Loop Erased Walks and Uniform Spanning Trees

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1 Introduction

The uniform spanning tree has had a fruitful history in probability theory. Most notably, it was the study of the scaling limit of the UST that led Oded Schramm [Sch00] to introduce the SLE process, work which has revolutionised the study of two dimensional models in statistical physics. But in addition, the UST relates in an intrinsic fashion with another model, the *loop erased random walk* (or LEW), and the connections between these two processes allow each to be used as an aid to the study of the other.

These notes give an introduction to the UST, mainly in $\mathbb{Z}^d$. The later sections concentrate on the UST in $\mathbb{Z}^2$, and study the relation between the intrinsic geometry of the UST and Euclidean distance. As an application, we study random walk on the UST, and calculate its asymptotic return probabilities.

This survey paper contains many results from the papers [Lyo98, BLPS, BKPS04], not always attributed.

Finite graphs. A graph G is a pair $G = (V, E)$. Here V is the set of vertices (finite or countably infinite) and E is the set of edges. Each edge e is a two element subset of V – so we can write $e = \{x, y\}$. We think of the vertices as points, and the edges as lines connecting the points. It will sometimes be

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useful to allow multiple edges between points. If $\{x, y\} \in E$ we write $x \sim y$, and say that x, y are neighbours, and that x is connected to y.

Now for some general definitions. Write $|A|$ for the number of elements in the set A.

(1) We define $d(x, y)$ to be the length n of the shortest path $x = x_0, x_1, \ldots, x_n = y$ with $x_{i-1} \sim x_i$ for $1 \le i \le n$. If there is no such path then we set $d(x, y) = \infty$. We also write for $x \in \mathcal{G}$, $A \subset \mathcal{G}$,

$$d(x, A) = \min\{d(x, y) : y \in A\}. \tag{1.1}$$

(2) G is *connected* if $d(x, y) < \infty$ for all x, y.
(3) G is *locally finite* if $N(x) = \{y : y \sim x\}$ is finite for each $x \in G$, – i.e. every vertex has a finite number of neighbours.
(4) Define balls in G by

$$B_d(x, r) = \{y : d(x, y) \le r\}, \quad x \in G, \quad r \in [0, \infty).$$

(5) For $A \subset G$ write $|A|$ for the cardinality of A. We define the *exterior boundary of* A by

$$\partial A = \{y \in A^c : \text{ there exists } x \in A \text{ with } x \sim y\}.$$

Set also

$$\partial_i A = \partial(A^c) = \{y \in A : \text{ there exists } x \in A^c \text{ with } x \sim y\}.$$

We use the notation $A_n \uparrow\uparrow G$ to mean that A_n is an increasing sequence of finite sets with $\cup_n A_n = G$.

From now on we will always assume:

$$G \text{ is locally finite and connected.}$$

Set

$$\mu_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{if } \{x, y\} \notin E, \end{cases}$$

and let

$$\mu_x = \sum_{y \in V} \mu_{xy},$$

so that μ_x is the degree of x. (Later we will allow more general edge weights.)

The adjacency matrix A^G of G is the $V \times V$ matrix defined by:

$$A_{xy} = \delta_{xy} - \mu_{xy}, \quad (x, y) \in V \times V.$$

Spanning Trees. A *cycle* in a graph is a closed finite loop, with no intersections. More precisely a cycle consists of a sequence of edges $e_i = \{x_{i-1}, x_i\}$, $i = 1, \dots, k$, with $x_k = x_0$, and $\{x_i, 0 \le i \le k-1\}$ all distinct. (Note that $\{x_0, x_1\}, \{x_1, x_0\}$ is a cycle.) A graph G is a *tree* if it contains no cycles of length greater than 2. A *subgraph* $G' = (V', E') \subset G$ is exactly what one would expect: that is G' is a graph, $V' \subset V$ and $E' \subset E$. A *spanning tree* T for G is a subgraph $T = (V_T, E_T)$ which is a tree, and which spans G, so that $V_T = V$. Write $\mathcal{T}(G)$ be the set of spanning trees.

Spanning trees have a long history in combinatorics, going back at least as far as Kirchoff in 1847:

Theorem 1.1 *(Matrix Tree Theorem [Kir47].) Let G be a finite graph, $x \in V = V_G$, and let $A^G[x]$ be the adjacency matrix A^G with the row and column associated with x deleted. Then the number of spanning trees is*

$$N(G) = \det A^G[x].$$

Kirchoff is familiar for his laws for electrical currents, and in fact the same (short) paper also introduces his network laws, and obtains an expression for effective resistance in terms of the number of spanning trees. An English translation can be found in [Kir47E].

Definition 1.2 Given a graph, we can turn it into an electrical circuit by replacing each edge by a unit resistor. Let A_0, A_1 be (disjoint) subsets of a finite graph V. Suppose that A_1 is placed at electrical potential 1, and A_0 at potential 0. Then current flows in the network according to Ohms/Kirchoff's laws; let I be the current which flows from A_0 to A_1. We define the *effective resistance between A_0 and A_1* to be

$$R_{\text{eff}}(A_0, A_1) = R_{\text{eff}}(A_0, A_1; G) = I^{-1}. \tag{1.2}$$

We also write $R_{\text{eff}}(x, y) = R_{\text{eff}}(\{x\}, \{y\})$.

The inverse of $R_{\rm eff}$ is effective conductance. We have the following useful characterization of $C_{\rm eff}(A_0, A_1)$ as the solution to a variational problem. Define the energy form

$$\mathcal{E}(f,f) = \tfrac{1}{2} \sum_x \sum_y \mu_{xy} (f(x) - f(y))^2. \tag{1.3}$$

In electrical terms, $\mathcal{E}(f,f)$ is the energy dissipation in the circuit if a potential f is imposed. Let $\mathcal{F}(A_0, A_1) = \{f : V \to \mathbb{R} \text{ s.t. } f|_{A_k} = k, k = 0,1\}$. Then

$$C_{\rm eff}(A_0, A_1) = R_{\rm eff}(A_0, A_1)^{-1} = \inf\{\mathcal{E}(f,f) : f \in \mathcal{H}(A_0, A_1)\}. \tag{1.4}$$

The definition (1.4) extends to infinite graphs, but there are difficulties with (1.2), in that we need to exclude currents which flow out to infinity and back again.

