MATH 223 — Midterm 1 — 45 minutes

4th October 2024

- The test consists of 7 pages and 5 questions worth a total of 25 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

These are the most important definitions that we encountered: (V is a vector space over a field $\mathbb{F}$).

- A list $\vec{v}_1, \ldots \vec{v}_m$ of vectors is **linearly independent** if

  $$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = \vec{0}$$

  has only the trivial solution $a_1 = \ldots = a_m = 0$.
- The **span** of a list $\vec{v}_1, \ldots \vec{v}_m$ is the set

  $$\{a_1\vec{v}_1 + \ldots + a_m\vec{v}_m \in V \mid a_i \in \mathbb{F}\}$$

- A **basis** of a vector space is a linearly independent spanning list.
- The **dimension** of a vector space is the number of elements in a basis.

1. 5 marks Find all values of a, b such that the list

$$\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}a\\b\\2\end{pmatrix}$$

is linearly dependent.

Solution: The vectors are linearly dependent if and only if the vector equation

$$x\begin{pmatrix}1\\2\\1\end{pmatrix} + y\begin{pmatrix}0\\-1\\1\end{pmatrix} + z\begin{pmatrix}a\\b\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a nontrivial solution. This happens if RREF of the matrix

$$\begin{pmatrix}1 & 0 & a\\2 & -1 & b\\1 & 1 & 2\end{pmatrix}$$

has fewer than three pivots. We row reduce

$$\begin{pmatrix}1 & 0 & a\\2 & -1 & b\\1 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & 0 & a\\0 & -1 & b-2a\\0 & 1 & 2-a\end{pmatrix} \to \begin{pmatrix}1 & 0 & a\\0 & 1 & 2a-b\\0 & 0 & 2+b-3a\end{pmatrix}$$

From here we see that the first two columns will always have pivots, and the third column has a pivot if and only if

$$2 + b - 3a \neq 0.$$

Therefore the list is linearly dependent if and only if $b = 3a - 2$ (and a is arbitrary).

2. 4 marks Let V be a vector space over the field $\mathbb{F}$. Let $\lambda \in \mathbb{F}$ and $\vec{v} \in V$. Prove that if $\lambda\vec{v} = \vec{0}$ then either $\lambda = 0$ or $\vec{v} = 0$.

Solution: We proceed by cases. If $\lambda = 0$, then we are done, so we can assume $\lambda \neq 0$. Then λ has a multiplicative inverse $\frac{1}{\lambda}$. Multiplying both sides of the equation

$$\lambda\vec{v} = \vec{0}$$

by $\frac{1}{\lambda}$ yields

$$\begin{aligned} \frac{1}{\lambda}\lambda\vec{v} &= \frac{1}{\lambda}\vec{0} \\ 1\vec{v} &= \vec{0} \\ \vec{v} &= \vec{0} \end{aligned}$$

and we are done.

3. 6 marks Recall that $\mathcal{P}_m(\mathbb{R})$ denotes the vector space of polynomials of degree at most m with real coefficients (we'll use x to denote the indeterminate).

(a) Prove that $U = \{p(x) \in \mathcal{P}_m(\mathbb{R}) \mid p(0) = 0\}$ is a subspace of $\mathcal{P}_m(\mathbb{R})$.

Solution: We check the three conditions for a subspace:

1. Since $\vec{0}(0) = 0$, we have $\vec{0} \in U$.
2. Suppose $p, q \in U$, i.e. $p(0) = q(0)$. Then $(p + q)(0) = p(0) + q(0) = 0$, so $p + q \in U$.
3. Suppose $p \in U$ and $\lambda \in \mathbb{R}$. Then $(\lambda p)(0) = \lambda(p(0)) = \lambda(0) = 0$ so $\lambda p \in U$.

(b) Find a basis for U.

Solution: We claim that the list $x, x^2, \ldots, x^m$ is a basis for U.

- Since the list consists of polynomials of different degrees it is LI.
- To see that the list spans U, note that if $p(x) = a_0 + a_1 x + a_2 x^2 \ldots + a_m x^m \in \mathcal{P}_m(\mathbb{R})$, then

  $$(p(0) = 0) \Leftrightarrow \big(a_0 + a_1(0) + a_2(0)^2 + \ldots + a_m(0^m) = 0\big) \Leftrightarrow a_0 = 0.$$

  Which means that p has zero constant term, which is the same as being a linear combination of monomials of strictly positive degree, i.e. $p \in \operatorname{Span}(x, x^2, \ldots, x^m)$.

4. 5 marks Suppose U and W are both four-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Solution: We have

$$\begin{aligned}\dim(U \cap W) &= \dim(U) + \dim(W) - \dim(U + W) \\ &= 4 + 4 - \dim(U + W) \\ &\geq 8 - \dim(\mathbb{C}^6) = 2.\end{aligned}$$

Pick a basis for $U \cap W$, since $\dim(U \cap W) \geq 2$, it has at least two elements $\vec{u}_1, \vec{u}_2$. Since a basis is linearly independent, in particular, the list $(\vec{u}_1, \vec{u}_2)$ is LI, which implies that neither of the two vectors $\vec{u}_1$ and $\vec{u}_2$ is a scalar multiple of the other.

5. 5 marks Given $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}$, let $U = \text{Span}(\vec{v}_1, \ldots, \vec{v}_m)$ and $W = \text{Span}(\vec{v}_1, \ldots, \vec{v}_m, \vec{w})$. Prove that either $\dim W = \dim U$ or $\dim W = \dim U + 1$.

Solution: Since $(\vec{v}_1, \ldots, \vec{v}_m)$ is a spanning list for U, we can reduce it to a basis. Let this basis be $(\vec{u}_1, \ldots, \vec{u}_k)$ (where each $\vec{u}_i$ is equal to some $\vec{v}_j$ and $k = \dim U$). We know that

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_m) = \text{Span}(\vec{u}_1, \ldots, \vec{u}_k)$$

so

$$W = \text{Span}(\vec{v}_1, \ldots, \vec{v}_m, \vec{w}) = \text{Span}(\vec{u}_1, \ldots, \vec{u}_k, \vec{w}).$$

If the list $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{w})$ is LI, then it is a basis for W, and $\dim W = k + 1 = \dim U + 1$. Otherwise, the list is LD. In this case, since it is a spanning list, we can reduce it to a basis. Note that $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$ is LI (since it is a basis for U), so the reduction has to remove $\vec{w}$ from the end. Then $(\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k)$ is a basis for W and therefore $\dim W = k = \dim U$.