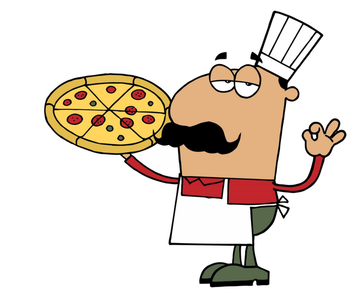
Pizzas and Kazhdan-Lusztig atlases in dimension 2



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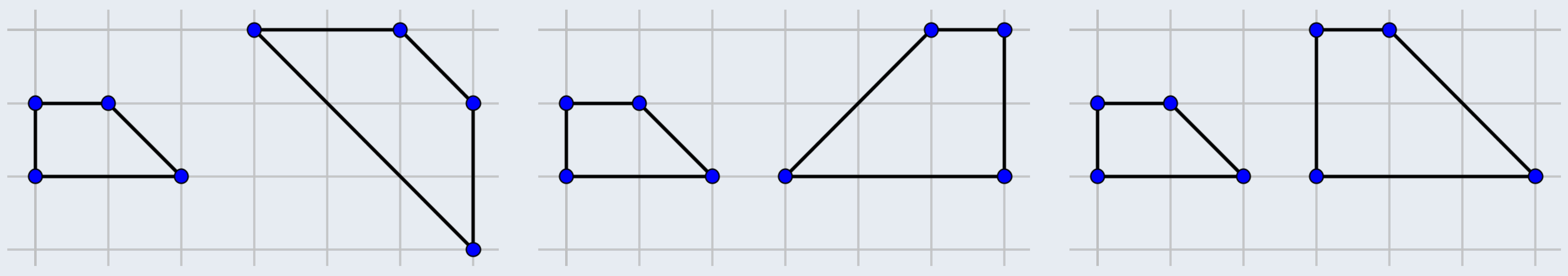
An equivalence relation on lattice polygons

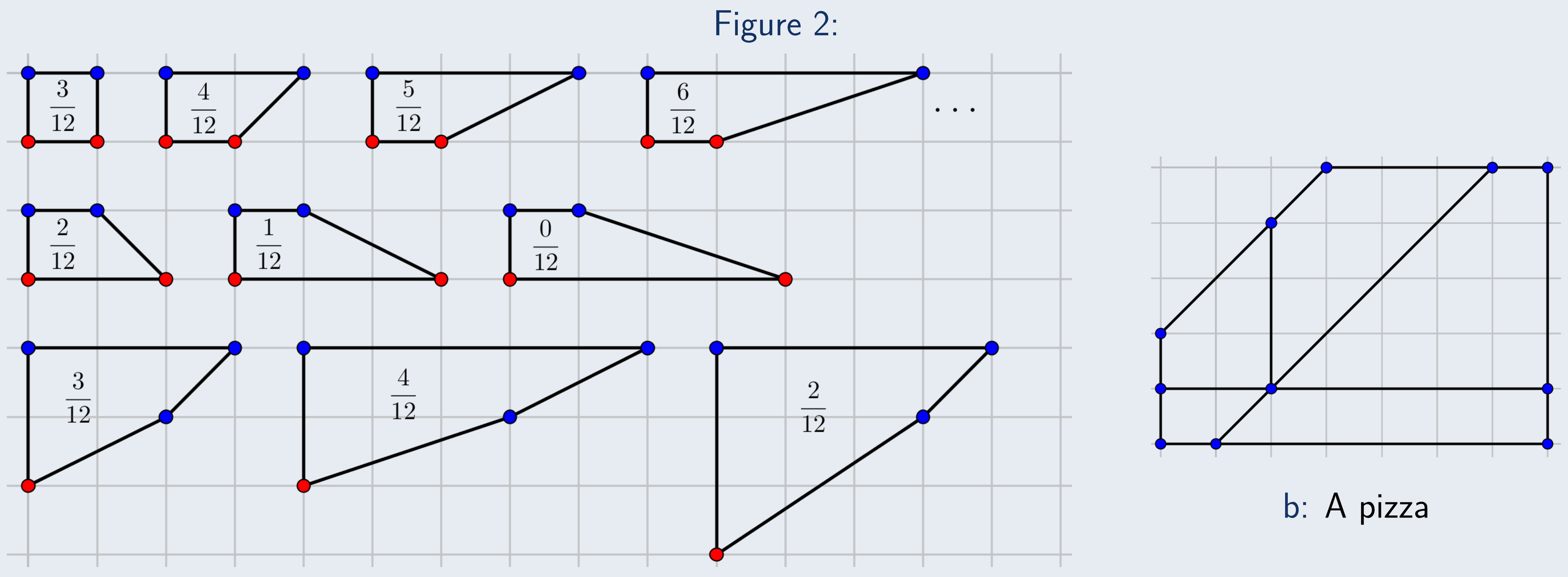
Two lattice polygons in the plane are **equivalent** if there is a continuous bijection between their edges and vertices such that, up to $GL(2,\mathbb{Z})$-transformations, the angles between the corresponding edges match simultaneously. For example, the following are equivalent:



Pizza slices and pizzas

A **pizza slice** is a quadrilateral equivalent to one of the quadrilaterals in figure 2a:

Figure 2:



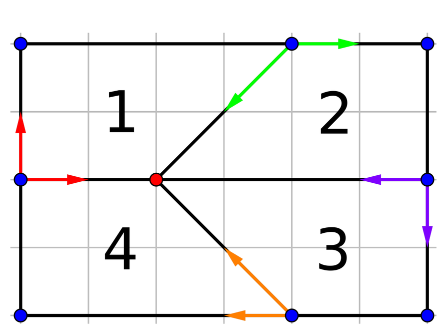
a: Pizza slices

b: A pizza

A **pizza** is a polygon subdivided into pizza slices in such a way that each pizza slice attaches to the center of the pizza at one of its red vertices, and each slice has exactly one vertex matching with a vertex of the polygon (its vertex opposite to the central one).

Baking Pizzas, Step 1: The crust

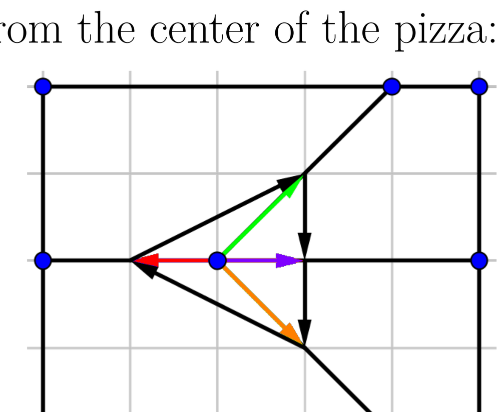
To bake a pizza, consider the ordered bases for $\mathbb{Z}^2$ found at the corners of the slices. Pizza slice 1 changes the standard (red) basis to the green one:



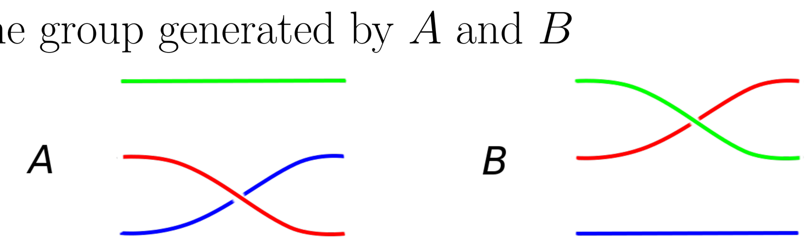
This lets us associate a matrix in $SL(2,\mathbb{Z})$ to a pizza slice; for instance, slice 1 above has matrix $\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$. And if the first pizza slice changes the standard basis to M and the second one to N, then the two pizza slices consecutively change it to $(MNM^{-1})M = MN$. If the slices form a pizza, then we should get back to the standard basis, as in the figure above or, in terms of matrices,

$$\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We want our pizzas to be polygons, not branched covers thereof, so we lift these matrices to the universal cover $\widetilde{SL(2,\mathbb{R})}$. This amounts to keeping track of the winding number of the path traced by the primitive vectors from the center of the pizza:



The preimage of $SL(2,\mathbb{Z})$ inside $\widetilde{SL(2,\mathbb{R})}$ is Br_3, the braid group on 3 strands, which is the group generated by A and B



satisfying the braid relation $ABA = BAB$.

The map to $SL(2,\mathbb{Z})$ is given by

$$A \mapsto \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, B \mapsto \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