Theorem 1.3 *([Kir47].) Let $G = (V, E)$ be a graph, and $e = \{x, y\} \in E$. Let N_G be the number of spanning trees in G, and N_G^e be the number of spanning trees which contain the edge e. Then*

$$R_{\rm eff}(x, y) = \frac{N_G^e}{N_G}. \tag{1.5}$$

Definition 1.4 A *uniform spanning tree* (UST) on a finite graph is a spanning tree chosen randomly (with uniform distribution) from the set of spanning trees. More precisely, let $N = |\mathcal{T}(G)|$ be the number of spanning trees, and $\mathbb{P}$ be the probability measure on $\mathcal{T}(G)$ which assigns mass $1/N$ to each element (tree) in $\mathcal{T}(G)$. The UST is a random element $\mathcal{U}$ of $\mathcal{T}(G)$ selected by the probability measure $\mathbb{P}$.

Simulation of the UST. One is interested in simulating the UST on a finite graph G. That is, given a graph G, and a supply of independent uniform $[0,1]$ numbers (i.e. random variables), $\xi = (\xi_1, \xi_2, \dots)$, one wants an algorithm which produces a spanning tree $\mathcal{U} = \mathcal{U}(\xi)$ with the property that for any fixed spanning tree $T_0 \in \mathcal{T}(G)$, $\mathbb{P}(\mathcal{U} = T_0) = 1/N$, where $N = |\mathcal{T}(G)|$.

A crude algorithm. Make a list of all N spanning trees in $\mathcal{T}(G)$, call these $T_1, T_2, \dots, T_N$. Then choose T_k with probability $1/N$.

This algorithm is simple and obviously correct – i.e. the tree it produces has the right distribution. However, it is slow and requires lots of memory, since one has to make a list of all the trees. In fact it is very slow: if G is a 31×31 square in $\mathbb{Z}^2$, then $|\mathcal{T}(G)| \approx 10^{500}$.

A better algorithm using the Matrix Tree Theorem. Fix an edge e. The MTT gives the number N_G of spanning trees of G, and also the number $N_{G'}$ of spanning trees in the graph $G' = (V, E - \{e\})$; each of these is a spanning tree of G which does not contain e. So

$$\mathbb{P}(e \notin \mathcal{U}) = \frac{N_{G'}}{N_G}.$$

Use this to make a decision about whether e is in the random UST or not. Then remove e from the graph, and continue with the reduced graph. Since determinants, and so N_G, can be computed in polynomial time, this algorithm enables the UST to be simulated in polynomial time.

In the late 1980s, as a result of conversations between Doyle, Diaconis, Aldous and Broder, it was realised that paths in the UST are also paths of loop erased walks. This led Aldous and Broder to produce improved algorithms for the UST – see [Ald90, Bro]. I will jump over these, and go on to describe the current best algorithm, Wilson's algorithm (WA) – see [Wil96].

Random walk on a graph. Let $G = (V, E)$ be a graph (finite or infinite). The (discrete time) simple random walk on G is a random process $X = (X_n, n \in \mathbb{Z}_+)$ such that

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \frac{\mu_{xy}}{\mu_x} = P(x, y).$$

We write $\mathbb{P}^x(\cdot)$ to denote probabilities if $X_0 = x$.

We also define the stopping times

$$\begin{aligned} T_A &= \min\{n \geq 0 : X_n \in A\}, \\ T_A^+ &= \min\{n \geq 1 : X_n \in A\}, \\ \tau_A &= T_{A^c}, \end{aligned}$$

and as usual write $T_x = T_{\{x\}}$.

Loop erased random walk. This model was introduced by Lawler in 1980 – see [Law80]. Let $G = (V, E)$ be a finite graph, and $\gamma = (x_0, x_1, \ldots, x_n)$ be a finite path in G. We say γ is *self avoiding* if the points $x_0, \ldots, x_n$ are distinct. The *loop erasure of* γ, denoted $\mathfrak{L}(\gamma)$ is defined as follows.

Step 1. Set $\gamma_0 = \gamma$, and set $k = 0$.

Step 2. If γ_k is a self-avoiding path then set $\mathfrak{L}(\gamma) = \gamma_k$.
Step 3. If not, write $\gamma_k = (y_0, y_1, \dots, y_m)$ and let

$$j_2 = \min\big\{i : y_i \in \{y_0, \dots, y_{i-1}\}\big\}, \quad j_1 = \min\{i : y_i = y_{j_2}\}.$$

Then set $\gamma_{k+1} = (y_0, \dots, y_{j_1}, y_{j_2+1}, \dots, y_m)$. (So γ_{k+1} is the path γ_k with the loop $\{y_i,\ j_1 < i \le j_2\}$ erased.) Now continue with Step 2.

Since γ is finite, and each erased loop contains at least one point, it it is clear that this procedure terminates, and defines a self avoiding path.

Now let x be a vertex in G, and $A \subset V$, and let $X_0 = x, X_1, X_2, \dots$ be a SRW on G started at x. The *loop erased walk* from x to A, denoted $\mathrm{LEW}(x, A)$ is defined to be $\mathfrak{L}(\gamma)$, where $\gamma = (X_0, X_1, \dots, X_{T_A})$. This is clearly a random self-avoiding path from x to A.

Given a process or path X we write $\mathfrak{L}(X)$ for the chronological loop erasure of X. It is easy to find examples which show that if the loops are erased in reverse order than one can end up with a different path. Originally it was thought that LEW might provide a model for SAW, but subsequently physicists have decided that SAW and LEW are in different 'universality classes'.

Wilson's algorithm. This proceeds as follows:
(0) Choose an ordering $\{z_0, z_1, \dots, z_m\}$ of V.
(1) Let $\mathcal{U}_0 = \{z_0\}$.
(2) Given $\mathcal{U}_k$ run $\mathrm{LEW}(z_{k+1}, \mathcal{U}_k)$ (independently of everything before), and let $\mathcal{U}_{k+1} = \mathcal{U}_k \cup \mathrm{LEW}(z_{k+1}, \mathcal{U}_k)$. (If $z_{k+1} \in \mathcal{U}_k$ already then $\mathrm{LEW}(z_{k+1}, \mathcal{U}_k)$ just consists of the point z_{k+1}, and $\mathcal{U}_{k+1} = \mathcal{U}_k$.)
(3) Stop when there are no vertices left to add – i.e. when $\mathcal{U}_k$ has vertex set V.

