, start with a single pizza slice sheared in a way that the bottom left basis of $\mathbb{Z}^2$ is the standard basis:



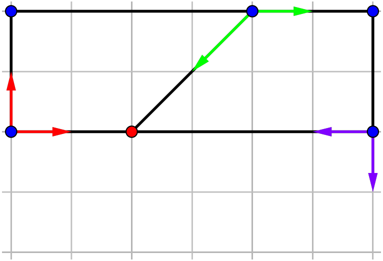
We know that the (clockwise) next slice will have to attach to the green basis



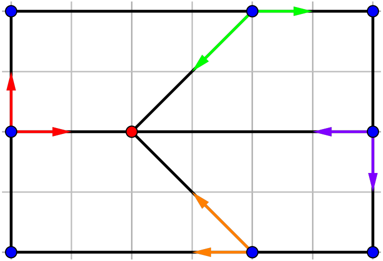
For instance,



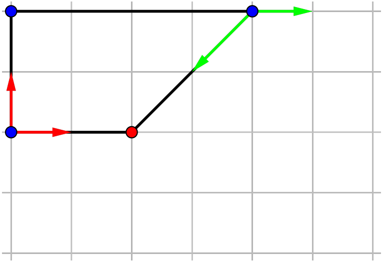
And the next slice will have to attach to the purple basis:



And if a pizza is formed, we must get back to the standard basis after some number of pizza slices

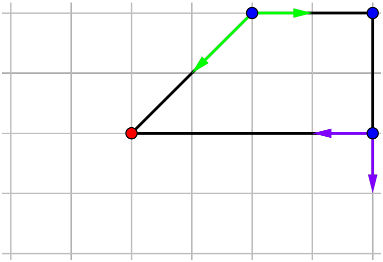


So we assign a matrix (in $SL_2(\mathbb{Z})$) for each pizza slice that records how it transforms the standard basis, for example



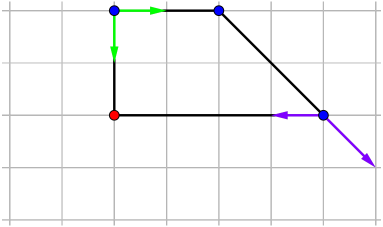
is assigned the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

And the second pizza slice



is assigned the matrix $\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$.

And the second pizza slice



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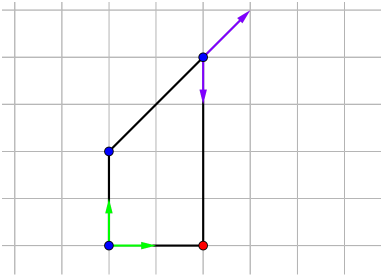


is assigned the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

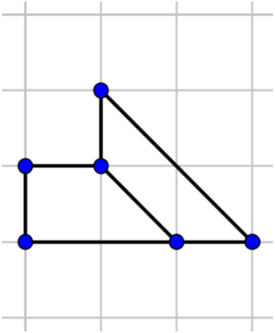
So if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to

So if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to $(MNM^{-1})M = MN$.

Theorem 10

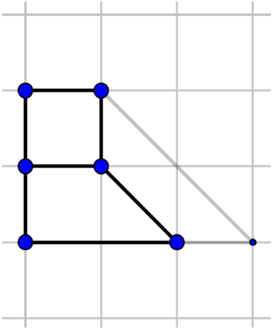
Let $M_1, M_2, \ldots, M_l$ be the matrices associated to a given list of pizza slices. If they form a pizza, then $\prod_{i=1}^{l} M_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

What is wrong with the following pizza?



The current matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

What is wrong with the following pizza?