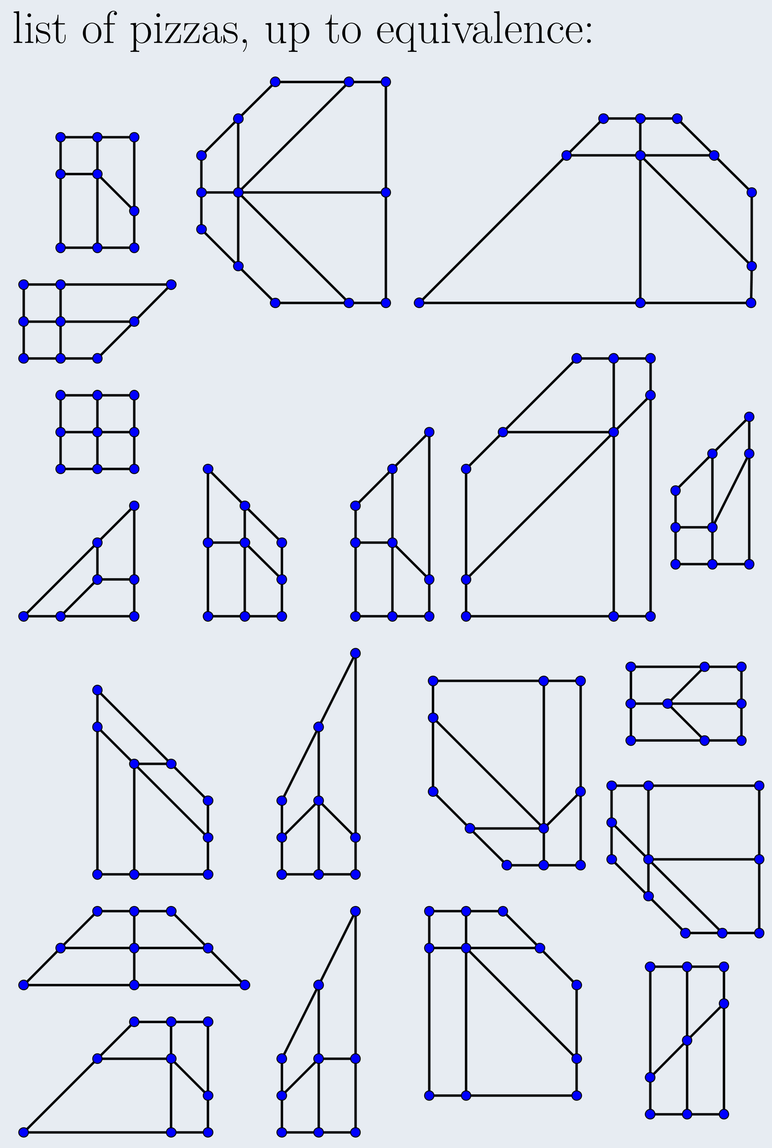
A (properly one-sheeted) pizza is then a sequence of pizza slices (now interpreted as braids), whose product is the "double twist braid" $(AB)^6$.

Nutrition

The map $Br_3 \to SL_2(\mathbb{Z}) \times \mathbb{Z}$, with second factor ab given by abelianization, is injective, and the **nutritive value** ν of a pizza slice S is the rational number $\frac{\mathrm{ab}(S)}{12}$ (these are the numbers displayed on the pizza slices in figure 2a). Looping around the origin exactly once now corresponds to a total nutritive value of 1.

Simply laced pizzas

If we restrict our attention to "simply laced" pizza slices (this is all of the first row and the first pizza slice of the second row in figure 2a), then the following is the complete list of pizzas, up to equivalence:



Geometric motivation

A **Kazhdan-Lusztig atlas** (He, Knutson, Lu, [2]) on a stratified T_V-variety $(V, \mathcal{Y})$ is:

1. A Kac-Moody group H with $T_V \hookrightarrow T_H$,
2. A ranked poset injection $w_M : \mathcal{Y}^{\mathrm{opp}} \to W_H$ whose image is
$$\bigcup_{f \in Y^T} [w(V), w(f)],$$
3. An open cover of V consisting of affine varieties around each $f \in Y^T$ and choices of T_V-equivariant stratified isomorphisms
$$V = \bigcup_{f\in Y^T} U_f \cong X_o^{w(f)} \cap X_{w(V)},$$
4. A T_V-equivariant degeneration $V \rightsquigarrow V' = \bigcup_{f\in Y^T} X^{w(f)} \cap X_{w(V)}$ carrying some ample line bundle on V to $\mathcal{O}(\rho)$.

Why would you care about such a structure? It turns out that a lot of interesting varieties have them:

Examples of Bruhat atlases

Theorem: (He, Knutson, Lu, [2]) Let G be a semisimple linear algebraic group. There are Bruhat atlases (these are Kazhdan-Lusztig atlases with $w(V) = 1$) on every G/P, and on the wonderful compactification $\widehat{G}$ of a group G.

What pizzas are about

We are trying to classify smooth toric surfaces (which are basically classified by their moment polygons) with Kazhdan-Lusztig atlases. Roughly speaking, these consist of the following:

- A subdivision of M's moment polygon into a pizza (this corresponds to the degeneration to a union of Richardson varieties).
- A Kac-Moody group H with $T_M \hookrightarrow T_H$ (a topping corresponds to a simple root of H).
- An assignment w of elements of W_H to the vertices of the pizza, compatible with the stratifications.