It is clear that Wilson's algorithm (WA) gives a random spanning tree. It is not so obvious that it gives the *uniform spanning tree* – i.e. that each fixed spanning tree $T_0 \in \mathcal{T}(G)$ has the same probability of being chosen.

Correctness of Wilson's algorithm. The proof uses a clever embedding of the SRW and LEW in a more complicated structure.

Fix the tree root $z_0 \in V$. For all $x \ne z_0$ define *stacks* $(\xi_{x,i}, x \in V - \{z_0\}, i \in \mathbb{N})$ as follows. The stack r.v. are all independent, and

$$\mathbb{P}(\xi_{x,i} = y) = P(x, y) = \frac{\mu_{xy}}{\mu_x}, \quad i \ge 1.$$

So each element in a stack at $x \in V$ points to a random neighbour of x.

We will see that the stacks determine:

(a) A SRW on G which ends at z_0,

(b) An implementation of WA which leads to a directed spanning tree $\mathcal{U}$.

The stacks give a SRW on G as follows. Let $z_1 \neq z_0$. Let $X_0 = z_1$, and use $\xi_{z_1,1}$ to find a random neighbour of z_1, y say, and set $X_1 = y$. Then 'pop' the value $\xi_{z_1,1}$ from the z_1-stack, so that the top element of the stack is now $\xi_{z_1,2}$. Then use $\xi_{y,1}$ to choose X_2, and continue until the SRW reaches z_0, where it ends.

Since we can define SRW from the stacks, we can also use the stacks to run WA. Use the stacks to define a SRW starting at z_1, which gives $\mathrm{LEW}(z_1, \{z_0\})$, and continue. (At this point in the argument we assume that the sequence of start points $z_1, \dots, z_n$ has been chosen in advance. Ultimately we will find that we get the same tree whatever the choice of (z_i) is, so that the tree we get only depends on the stack r.v.)

Suppose that at some point the SRW is about to start a cycle, say from $y_0 \to y_1 \to y_2 \to \cdots \to y_k = y_0$. Let ξ_{x,n_x} be the top values in the stacks at this time. Then we must have

$$\xi_{y_i,n_{y_i}} = y_{i+1}, \qquad \text{for } i = 0, \dots, k-1;$$

i.e. we see a cycle "sitting at the top of the stacks". As the SRW goes round the cycle the values at the top of each stack on the cycle get popped. We call this 'popping the cycle', and can regard this as a single step of the algorithm.

We can therefore rephrase the algorithm as follows: starting with the collection of stack r.v., we pop the cycles in some order, until no cycles remain. At that point the stacks give a self-avoiding path from each $x \in V$ to the root z_0. The sequence $z_1, \dots, z_n$ will determine the order of cycle popping, and one key point to be proved is that in fact the order in which cycles are popped makes no difference.

Lemma 1.5 *With probability 1, only finitely many cycles are popped.*

Proof. For a graph $G = (V, E)$ let $\Omega(G)$ be the space of stack variables, and $\Omega_0(G)$ be the set of $\omega \in \Omega(G)$ such that for each $x \in V$ and $y \sim x$, $\xi_{x,k} = y$ for infinitely many k. Clearly we have $\mathbb{P}(\Omega_0(G)) = 1$. We prove a slightly stronger result: for any graph G and any $\omega \in \Omega_0(G)$, only finitely many cycles can be popped.

Suppose that this is false, and let $G = (V, E)$ with root z_0 be a smallest graph which is a counterexample. So there exists $\omega \in \Omega_0(G)$, such that the stack variables $\xi_{x,i}(\omega)$ allow an infinite sequence of poppable cycles. Denote these $C_1, C_2, \dots$ Let $x \sim z_0$. Since $\omega \in \Omega_0(G)$ there exists a smallest k such that $\xi_{x,k} = z_0$. Then x can be contained in at most k of the cycles C_i, since after x is popped k times the r.v. $\xi_{x,k} = z_0$ will be on top of the stacks, and so will never be popped again. Let C_m be the last cycle containing x to be popped. Then $C_{m+1}, \dots$ is an infinite sequence of poppable cycles for the graph obtained by collapsing $\{x, z_0\}$ to a single point z_0', which contradicts the minimality of G. □

Suppose initially that there is no cycle in the top stacks. Then whatever order we choose the points $z_i, i \geq 1$ in, WA will give the same tree. Given a tree T_0 with root z_0, for each point x there is a point $D(x, T_0)$ (descendant of x in the tree T_0) which is the next point on the unique path from x to z_0. Let us calculate the probability of seeing T_0 'on top of the stacks'. For each $x \in V$ we must have $\xi_{x,1} = D(x, T_0)$. So this probability is

$$\mathbb{P}(T_0) = \prod_{x \neq z_0} \mathbb{P}\big(\xi_{x,1} = D(x, T_0)\big) = \prod_{x \neq z_0} \frac{1}{\mu_x} = \mu_{z_0} \prod_{x \in V} \frac{1}{\mu_x}.$$

The final term does not depend on T_0; write $p_G(z_0)$ for this probability.

Define a *labelled cycle* in the stacks to be a sequence

$$C = \Big((z_0, i_0), (z_1, i_1), \dots, (z_k, i_k) = (z_0, i_0)\Big)$$

such that

$$\xi_{z_j, i_j} = z_{j+1} \text{ for each } j = 0, \dots, k-1.$$

Note that two distinct labelled cycles can have a non-empty intersection. The probability a particular labelled cycle C is in the stacks is

$$\mathbb{P}(C) = \prod_{j=0}^{k} \mathbb{P}(\xi_{z_j, i_j} = z_{j+1}) = \prod_{j=1}^{k} P(z_j, z_{j+1}) = \prod_{j=1}^{k} \mu_{z_j}^{-1}.$$

The algorithm proceeds by popping labelled cycles until there are no cycles left to pop and the stacks show a UST. We call a *popping procedure* $\mathcal{P}$ an order of popping labelled cycles, which ends with a tree, and write $(C_1, \ldots, C_m)$ for the (labelled) cycles which are popped in order by $\mathcal{P}$. We also write $\mathcal{P} = (C_1, \ldots, C_m)$.

