MATH 223 — Midterm 2 — 45 minutes

9th November 2024

- The test consists of 6 pages and 4 questions worth a total of 0 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Signature								
Name								

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

These are the most important definitions that we encountered:

- For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null(T) is

$$\text{null}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$

- For $T \in \mathcal{L}(V, W)$, the range of T, denoted range(T) is

$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

- A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.
- Suppose $T \in \mathcal{L}(V)$. A nonzero vector $\vec{v} \in V$ is called an eigenvector of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$ if

$$T(\vec{v}) = \lambda\vec{v}.$$

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + b \\ cz + dx \\ ay \end{pmatrix}$$

(a) Find all the values of a, b, c, d for which T is a linear map. Justify your answer.

(b) Compute the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

2. Suppose $S, T \in \mathcal{L}(V)$.

   (a) Prove that $\text{null}(T)$ is a subspace of $\text{null}(ST)$.

   (b) Give a concrete example where $\text{null}(T) \neq \text{null}(ST)$.

3. Prove that there is no linear map $T : \mathcal{P}_5(\mathbb{F}) \to \mathcal{P}_3(\mathbb{F})$ that is surjective and satisfies

$$T(x^2 + 1) = T(x - 4) = T(x^2 + x + 1) = 0.$$

4. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and that there are nonzero vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that satisfy

$$T(\vec{v}_1) = \vec{v}_1, \quad T(\vec{v}_2) = 2\vec{v}_2, \qquad T(\vec{v}_3) = 3\vec{v}_3.$$

Prove that T is invertible.