MATH 223 — Midterm 2 — 45 minutes

9th November 2024

- The test consists of 6 pages and 4 questions worth a total of 0 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, phones, smart watches, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

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Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

Cheat sheet

These are the most important definitions that we encountered:

- For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null(T) is

$$\text{null}(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$

- For $T \in \mathcal{L}(V, W)$, the range of T, denoted range(T) is

$$\text{range}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

- A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V and TS is the identity on W.
- Suppose $T \in \mathcal{L}(V)$. A nonzero vector $\vec{v} \in V$ is called an eigenvector of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$ if

$$T(\vec{v}) = \lambda\vec{v}.$$

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + b \\ cz + dx \\ ay \end{pmatrix}$$

(a) Find all the values of a, b, c, d for which T is a linear map. Justify your answer.

Solution: If T is linear then $T(\vec{0}) = \vec{0}$, so we see immediately that T can not be linear if $b \neq 0$. So if T is linear then

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay \\ cz + dx \\ ay \end{pmatrix}.$$

We check if T is additive and homogeneous

$$T \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} ay_1 + ay_2 \\ cz_1 + dx_1 + cz_2 + dx_2 \\ ay_1 + ay_2 \end{pmatrix} = T \left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right)$$

and

$$T \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} \lambda ay \\ \lambda cz + \lambda dx \\ \lambda ay \end{pmatrix} = \lambda \begin{pmatrix} ay \\ cz + dx \\ ay \end{pmatrix} = \lambda T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so in this case T is linear, so T is linear if $b = 0$ for all values of a, c, d.

(b) Compute the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

Solution: We have

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} \qquad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \qquad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}.$$

So the matrix of T with respect to the standard basis is

$$\begin{pmatrix} 0 & a & 0 \\ d & 0 & c \\ 0 & a & 0 \end{pmatrix}.$$

2. Suppose $S, T \in \mathcal{L}(V)$.

(a) Prove that $\text{null}(T)$ is a subspace of $\text{null}(ST)$.

Solution: Note that $\text{null}(T)$ is always a subspace, so we just need to show $\text{null}(T) \subseteq \text{null}(ST)$. So assume $\vec{v} \in \text{null}(T)$. Then $T\vec{v} = \vec{0}$, and therefore

$$\vec{0} = S(\vec{0}) = S(T(\vec{v})) = (ST)(\vec{v}),$$

so $\vec{v} \in \text{null}(ST)$, so $\text{null}(T)$ is a subspace of $\text{null}(ST)$.

(b) Give a concrete example where $\text{null}(T) \neq \text{null}(ST)$.

Solution: Let $V = \mathbb{C}^1$, $S = 0$ and $T = 1$. Then $\text{null}(ST) = V$ and $\text{null}(T) = \{0\} \neq V$.

3. Prove that there is no linear map $T : \mathcal{P}_5(\mathbb{F}) \to \mathcal{P}_3(\mathbb{F})$ that is surjective and satisfies
$$T(x^2+1) = T(x-4) = T(x^2+x+1) = 0.$$

Solution: First we show that the list $x^2+1, x-4, x^2+x+1$ is LI. Assume we have

$$0 = a(x^2+1)+b(x-4)+c(x^2+x+1) = x^2(a+c)+x(b+c)+1(a-4b+c).$$

This is equivalent to the system

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -4 & 1 & 0 \end{array}\right)$$

We row reduce

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -4 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

and we see that the only solution is $a = b = c = 0$, so our list of polynomials is LI. Hence $\dim(\text{null}(T)) \geq 3$, so by the FTLM,

$$\dim \text{range}(T) = \dim \mathcal{P}_5(\mathbb{F}) - \dim \text{null}(T) \leq 6 - 3 = 3.$$

Since $\dim \mathcal{P}_3(\mathbb{F}) = 4$, the map T can not be surjective.

4. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and that there are nonzero vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ that satisfy

$$T(\vec{v}_1) = \vec{v}_1, \quad T(\vec{v}_2) = 2\vec{v}_2, \qquad T(\vec{v}_3) = 3\vec{v}_3.$$

Prove that T is invertible.

Solution: Note that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for T corresponding to the eigenvalues $1, 2, 3$, respectively. As they are eigenvectors corresponding to distinct eigenvalues, the list $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is linearly independent. Since $\dim V = 3$, the list $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis for V, and with respect to this basis, we have

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

and we see that this matrix is invertible with

$$(\mathcal{M}(T))^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix},$$

and therefore T is invertible.