Lemma 1.6 *Let G be a finite graph, and $\omega \in \Omega_0(G))$. Let C_1, C_1' be cycles which are initially on top of the stacks, and $\mathcal{P} = (C_1, \ldots, C_m)$ be a popping procedure for which the first cycle popped is C_1.*
(a) We have that $C_1' = C_k$ for some $k \in \{2, \ldots, m\}$.
(b) $\mathcal{P}' = (C_1', C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_m)$ is a popping procedure, and $\mathcal{P}$ and $\mathcal{P}'$ lead to the same final tree.

Proof. (a) Under the procedure the cycle C_1' will remain on top of the stacks until some r.v. in C_1' is popped. However, if at some point distinct labelled cycles C_j and C_1' are on top of the stacks, then the two cycles are disjoint, so at most one can be popped. Thus C_1' will remain until the whole cycle is popped, and thus $C_1' = C_k$ for some $2 \leq k \leq m$.
(b) Under the procedure $\mathcal{P}$, the cycle C_1' will still be on top when the cycle C_{k-1} is popped. After k steps each of $\mathcal{P}$ and $\mathcal{P}'$ will have popped the cycles $C_1', C_1, \ldots, C_{k-1}$, and so at that point will show the same r.v. on top of the stacks. It follows that they will then end with the same tree. □

Proposition 1.7 *If $\omega \in \Omega_0(G))$ then any two popping procedures will end with the same tree.*

Proof. If not, then there exists a minimal graph for which this can fail; call this G. Given any popping procedure $\mathcal{P}$, define the weight $w(\mathcal{P})$ to be the number of r.v. popped. If there are popping procedures $\mathcal{P}_1$ and $\mathcal{P}_2$ which give different outcomes, choose a pair $(\mathcal{P}_1, \mathcal{P}_2)$ such that the sum of the weights $w(\mathcal{P}_1) + w(\mathcal{P}_2)$ is minimal. Write also $T(\mathcal{P})$ for the tree given by the popping procedure $\mathcal{P}$.

Now look at the r.v. initially on top of the stacks, $\xi_{x,1} = \xi_{x,1}(\omega)$. These cannot form or a tree, or we would have $\mathcal{P}_1 = \mathcal{P}_2 = \emptyset$. There must therefore be some cycles on initially on top: denote these $C_1, \ldots, C_j$. We can assume that $\mathcal{P}_i$ pops C_i first, for $i = 1, 2$. Now let $\mathcal{P}_1'$ be the procedure given by applying the previous lemma to $\mathcal{P}_1$ and the cycle C_2, so that $\mathcal{P}_1'$ first pops C_2 and then C_1, and then the remaining cycles (except C_2) popped by $\mathcal{P}_1$. By Lemma 1.6

we have $T(\mathcal{P}_1') = T(\mathcal{P}_1)$, while by hypothesis we have $T(\mathcal{P}_1) \neq T(\mathcal{P}_2)$. But then if we consider the stacks given by popping C_2, then the cycles $\mathcal{P}_1' - C_2$ and $\mathcal{P}_2 - C_2$ will give different trees, and will have smaller total weight than the pair $(\mathcal{P}_1, \mathcal{P}_2)$, a contradiction. □

It follows from this Proposition that (on the set $\Omega_0(G)$), for any labelled cycle C, whatever popping procedure is used, either C will ultimately be popped, or else C will never be popped. If C will ultimately be popped we call C a *poppable cycle*.

Now look at a fixed set of labelled cycles $\mathcal{C} = \{C_1, \dots, C_n\}$, and a spanning tree T_0. By the above the probability we will have the C_i as poppable cycles, and then have the tree T_0 underneath is

$$\Big(\prod_{i=1}^{n} \mathbb{P}(C_i) \Big) \times \mathbb{P}(T_0) = \mathbb{P}(\mathcal{C}) p_G(z_0).$$

Since this is the same for all trees T_0, it follows that WA is correct, i.e. it produces each spanning tree with equal probability.

Remark 1.8 The initial statement of the algorithm depends on fixing a sequence of vertices $z_0, z_1, \dots, z_m$. However, once the stack r.v. are fixed, the order of popping cycles does not affect the final tree obtained. So, any reordering of $\{z_1, \dots, z_m\}$ will give the same outcome. Further, the choice of z_k can be allowed to be random, and depend on the tree T_{k-1}.

Remark 1.9 Strictly speaking, WA produces a directed tree with a root z_0, and the argument above proves that once z_0 is fixed than any two directed trees have the same probability $p_G(z_0)$. Once we fix the root, there is an isomorphism between the set of directed and undirected spanning trees. Now fix two undirected spanning trees T_1 and T_2. If we choose a root z_0, write $\overrightarrow{T_i}$ for the associated directed trees. Then WA with root z_0 has the same probability of producing $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$, and hence the same probability of producing T_1 and T_2. It follows that the choice of root will not affect the probability of obtaining a particular spanning tree.

In a paper on LEW in $\mathbb{Z}^3$ [Koz] , G. Kozma comments:
"Of all the non-Gaussian models in statistical mechanics, LEW is probably

the most tractable".
Because of Wilson's algorithm UST is the next easiest.

Finally, we remark that the UST can be obtained from the FK random-cluster model. This connection is due to Häggstrom [Hag95], and shows that the uniform spanning tree belongs to the same family of models in statistical physics as percolation, the Ising and Potts models. Further, the UST is in some sense critical, since the tree has just enough bonds to form large connected sets. For more on the random-cluster model see [Gr].

Let $p \in (0, 1)$ and $q > 0$, and define a probability measure on subsets $E' \subset E$ as follows. Let $n(E')$ be the number of edges in E', and $k(E')$ be the number of connected components of the graph (V, E'). Then set

$$\mathbb{P}_{p,q}(E') = Z_{p,q}^{-1}\, p^{n(E')}(1-p)^{|E|-n(E')}q^{k(E')}. \tag{1.6}$$

Here $Z_{p,q}$ is a normalising constant. If $q = 1$ then the term with q^k disappears, and one obtains bond percolation with probability p on G. The Ising model relates to $q = 2$.