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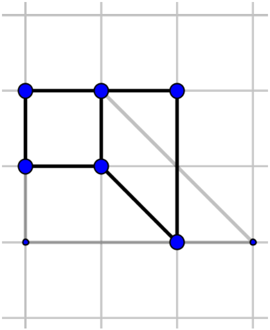

The current matrix is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

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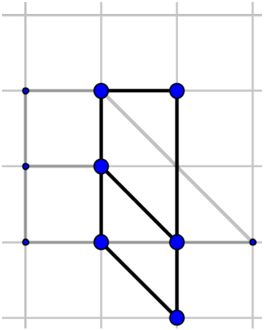
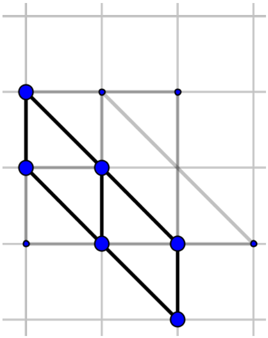


The current matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

What is wrong with the following pizza?



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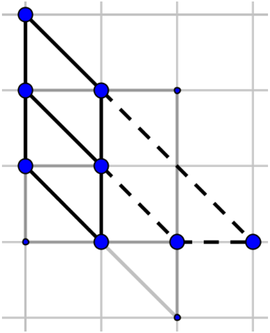
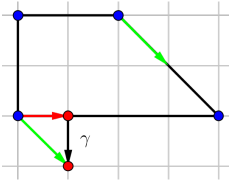


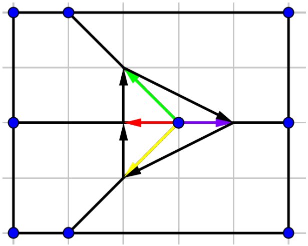
The current matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

To make sure our pizza is single-layered, we want to think of pizza slices not living in $SL(2,\mathbb{R})$ but in its universal cover $\widetilde{SL_2(\mathbb{R})}$. We will represent this by assigning the slice its matrix and the homotopy class of the straight line path connecting $\begin{pmatrix}1\\0\end{pmatrix}$ to $M\begin{pmatrix}1\\0\end{pmatrix}$, i.e.



and we think of multiplication in $\widetilde{SL_2(\mathbb{R})}$ as multiplication of the matrices and appropriate concatenatenation of paths.

Then for a pizza, we will have a closed loop around the origin. Also, as this path is equivalent to the path consisting of following the primitive vectors of the spokes of the pizza, its winding number will coincide with the number of layers of our pizza, as demonstrated by the following picture:

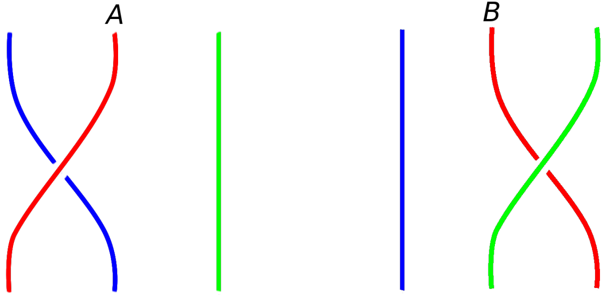


A fun fact about this lifting of pizza slices to $\widetilde{SL_2(\mathbb{R})}$:

Theorem 11

(Wikipedia) The preimage of $SL_2(\mathbb{Z})$ inside $\widetilde{SL_2(\mathbb{R})}$ is Br_3, the braid group on 3 strands.

The braid group Br_3 is generated by the braids A and B (and their inverses):



with (vertical) concatenatenation as multiplication, satisfying the braid relation $ABA = BAB$.

The homomorphism $Br_3 \to SL(2,\mathbb{Z})$ is given by:

$$A \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$B \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Exercise: Check what the braid relation corresponds to via this mapping.

There is a very special element of Br_3, the "full twist" braid $(AB)^3$, who gets sent to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In fact, the kernel of the homomorphism is generated by $(AB)^6$.

Braids are really cool, but for computational reasons we would prefer to work with matrices:

Lemma 12

The map $Br_3 \to SL_2(\mathbb{Z}) \times \mathbb{Z}$, with second factor ab given by abelianization, is injective.

So for each pizza slice, we want to specify an integer.

