MATH 223 — Practice Midterm 1 — 45 minutes

4th October 2024

- The test consists of 8 pages and 5 questions worth a total of 0 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

(you will have the same cheat sheet on the midterm) These are the most important definitions that we encountered: (V is a vector space over a field $\mathbb{F}$).

- A list $\vec{v}_1, \ldots v_m$ is **linearly independent** if

$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$

  has only the trivial solution $a_1 = \ldots = a_m = 0$.

- The **span** of a list $\vec{v}_1, \ldots v_m$ is the set

$$\{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m \in V \mid a_i \in \mathbb{F}\}$$

- A **basis** of a vector space is a linearly independent spanning list.
- The **dimension** of a vector space is the number of elements in a basis.

1. Find all values of $a, x \in \mathbb{R}$ such that

$$\begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 3 \end{pmatrix}\right)$$

Note that the statement is equivalent to the statement that $\begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$ is a LC of $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 3 \end{pmatrix}$. This is in turn equivalent to the system

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ -1 & x & 2 \\ 3 & 3 & 1 \end{array}\right)$$

being consistent. We row reduce to obtain

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & x+1 & a+2 \\ 0 & 0 & 1-3a \end{array}\right)$$

From here we see that the system is inconsistent unless $1 - 3a = 0$, or, $a = \frac{1}{3}$. Then our matrix becomes

$$\left(\begin{array}{cc|c} 1 & 1 & 1/3 \\ 0 & x+1 & 7/3 \\ 0 & 0 & 0 \end{array}\right)$$

For this system to be consistent, the second row should have a pivot, which is equivalent to $x \neq -1$. So

$$\begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 3 \end{pmatrix}\right)$$

if and only if $a = \frac{1}{3}$ and $x \neq -1$.

2. Let V be a vector space over a field $\mathbb{F}$. Prove that $U \subseteq V$ is a subspace if and only if the following two conditions hold:

   - $\vec{0} \in U$,
   - For $\lambda \in \mathbb{F}, \vec{u}, \vec{v} \in U$, we have $\lambda\vec{u} + \vec{v} \in U$.

   Recall that U is a subspace if and only if

   1. $\vec{0} \in U$,
   2. For $\vec{u}, \vec{v} \in U$, we have $\vec{u} + \vec{v} \in U$,
   3. For $\vec{u} \in U, \lambda \in \mathbb{F}$, we have $\lambda\vec{u} \in U$.

   Note that $\vec{0} \in U$ is in both sets of conditions. Assume that for $\lambda \in \mathbb{F}, \vec{u}, \vec{v} \in U$, we have $\lambda\vec{u}+\vec{v} \in U$. Then choosing $\lambda = 1$, we get that $\vec{u}+\vec{v} \in U$, and choosing $\vec{v} = \vec{0}$ we get $\lambda\vec{u} \in U$. So U is a subspace.

   For the converse, assume that U is a subspace. Let $\vec{u}, \vec{v} \in U, \lambda \in \mathbb{F}$. Since U is closed under scalar multiplication, we have $\lambda\vec{u} \in U$. Since U is closed under addition, we have $(\lambda\vec{u}) + (\vec{v}) = \lambda\vec{u} + \vec{v} \in U$, and we are done.

3. Let V be a finite-dimensional vector space. Let $U \subseteq V$ be a subspace. Prove that if

$$\dim U = \dim V$$

then $U = V$.

Let $n = \dim U$. Pick a basis for U, say $\vec{u}_1, \ldots, \vec{u}_n$. Then $\vec{u}_1, \ldots, \vec{u}_n$ is a linearly independent list of n vectors in V. We know that any list of $n = \dim V$ linearly independent vectors in V is a basis for V. So $\vec{u}_1, \ldots, \vec{u}_n$ is a basis of V. Then $V = \text{Span}(\vec{u}_1, \ldots, \vec{u}_n)$. But $\vec{u}_1, \ldots, \vec{u}_n$ is also a basis for U, so $U = \text{Span}(\vec{u}_1, \ldots, \vec{u}_n) = V$.

4. Let f, g be functions in the vector space $\mathbb{R}^{[0,1]}$. Assume that $f(0) = 1 = g(1)$ and $f(1) = 0 = g(0)$. Show that

$$(f, g)$$

is a linearly independent list in $\mathbb{R}^{[0,1]}$.