Now consider the limit of (1.6) as $p, q \to 0$ with $q/p \to 0$. Let E_C be a smallest subset of E' so that every component of $(V, E' - E_C)$ is a tree, and set $c(E') = |E_C|$, $n_1(E') = n(E) - c(E)$. We have $n_1(E') + k(E') = |V|$, so we can write

$$\begin{aligned}\mathbb{P}_{p,q}(E') &= Z_{p,q}^{-1}(1-p)^{|E|-n(E')}p^{|V|}p^{c(E')}(q/p)^{k(E')}\\ &= Z'_{p,q}(1-p)^{-n(E')}p^{c(E')}(q/p)^{k(E')}.\end{aligned}$$

Since p is small, the measure $\mathbb{P}_{p,q}$ will concentrate on configurations with $c(E') = 0$ – that is on subgraphs which are trees. Since q/p is also small, it also concentrates on configurations which have only one component – so on spanning trees. Finally, if E' is any spanning tree then $k(E') = 1$, $c(E') = 0$ and $n_1(E') = |V| - 1$, so $\mathbb{P}_{p,q}(E')$ will be the same for any E'. Hence the limiting measure is uniform on spanning trees.

2 Infinite graphs

An infinite graph (such as $\mathbb{Z}^d$) will have infinitely many spanning trees, so it is no longer clear how to define one uniformly. Let $B_n = [-n, n]^d \subset \mathbb{Z}^d$, and write $\mathbb{P}_n$ for the probability law of the UST on B_n. Then following [Pem91] we wish to show that the laws $\mathbb{P}_n$ have a limit $\mathbb{P}$.

We will use Wilson's algorithm, and begin with a calculation for finite graphs. We write $\mathcal{U} = \mathcal{U}_G$ for the UST on a graph G.

Lemma 2.1 *Let $G = (V, E)$ be finite, and $e = \{x, y\} \in E$. Then*

$$\mathbb{P}(e \in \mathcal{U}) = \mu_{xy} R_{\text{eff}}(x, y). \tag{2.1}$$

Proof. Calculations with random walks and electrical resistance give

$$\mathbb{P}^x(T_y < T_x^+) = \frac{1}{\mu_x R_{\text{eff}}(x, y)}. \tag{2.2}$$

We construct $\mathcal{U}$ by using WA with root y and then adding LEW(x, y). Then $e \in \mathcal{U}$ if and only if this LEW just consists of e. (If the LEW is not e, then e can never be added during the later stages of the construction, since adding e would create a loop.)

Write $p_e = \mathbb{P}(e \in \mathcal{U})$. Then p_e is the probability that X first hits y by a step from x. Using the Markov property we have

$$p_e = \mathbb{P}^x(X_1 = y) + \mathbb{P}^x(T_x^+ < T_y) p_e,$$

and so

$$p_e = \frac{\mathbb{P}^x(X_1 = y)}{\mathbb{P}^x(T_y < T_x^+)} = \frac{\mu_{xy}/\mu_x}{1/(\mu_x R_{\text{eff}}(x, y))} = \mu_{xy} R_{\text{eff}}(x, y).$$

□

Remark. In an unweighted graph we have $\mu_{xy} = 1$ if $x \sim y$, so Kirchoff's Theorem 1.3 follows immediately.

Now let $G = (V, E)$ be an infinite graph. (As usual we will assume G is locally finite and connected.) Let V_n be an increasing sequence of connected finite sets, with $\cup V_n = V$. We define the subgraphs $G_n = (V_n, E_n)$ in the obvious way: $E_n = \big\{\{x, y\} : x, y \in V_n\big\}$. We also define the *wired* subgraphs $G_n^W = (V_n^W, E_n^W)$ as follows. Let ∂_n be an additional point, $V_n^W = G_n \cup \{\partial_n\}$, and let $E_n^W = E_n \cup \big\{\{x, \partial_n\}, x \in \partial_i V_n\big\}$. So G_n and G_n^W are the same away from the boundary $\partial_i V_n$, but for the graph G_n^W every point on the boundary $\partial_i V_n$ is connected to the extra point ∂_n. We write $\mathcal{U}_{G_n}, \mathcal{U}_{G_n^W}$ for the USTs on G_n and G_n^W, and write $\mathbb{P}_n^F, \mathbb{P}_n^W$ for their laws. (Here 'F' stands for *free*.)

The key to the existence of a limit of the laws $\mathbb{P}_n^{\cdot}$ is the following.

Lemma 2.2 *Let $e \in G_1$. Then for any $n \geq 1$*

$$\mathbb{P}(e \in \mathcal{U}_{G_n^W}) \leq \mathbb{P}(e \in \mathcal{U}_{G_{n+1}^W}) \leq \mathbb{P}(e \in \mathcal{U}_{G_{n+1}}) \leq \mathbb{P}(e \in \mathcal{U}_{G_n}). \tag{2.3}$$

Proof. This is easy from Lemma 2.1 and standard monotonicity properties of electrical networks using the variational characterisation of effective conductance (1.4). □

Now let $G = (V, E)$ be a finite graph, and F be a subset of E with no cycles. We define the *collapsed graph* G/F by collapsing any pair of vertices connected by edges in F to a single point. (This graph may have loops and multiple edges.)

Lemma 2.3 *Let G be finite, $F \subset E$ with no cycles. Let $T \in \mathcal{T}(G)$. Then*

$$\mathbb{P}(\mathcal{U}_G = T | F \subset \mathcal{U}_G) = \mathbb{P}(U_{G/F} \cup F = T). \tag{2.4}$$

Proof. By induction, it is enough to consider the case when $F = \{e\}$. Let $T_1, T_2 \in \mathcal{T}(G)$ be trees containing e, and let T_1', T_2' be their contractions in $G' = G/F$. Consider the following construction of a spanning tree for G:
(i) Construct a UST $\mathcal{U}'$ for G'.
(ii) Add the edge e to create a spanning tree $\mathcal{U} = \mathcal{U}' \cup \{e\}$ for G.

