MATH 223 — Midterm 2 — 45 minutes

9th November 2024

- The test consists of 7 pages and 4 questions worth a total of 20 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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Section								
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Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

These are the most important definitions that we encountered:

- For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null(T) is

$$\text{null}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$

- For $T \in \mathcal{L}(V, W)$, the range of T, denoted range(T) is

$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

- A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.
- Suppose $T \in \mathcal{L}(V)$. A nonzero vector $\vec{v} \in V$ is called an eigenvector of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$ if

$$T(\vec{v}) = \lambda \vec{v}.$$

1. 6 marks Let $S, T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear maps given by

$$T(p(x)) = x\left(\frac{d}{dx}p(x)\right)$$
$$S(p(x)) = 2p(0)$$

(the second line is not a typo, S sends p to a constant function).

(a) Compute the matrix of S with respect to the basis $\{1, x, x^2, x^3\}$

Solution: We compute

$$S(1) = 2$$
$$S(x) = 0$$
$$S(x^2) = 0$$
$$S(x^3) = 0$$

So the matrix of S is

$$\mathcal{M}(S) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall that $S, T : \mathbb{P}_3 \to \mathbb{P}_3$ are linear maps given by

$$\begin{aligned} T(p(x)) &= x\left(\frac{d}{dx}p(x)\right) \\ S(p(x)) &= 2p(0) \end{aligned}$$

(the second line is not a typo, S sends p to a constant function).

(b) Compute the matrix of T with respect to the basis $\{1, x, x^2, x^3\}$

Solution: We compute

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2 \\ T(x^3) &= 3 \end{aligned}$$

So the matrix of T is

$$\mathcal{M}(S) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(c) Is $S + T$ invertible? Justify your answer.

Solution: Notice that $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$, so

$$\mathcal{M}(S+T) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and since this is an upper-triangular matrix, the determinant is the product of the diagonal entries, which is 12, so the matrix is invertible and therefore the linear transformation $S+T$ is invertible.

2. 4 marks Suppose $S, T \in \mathcal{L}(V)$ are such that $\text{range}(S) \subseteq \text{null}(T)$. Prove that $(ST)^2 = 0$.

Solution: Since $\text{range}(S) \subseteq \text{null}(T)$, we have that $TS = 0$ is the zero transformation. Then using associativity, we have

$$(ST)^2 = (ST)(ST) = S(TS)T = S(0)T = 0.$$

3. [5 marks] Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map. Suppose that

$$(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1)$$

are in range(T). Is is possible that the list

$$(1, 0, 0, 0), (1, 1, 0, 0)$$

is a basis for null(T)? Justify your answer.

Solution: We claim that the list $(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1)$ is linearly independent. To check this, we use row reduction

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So the vectors are linearly independent. Therefore $\dim \operatorname{range}(T) \geq 3$, and by the FTLM, we have $\dim \operatorname{null}(T) \leq 1$. So any basis of null$(T)$ has at most one element, and therefore it is not possible that $(1, 0, 0, 0), (1, 1, 0, 0)$ is a basis for null(T).

4. [5 marks] Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Solution: Let $\vec{v}$ be an eigenvector of P with eigenvalue λ. Then $P\vec{v} = \lambda v$. Applying P to both sides yields $P^2\vec{v} = P\lambda\vec{v} = \lambda^2\vec{v}$. Since $P^2 = P$, we also have $P^2\vec{v} = P\vec{v} = \lambda\vec{v}$. So we have $\lambda^2\vec{v} = \lambda\vec{v}$, or, equivalently, $(\lambda^2 - \lambda)\vec{v} = \vec{0}$, and since $\vec{v} \neq \vec{0}$ this implies that $\lambda^2 - \lambda = \lambda(\lambda - 1) = 0$, and therefore $\lambda = 0$ or $\lambda = 1$.