This integer should be compatible with the abelianization maps:

Lemma 13

([3]) The abelianization of $SL_2(\mathbb{Z})$ is $\mathbb{Z}/12\mathbb{Z}$. Moreover, for

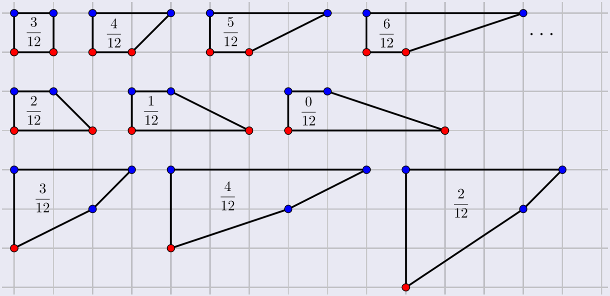
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

the image in $\mathbb{Z}/12\mathbb{Z}$ can be computed by taking

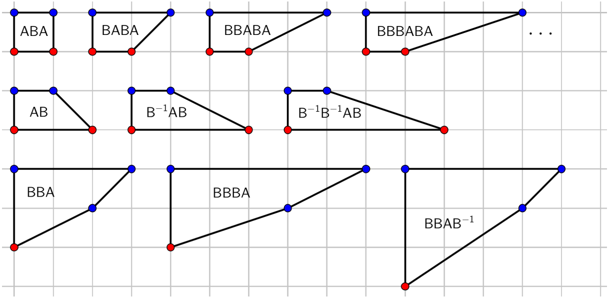
$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((1-c^2)(bd+3(c-1)d+c+3)+c(a+d-3))/12\mathbb{Z}.$$

Definition 14

The **nutritive value** ν of a pizza slice S is the rational number $\frac{\mathrm{ab}(S)}{12}$. They are given by



Assigning the nutritive value of pizza slices is equivalent to lifting their matrices to Br_3:



(Notice: $\nu(S)$ is equal to the number of As and Bs, minus the number of A^{-1}s and B^{-1}s)

Now we can make sure our pizza is bakeable in a conventional oven by requiring that the product of the matrices is the identity, and the sum of the nutritive values of the slices in the pizza is $\frac{12}{12}$. This almost reduces the classification to a finite problem. Rephrasing this in terms of braids, a pizza is a list of the words of the slices whose product is equal to the double full twist element $(AB)^6$.

Using the nutritive values of the Richardson quadrilaterals, we force this through Sage [4] to obtain all possible pizzas.

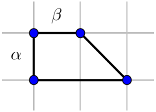
- ☑ A subdivision of M's moment polygon into a pizza.
- ☐ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

Recall that in order to have a Kazhdan-Lusztig atlas on a toric surface, we need a Kac-Moody group H and a map $w : \mathcal{Y}^{\text{opp}} \to W$, i.e. we need a map from the vertices of the pizza to W, where vertices should be adjacent when there is a covering relation between them.

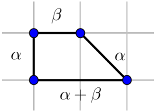
Lemma 15

All covering relations $v \lessdot w$ are of the form $vr_\beta = w$ for some positive root β, and we will label the edges in the pizza by these positive roots of H. The lattice length of an edge in a pizza equals the height of the corresponding root.

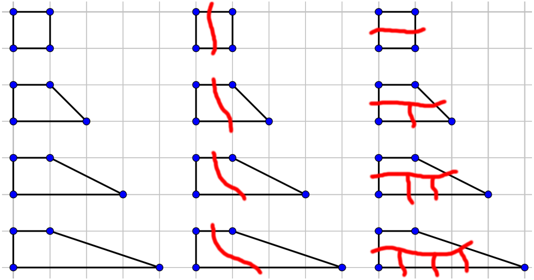
Note that the covering relations in W correspond to T-invariant $\mathbb{CP}^1$'s in H/B_H, and the edge labels are determined by the cohomology classes of these. For instance, if we know the labels on two edges of a pizza slice:

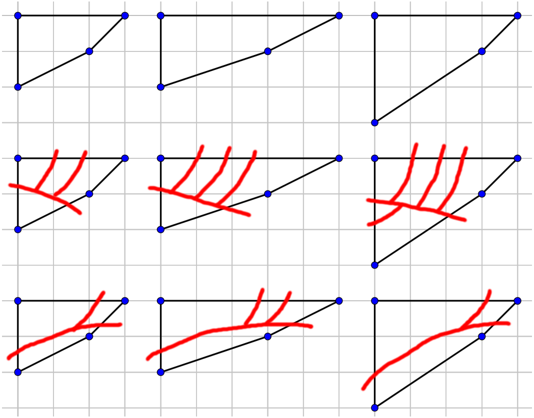


Then we can deduce the other two:



To keep track of the ways a root can appear as a summand for the class of an edge, we introduce toppings. A topping is a curve drawn across the edges of the pizza, and the possible toppings on the individual pizza slices are:



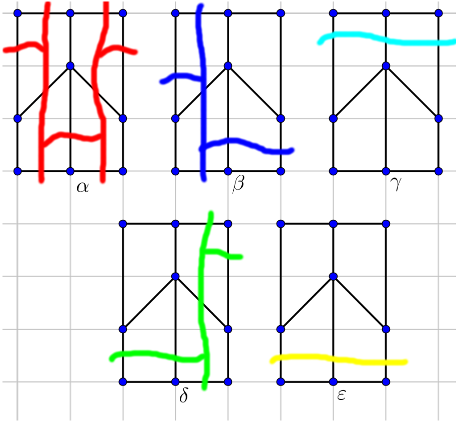


Also, even though we specified the allowed topping configurations on the individual slices, they should of course be consistent across the pizza:

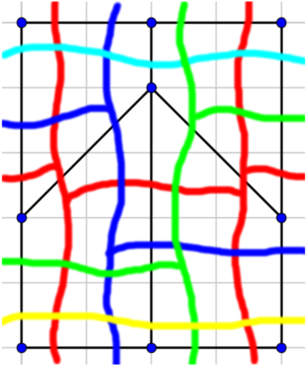
- Every edge of the pizza must have the number of toppings equal to its lattice length going across it.
- Toppings can only end at the edge of the pizza, not between slices.
- No two spokes (edges adjacent to the center vertex) should have the same set of toppings over them.
- No two spokes should have a combined amount of toppings on them equal to the toppings on a third spoke.

If these conditions are satisfied, then we call this configuration a **topping arrangement** (secretly these weird conditions are there because of the structure of Bruhat order, and part (2) of definition 5).

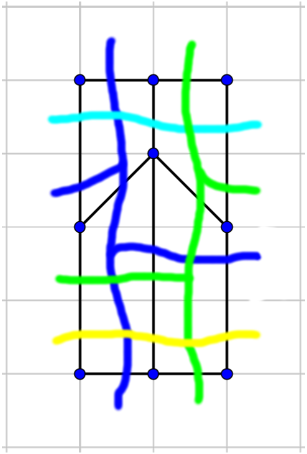
For example, these are all the possible toppings on this pizza:



And here is a topping arrangement:



Sometimes we do not need to use all available toppings to get an arrangement:



Considering the topping configurations on the pizzas, we can find out how the simple roots should be arranged, i.e. identify potential H's.

- ☑ A subdivision of M's moment polygon into a pizza.
- ☑ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☐ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

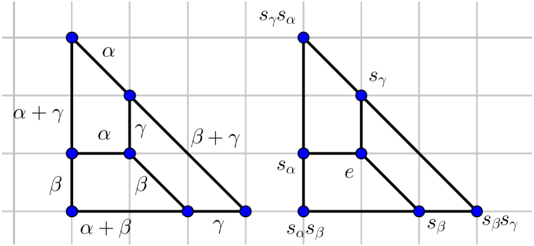
For a given H, finding W_H-elements labeling the vertices of the pizza can (usually) be done by computer.