Then we have $\mathbb{P}(\mathcal{U}' = T_1') = \mathbb{P}(\mathcal{U}' = T_2')$, and so $\mathbb{P}(\mathcal{U} = T_1) = \mathbb{P}(\mathcal{U} = T_2)$. Thus $\mathcal{U}$ is uniform on the set of all spanning trees containing e, which proves (2.4). □

Proposition 2.4 *Let G_n be as above, and $F = \{e_1, \dots, e_m\} \in E$. Then for all $n \geq 1$,*

$$\mathbb{P}(F \subset \mathcal{U}_{G_n^W}) \leq \mathbb{P}(F \subset \mathcal{U}_{n+1}^W) \leq \mathbb{P}(F \subset \mathcal{U}_{n+1}^F) \leq \mathbb{P}(F \subset \mathcal{U}_n^F). \tag{2.5}$$

Proof. Assume n is large enough so that $F \subset V_n - \partial_i V_n$. By Lemma 2.3 we have, writing $F_{m-1} = \{e_1, \dots, e_{m-1}\}$,

$$\begin{aligned}\mathbb{P}(e_1, \dots, e_m \in \mathcal{U}_{G_n^W}) &= \mathbb{P}(F_{m-1} \subset \mathcal{U}_{G_n^W} | e_m \in \mathcal{U}_{G_n^W})\mathbb{P}(F_{m-1} \subset \mathcal{U}_{G_n^W}) \\ &= \mathbb{P}(F_{m-1} \subset \mathcal{U}_{G_n^W/\{e_m\}})\mathbb{P}(F_{m-1} \subset \mathcal{U}_{G_n^W}).\end{aligned}$$

Using induction and Lemma 2.2 the conclusion then follows. □

Although we will not need this, we remark that we do have an expression for the law of the UST in a graph finite G. Define the *transfer current matrix* $Y(e, e')$ on $E \times E$ by taking $Y(e, e')$ to be the current which flows in the (directed) edge e' if a current of size 1 is imposed on e.

Theorem 2.5 *(Transfer Current Theorem, [BuPe].)*

$$\mathbb{P}(e_1, \dots, e_m \in \mathcal{U}) = det(Y(e_i, e_j)_{1 \le i,j \le m}).$$

Definition 2.6 A *spanning forest* of a graph G is a subgraph $\mathcal{F} = (V, E_{\mathcal{F}})$ with vertex set V such that each connected component of $\mathcal{F}$ is a tree.

Using Proposition 2.4 one obtains:

Theorem 2.7 *Let $G = (V, E)$ and $V_n \uparrow\uparrow V$. Then the limits $\mathbb{P}^F$ and $\mathbb{P}^W$ of the laws $\mathbb{P}_n^F$ and $\mathbb{P}_n^W$ exist.*

We call $\mathbb{P}^W$ the *wired spanning forest* and $\mathbb{P}^F$ the *free spanning forest*, and write $\mathcal{U}^F = \mathcal{U}_G^F$, $\mathcal{U}^W = \mathcal{U}_G^W$ for the associated random variables.

The following properties of the wired and free spanning forests follow from this construction.

Theorem 2.8 *(1) All the components of $\mathcal{U}^F$ and $\mathcal{U}^W$ are infinite trees.*
(2) If A is any increasing event then

$$\mathbb{P}^W(A) \subset \mathbb{P}^F(A). \tag{2.6}$$

(3) There exists a coupling of $\mathcal{U}^F$ and $\mathcal{U}^W$ such that $\mathcal{U}^W \subset \mathcal{U}^F$.
(4) The laws of $\mathcal{U}^W$ and $\mathcal{U}^F$ on $\mathbb{Z}^d$ are translation invariant.

Corollary 2.9 *Suppose that $\mathbb{P}(e \in \mathcal{U}^F) = \mathbb{P}(e \in \mathcal{U}^W)$ for all edges e. Then $\mathcal{U}^F = \mathcal{U}^W$.*

We now wish to explore the properties of the wired and free UST. Our first remark is that they can be different.

Example. Let G be the binary tree. Fix a vertex $o \in V$, and set $V_n = B(0, n)$. Then clearly $\mathcal{U}_{G_n} = G_n$. However, a calculation using (2.2) in G_n^W gives

$P^o(T_{\partial_n} < T_o^+) \simeq \frac{1}{2}$, so that if $x \sim o$ then $P^x(T_o < T_{\partial_n}) \simeq \frac{1}{2}$. Using WA with root ∂_n it then follows that with positive probability the edge $e = \{o, x\}$ is not in $\mathcal{U}_{G_n^W}$. Taking limits we have $\mathbb{P}(e \in \mathcal{U}^W) < 1$, so with positive probability $\mathcal{U}^W$ is not connected – i.e. it is a forest and not a tree. With a little more work, one has that $\mathcal{U}^W$ is a forest with infinitely many components, a.s.

A function $h : V \to \mathbb{R}$ is *harmonic* if

$$\Delta h(x) = \sum_y \mu_{xy}(h(y) - h(x)) = 0 \text{ for all } x \in V. \tag{2.7}$$

We set $\mathcal{HD}$ to be the set of h which are harmonic and with $\mathcal{E}(h,h) < \infty$ – this is called the set of harmonic Dirichlet functions. Clearly $\mathcal{HD}$ contains constant functions.

Theorem 2.10 *([BLPS]) The following are equivalent:*
(1) $\mathcal{U}^W = \mathcal{U}^F$ *a.s.*
(2) The space $\mathcal{HD}(G)$ *is trivial, that is consists only of constants.*

It follows that $\mathcal{U}^W = \mathcal{U}^F$ for $\mathbb{Z}^d$. Here is another proof of that fact, from Section 6 of [BLPS].

Definition 2.11 A graph G is *amenable* if there exists $V_n \uparrow\uparrow V$ such that

$$\lim_n \frac{|\partial V_n|}{|V_n|} = 0.$$

Lemma 2.12 *Let G be amenable. Let $\mathcal{F}$ be a fixed forest in G, such that all components of $\mathcal{F}$ are infinite. Let $\mathcal{F}_n$ be the restriction of $\mathcal{F}$ to G_n. If* $\deg(x|\mathcal{F}_n)$ *is the degree of x in $\mathcal{F}_n$ then*

$$\lim_n |V_n|^{-1} \sum_{x \in V_n} \deg(x|\mathcal{F}_n) = 2. \tag{2.8}$$

Further, the limit above is uniform in $\mathcal{F}$.

