MATH 223 — Practice Midterm 1 — 45 minutes

4th October 2024

- The test consists of 7 pages and 5 questions worth a total of 0 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Signature								
Name								

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

(you will have the same cheat sheet on the midterm) These are the most important definitions that we encountered: (V is a vector space over a field $\mathbb{F}$).

- A list $\vec{v}_1, \dots v_m$ is **linearly independent** if

  $$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$

  has only the trivial solution $a_1 = \ldots = a_m = 0$.
- The **span** of a list $\vec{v}_1, \dots v_m$ is the set

  $$\{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m \in V \mid a_i \in \mathbb{F}\}$$

- A **basis** of a vector space is a linearly independent spanning list.
- The **dimension** of a vector space is the number of elements in a basis.

1. Find all values of $a, x \in \mathbb{R}$ such that

$$\begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 3 \end{pmatrix} \right)$$

2. Let V be a vector space over a field $\mathbb{F}$. Prove that $U \subseteq V$ is a subspace if and only if the following two conditions hold:

   - $\vec{0} \in U$,
   - For $\lambda \in \mathbb{F}, \vec{u}, \vec{v} \in U$, we have $\lambda\vec{u} + \vec{v} \in U$.

3. Let V be a finite-dimensional vector space. Let $U \subseteq V$ be a subspace. Prove that if

$$\dim U = \dim V$$

then $U = V$.

4. Let f, g be functions in the vector space $\mathbb{R}^{[0,1]}$. Assume that $f(0) = 1 = g(1)$ and $f(1) = 0 = g(0)$. Show that

$$(f, g)$$

is a linearly independent list in $\mathbb{R}^{[0,1]}$.

5. A polynomial $p(x) \in \mathcal{P}(\mathbb{R})$ is called **even** if it is an even function, i.e. $p(-x) = p(x)$ for all x and **odd** if $p(-x) = -p(x)$. Let E_m and O_m denote the sets of even and odd polynomials in $\mathcal{P}_m(\mathbb{R})$.

   (a) Show that E_n is a subspace of $\mathcal{P}_m(\mathbb{R})$ and find $\dim E_n$.

   (b) Show that O_n is a subspace of $\mathcal{P}_m(\mathbb{R})$ and find $\dim O_n$.