- ☑ A subdivision of M's moment polygon into a pizza.
- ☑ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☑ An assignment w of elements of W_H to the vertices of the pizza.
- ☐ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

Having the labels on the vertices, for H finite type, we may use the maps $H/B_H \twoheadrightarrow H/P_{\alpha_i^c}$ for simple roots α_i to find which Plücker coordinates should vanish on a potential m.

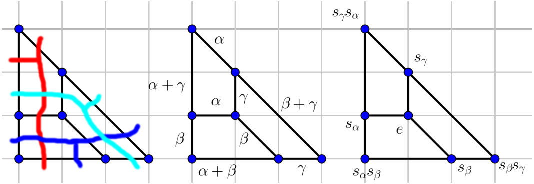
- ☑ A subdivision of M's moment polygon into a pizza.
- ☑ A Kac-Moody group H with $T_M \hookrightarrow T_H$.
- ☑ An assignment w of elements of W_H to the vertices of the pizza.
- ☑ A point $m \in H/B_H$ such that $\overline{T_M \cdot m} \cong M$.

Consider the example of $\mathbb{CP}^2$:

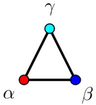


with α, β, γ the simple roots of $H = \widehat{SL_2(\mathbb{C})}$.

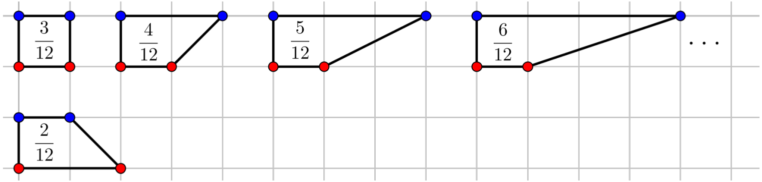
For $\mathbb{CP}^2$, the compatible topping arrangement leading to this atlas is:



with H's diagram being



Because of the low nutritive value of certain pizza slices, and to make the problem more amenable to computers, we decided to also consider pizzas made of the following pizza slices (a condition that we will refer to as “simply laced”):



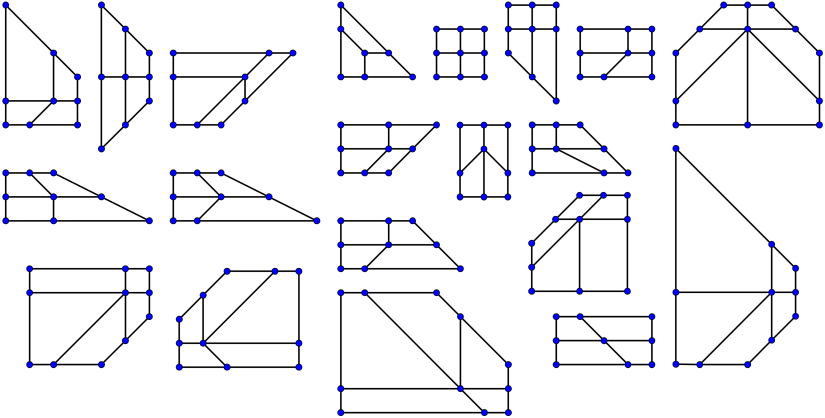
Our main results are the following:

Theorem 16

There are 20 *non-equivalent pizzas made of simply laced pizza slices, and at least* 19 *of those have Kazhdan-Lusztig atlases (a necessary condition for this is the existence of a topping arrangement). Moreover, in each of the cases where* H *is of finite type, the degeneration can be carried out inside* H/B_H.

Theorem 17

Without the simply-laced assumption, there are at most 7543 *pizzas with Kazhdan-Lusztig atlases.*



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Compos. Math. 149 (2013), no. 10, 17101752. 14M15 (05B35 05E05)

[3] Konrad, K., $SL(2, \mathbb{Z})$, notes posted online: `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/SL(2,Z).pdf`

[4] Sage Mathematics Software (Version 5.0.1), The Sage Developers, 2015, `http://www.sagemath.org`.

[5] Snider, M.B., *Affine patches on positroid varieties and affine pipe dreams.*
Thesis (Ph.D.)Cornell University. 2010. 70 pp.