Proof. Let k_n be the number of components in the subgraph $\mathcal{F}|_{V_n}$. Then $k_n \le |\partial V_n|$. If T is a finite tree, then T has $|T| - 1$ edges, and so the number of edges in $\mathcal{F}|_{V_n}$ is $|V_n| - k_n$. Since each edge contains 2 vertices,

$$\sum_{x \in V_n} \deg(x|\mathcal{F}_n) = 2(|V_n| - k_n),$$

and taking the limit in n gives (2.8). □

Theorem 2.13 *On $\mathbb{Z}^d$, $\mathcal{U}^W = \mathcal{U}^F$.*

Proof. Let $V_n = B_\infty(0,n) = [-n,n]^d$. As $\mathcal{U}^W$ is translation invariant, $a_W = \mathbb{E}(\deg(x|\mathcal{U}^W))$ does not depend on x. Taking expectation in (2.8) (and using the fact that the limit is uniform) we obtain

$$2 = \lim_n |V_n|^{-1} \sum_{x \in V_n} \mathbb{E}(\deg(x|\mathcal{U}^W|_{V_n})),$$

from which it follows that $a_W = 2$. Similarly $a_F = 2$. Using symmetry again we have for any edge e that

$$\mathbb{P}(e \in \mathcal{U}^F) = \mathbb{P}(e \in \mathcal{U}^W) = \frac{1}{d},$$

and hence $\mathcal{U}^W = \mathcal{U}^F$. □

The construction of the measures $\mathbb{P}^W$ and $\mathbb{P}^F$ do not give a straightforward random construction of the forests $\mathcal{U}^W$ and $\mathcal{U}^F$. However, for the wired spanning forest, one can extend Wilson's algorithm to the infinite graph.

First, if G is recurrent, then WA can be performed exactly as in the finite case. At each stage the SRW started at z_n will hit the tree T_{n-1} with probability 1.

If G is transient then since X will hit any point only finitely many times, the loop erasure of X, denoted $\mathrm{LEW}(x_0,\infty)$ can be defined. Further, if $V_n \uparrow\uparrow V$ then the paths $\mathrm{LEW}(x_0, V - V_n)$ converge to $\mathrm{LEW}(x_0,\infty)$.

Wilson's algorithm rooted at infinity is as follows. As before, list the set V as a sequence $V = \{z_0, z_1, \dots\}$. Then:
(1) Let $\mathcal{U}_0 = \mathrm{LEW}(z_0,\infty)$.
(2) Given $\mathcal{U}_{n-1}$, let $\mathcal{U}_n = \mathcal{U}_{n-1} \cup \mathrm{LEW}(\mathcal{U}_{n-1})$, and continue.

Theorem 2.14 *(a) If G is recurrent then Wilson's algorithm creates a tree with law equal to $\mathcal{U}^W$.*
(b) If G is transient then Wilson's algorithm rooted at infinity creates a forest with law equal to $\mathcal{U}^W$.

Proof. Let $V_n \uparrow\uparrow V$. For both parts, the proof proceeds by fixing a ball $B(o,R) \subset V$, and showing that, with high probability, for large enough n, the restrictions of the trees produced by WA in the infinite case, and in G_n^W, are the same.

Write X for the SRW on G, and $X^{(G_n^W)}$ for the SRW on G_n^W. The argument uses the fact that for $x \in B(o,R)$, $A \subset B(o,R)$, we have

$$\lim_n \mathbb{P}^x(X^{(G_n^W)} \text{ hits } A) = \mathbb{P}^x(X \text{ hits } A).$$

In the transient case this fails if we replace $X^{(G_n^W)}$ by $X^{(G_n)}$. □

Remark. If G is recurrent then $\mathcal{U}^F = \mathcal{U}^W$.

We now write $\mathcal{U}_d$ for the UST in $\mathbb{Z}^d$, and are interested in whether $\mathcal{U}_d$ is a tree or a forest.

Given a random process $X = (X_n, n \geq 0)$ in $\mathbb{Z}^d$ we define the range of X by

$$\mathcal{R}(X) = \{X_0, X_1, \dots\} \subset \mathbb{Z}^d. \tag{2.9}$$

We say that $X^{(0)}$ and $X^{(1)}$ intersect if $\mathcal{R}(X^{(0)}) \cap \mathcal{R}(X^{(1)}) \neq \emptyset$, and will need the following well known result on intersections of random walks.

Theorem 2.15 *Let $X^{(0)}$ and $X^{(1)}$ be independent SRW in $\mathbb{Z}^d$, with different initial points.*
(a) If $d \geq 5$ then $\mathbb{P}(\mathcal{R}(X^{(0)}) \cap \mathcal{R}(X^{(1)}) = \emptyset) > 0$, and $\mathbb{P}(|\mathcal{R}(X^{(0)}) \cap \mathcal{R}(X^{(1)})| = \infty) = 0$. Further,

$$\lim_{|x_0 - x_1| \to \infty} \mathbb{P}(\mathcal{R}(X^{(0)}) \cap \mathcal{R}(X^{(1)}) \neq \emptyset | X^{(0)} = x_0, X^{(1)} = x_1) = 0. \tag{2.10}$$

(b) If $d \leq 4$ then $\mathbb{P}(|\mathcal{R}(X^{(0)}) \cap \mathcal{R}(X^{(1)})| = \infty) = 1$.

Proof. (Sketch). The Green's function for SRW on $\mathbb{Z}^d$ with $d \geq 3$ satisfies

$$G(x,y) = \sum_{n=0}^{\infty} P^x(X_n = y) \simeq \frac{1}{1 \vee |x-y|^{d-2}}.$$

So if $X_0^{(i)} = x_i$, $i = 0, 1$ then straightforward computations give for $d \geq 5$ that

$$\mathbb{E} \sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 1_{(X_n^{(0)} = x, X_m^{(1)} = x)} = \sum_{x \in \mathbb{Z}^d} G(x_0, x) G(x_1, x) \asymp \frac{1}{|x_0 - x_1|^{d-4}}.$$

This establishes (2.10), and the other assertions in (a) follow.
The proof of (b) is a little harder, particularly when $d = 4$ – see [Law91]. □

Theorem 2.16 *Let $d \geq 5$. Then $\mathcal{U}_d$ consists of infinitely many components a.s.*

Proof. It is enough to prove that for any $\varepsilon > 0$ and $N \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{U}_d \text{ has at least } N \text{ components}) > 1 - \varepsilon. \tag{2.11}$$

By (2.10) we can find points $x_1, \dots, x_N$ such that if $X(i)$ denote independent SRW started at x_i then for each $i \neq j$, $\mathbb{P}(\mathcal{R}(X(i)) \cap \mathcal{R}(X^{(j)}) \neq \emptyset) < N^{-2}\varepsilon$. Now apply WA rooted at infinity, starting the SRW at the points x_i. The probability that any of the SRW paths meet is less than ε, and so the same holds for their loop erasures. So at stage N of the algorithm, we obtain, with probability greater than $1-\varepsilon$, N distinct paths, each going from a point x_i to infinity. The additional stages of the algorithm cannot join these components, so we have proved (2.11). □

It is harder for a SRW to hit an independent LEW than another SRW, so Theorem 2.15 is not enough on its own to handle the cases $d \leq 4$. However, we have the following remarkable result.