Suppose that $c_1 f + c_2 g = \vec{0}$ (as elements of $\mathbb{R}^{[0,1]}$). We need to show that $c_1 = c_2 = 0$. Since $c_1 f + c_2 g = \vec{0}$ holds as an equality of functions, we have

$$\begin{aligned} c_1 f(0) + c_2 g(0) &= \vec{0}(0) \\ c_1(1) + c_2(0) &= 0 \\ c_1 &= 0 \end{aligned}$$

and

$$\begin{aligned} c_1 f(1) + c_2 g(1) &= \vec{0}(1) \\ c_1(0) + c_2(1) &= 0 \\ c_2 &= 0 \end{aligned}$$

so $c_1 = c_2 = 0$ so f, g is LI.

5. A polynomial $p(x) \in \mathcal{P}(\mathbb{R})$ is called **even** if it is an even function, i.e. $p(-x) = p(x)$ for all x and **odd** if $p(-x) = -p(x)$. Let E_m and O_m denote the sets of even and odd polynomials in $\mathcal{P}_m(\mathbb{R})$.

   (a) Show that E_m is a subspace of $\mathcal{P}_m(\mathbb{R})$ and find $\dim E_m$.

   We check the conditions for a subspace:

   - Since $\vec{0}(x) = 0 = \vec{0}(-x)$, we have $\vec{0} \in E_n$.
   - Let $p, q \in E_m$, then

   $$(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x),$$

   so $p + q \in E_m$.
   - Let $p \in E_m, \lambda \in \mathbb{R}$, then

   $$(\lambda p)(-x) = \lambda p(-x) = \lambda p(x) = (\lambda p)(x),$$

   so $\lambda p \in E_m$.

   So E_m is a subspace. We claim that if m is even, then $(1, x^2, \ldots, x^m)$ is a basis of E_m, and if m is odd, then $(1, x^2, \ldots, x^{m-1})$ is a basis of E_m. To see that it spans, let $p(x) = a_0 + a_1 x + \ldots a_m x^m \in \mathcal{P}_m(\mathbb{R})$. Then $p \in E_m \Leftrightarrow p(-x) = p(x)$, i.e. if

   $$\begin{aligned} a_0 + a_1 x + a_2 x^2 \ldots a_m x^m &= a_0 + a_1(-x) + a_2(-x)^2 \ldots a_m(-x)^m \\ &= a_0 + (-a_1)x + a_2 x^2 \ldots + ((-1)^m a_m)x^m \end{aligned}$$

   which happens if and only if $a_i = 0$ for i even. So $p \in E_m$ must be of the form

   $$a_0 + a_2 x^2 + a_4 x^4 + \ldots$$

   which is a linear combination of the elements in our basis. To see that the list is linearly independent, note that it consists of monomials of different degrees. So indeed our list is a basis, in particular,

   $$\dim E_m = \begin{cases} m/2 + 1 & \text{if } m \text{ is even} \\ (m-1)/2 + 1 & \text{if } m \text{ is odd} \end{cases}$$

(b) Show that O_m is a subspace of $\mathcal{P}_m(\mathbb{R})$ and find $\dim O_m$. We check the conditions for a subspace:

- Since $\vec{0}(x) = 0 = -\vec{0}(-x)$, we have $\vec{0} \in O_m$.
- Let $p, q \in O_m$, then
$$(p+q)(-x) = p(-x) + q(-x) = -p(x) - q(x) = -(p+q)(x),$$
so $p + q \in O_m$.
- Let $p \in O_m, \lambda \in \mathbb{R}$, then
$$(\lambda p)(-x) = \lambda p(-x) = -\lambda p(x) = -(\lambda p)(x),$$
so $\lambda p \in O_m$.

So O_m is a subspace. We claim that if m is odd, then $(x, x^3, \ldots, x^m)$ is a basis of O_m, and if m is even, then $(x, x^3, \ldots, x^{m-1})$ is a basis of O_m. To see that it spans, let $p(x) = a_0 + a_1 x + \ldots a_m x^m \in \mathcal{P}_m(\mathbb{R})$. Then $p \in O_m \Leftrightarrow p(-x) = -p(x)$, i.e. if

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 \ldots a_m x^m &= a_0 + a_1(-x) + a_2(-x)^2 \ldots a_m(-x)^m \\ &= -a_0 - (-a_1)x - a_2 x^2 \ldots - ((-1)^m a_m) x^m \end{aligned}$$

which happens if and only if $a_i = 0$ for i even. So $p \in O_m$ must be of the form
$$a_1 x + a_3 x^3 + a_5 x^5 + \ldots$$
which is a linear combination of the elements in our basis. To see that the list is linearly independent, note that it consists of monomials of different degrees. So indeed our list is a basis, in particular,

$$\dim O_m = \begin{cases} (m-1)/2 + 1 & \text{if } m \text{ is odd} \\ m/2 & \text{if } m \text{ is even} \end{cases}$$