Baking Pizzas, Step 2: Toppings

Toppings

A **topping** on a pizza is a curve drawn across the edges of the pizza. Figure 4 shows the allowed toppings on a single slice and on a pizza:

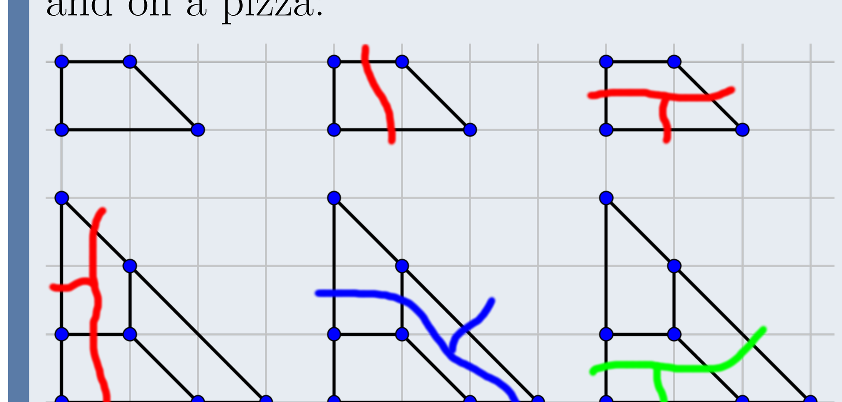


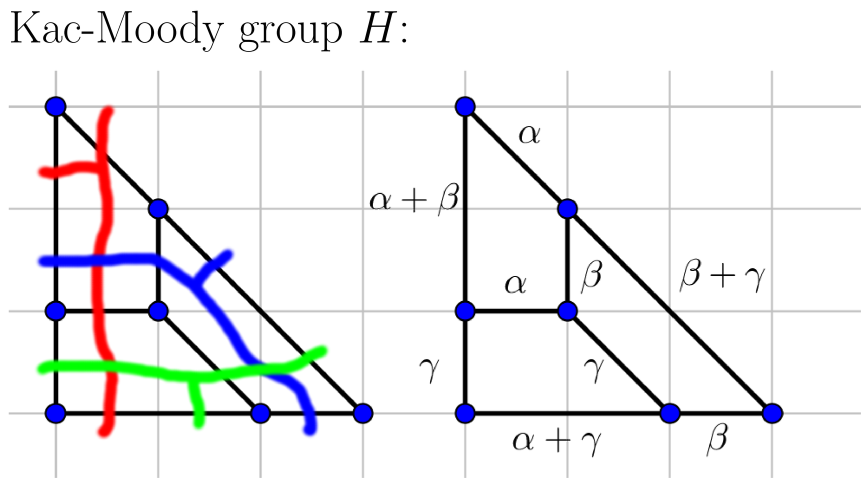
Figure 4: Toppings

Toppings on a pizza should satisfy the following:

- Every edge of the pizza must have the number of toppings going across it be equal to its lattice length.
- Toppings can only end at the edge of the pizza, not between slices.
- No two spokes (edges adjacent to the center vertex) should have the same set of toppings over them.
- No two spokes should have a combined amount of toppings on them equal to the toppings on a third spoke.

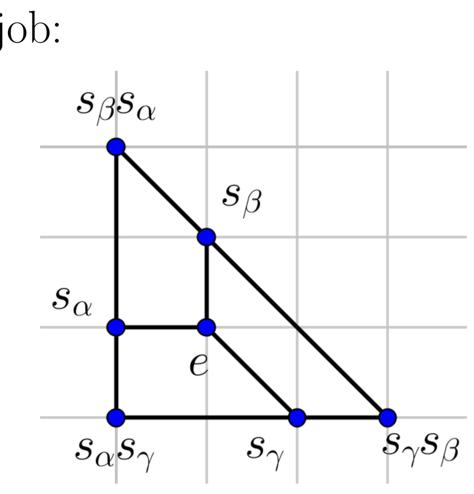
Baking Pizzas, Step 3: The oven

We need to find our oven H/B_H in which our pizza embeds. A topping arrangement gives us relations between the simple roots of the Kac-Moody group H:



In this example, we learn that $\alpha+\beta$, $\beta+\gamma, \alpha+\gamma$ should all be roots of H, so α, β, and γ should form a root system of type $\widetilde{A}_2$.

To find the W_H-labels of the vertices, we use the root labels on the edges, which we interpret as covering relations by reflections across the corresponding root, then look for an element for the central vertex which satisfy these constraints. In this case, the identity e does the job:



and we have found the Bruhat atlas on $\mathbb{CP}^2$ described in [2].

Computation

The computer search for pizzas with nutritive value 1 was done using Sage [3]. Using results of Dyer [1] and Sage, we were able to bound the total number of pizzas that have Kazhdan-Lusztig atlases. Our current best bound is 7543.

References

[1] Dyer, M., *On the "Bruhat graph" of a Coxeter system.* Compositio Math. 78 (1991), no. 2, 185-191.

[2] He, X.; Knutson, A.; Lu, J-H. *Bruhat atlases* preprint

[3] Sage Mathematics Software (Version 5.0.1), The Sage Developers, 2015, `http://www.sagemath.org`.