Theorem 2.17 *([LPS03]) Let G be a transient graph, and X, Y be independent SRW. Write $\mathfrak{L}(X)$ for the loop erasure of X. Then $|\mathcal{R}(\mathfrak{L}(X)) \cap \mathcal{R}(Y)| = \infty$ a.s. on $\{|\mathcal{R}(X) \cap \mathcal{R}(Y)| = \infty\}$.*

Proof. (Sketch of main idea). For $m \geq 0$ let $L^{(m)} = \mathfrak{L}(X_0, \dots, X_m)$, Define

$$\begin{aligned} I_{m,n} &= 1_{(X_m = Y_n)}, \\ T_m^X &= \min\{k \geq 0 : L_k^{(m)} \in X[m, \infty)\}, \\ T_{m,n}^Y &= \min\{k \geq 0 : L_k^{(m)} \in Y[n, \infty)\}, \\ J_{m,n} &= I_{m,n} 1_{(T_{m,n}^Y \leq T_m^X)}. \end{aligned}$$

Thus T_m^X is the first point on the path $L^{(m)}$ which is hit by X after m, and $T_{m,n}^Y$ is the first point on $L^{(m)}$ which is hit by Y after n. Let $\mathcal{F}^X$ and $\mathcal{F}^Y$ be the filtrations of the random walks X and Y. Then

$$\mathbb{E} I_{m,n} \geq \mathbb{E} J_{m,n} = \mathbb{E}\Big(I_{m,n} \mathbb{P}(T_{m,n}^Y \leq T_m^X | \mathcal{F}_m^X \vee \mathcal{F}_n^Y) \Big). \tag{2.12}$$

However, on $\{I_{m,n} = 1\}$ we have $X_m = Y_n$, and the path $L^{(m)}$ is $\mathcal{F}_m^X \vee \mathcal{F}_n^Y$ measurable. So the probability in (2.12) is the probability that one of two SRW stated at the same point hits a fixed path at an earlier point than the other, and so is at least $\frac{1}{2}$. Thus

$$\mathbb{E}I_{m,n} \geq \mathbb{E}J_{m,n} \geq \tfrac{1}{2}\mathbb{E}I_{m,n}. \tag{2.13}$$

Now set

$$I = \sum_n \sum_m I_{n,m}, \quad J = \sum_n \sum_m J_{n,m}.$$

If $J_{m,n} = 1$ then Y hits $L^{(m)}$ at a point which will not be erased later by $\mathfrak{L}(X)$, and so Y and $\mathfrak{L}(X)$ intersect. It follows that $\{\mathcal{R}(Y) \cap \mathcal{R}(\mathfrak{L}(X)) \neq \emptyset\} = \{J > 0\}$, while $\{\mathcal{R}(Y) \cap \mathcal{R}(X) \neq \emptyset\} = \{I > 0\}$. Finally,

$$\mathbb{P}(J > 0) \geq \frac{(\mathbb{E}J)^2}{\mathbb{E}J^2} \geq \frac{(\mathbb{E}I)^2}{4\mathbb{E}I^2}. \tag{2.14}$$

Thus we can control the intersections of Y and $\mathfrak{L}(X)$ in terms of the intersections of Y and X. The final part of the proof is to study intersections in balls to obtain a good lower bound in (2.14). □

Corollary 2.18 *For $d = 2, 3$ a SRW and an independent LEW intersect infinitely often.*

Theorem 2.19 *Let $d \leq 4$. Then $\mathcal{U}_d$ is a tree, a.s.*

Proof. For $d = 2$ this is clear, from the extension of WA to a recurrent infinite graph. For $d = 3, 4$ we use WA rooted at infinity: let γ be the initial path, which we can take to be LEW$(0, \infty)$. By Theorem 2.15 and Corollary 2.18, if X is a SRW in $\mathbb{Z}^d$ then X hits γ with probability 1. So at each stage of WA the new path connects with the existing component of the tree, and hence the resulting set has just one component. □

3 Loop erased walk in $\mathbb{Z}^d$

As mentioned above, this was introduced by Lawler in 1980. We begin with some general properties of LEW on any graph $G = (V, E)$. Consider the random walk X with transition probabilities given by

$$P_X(x,y) = \frac{\mu_{xy}}{\mu_x}, \quad \text{where } \mu_x = \sum_y \mu_{xy}. \tag{3.1}$$

Here μ_{xy} are edge weights; we assume $\mu_{xy} > 0$ if $x \sim y$, and $\mu_{xy} = 0$ whenever $x \not\sim y$. For $D \subset V$ let $G_D(x,y)$ be the Green function given by

$$G_D(x,y) = \mathbb{E}^x \sum_{i=0}^{\tau_D} 1_{(X_i = y)}. \tag{3.2}$$

We have $\mu_x G_D(x,y) = \mu_y G_D(y,x)$.

For simplicity we give the first result for a transient graph.

Theorem 3.1 *[Law99] Let G be transient and $\gamma = \{x_0, x_1, \dots, x_n\}$ be a self avoiding path. Set $D_k = V - \{x_0, \dots, x_k\}$, and $D_{-1} = V$. If $L = \mathrm{LEW}(x_0, \infty)$ then*

$$\mathbb{P}^{x_0}((L_0, \dots, L_n) = \gamma) = \prod_{j=1}^n P_X(x_{i-1}, x_i) \prod_{j=0}^n G_{D_{j-1}}(x_j, x_j) \mathbb{P}^{x_n}(\tau^+_{D_n} = \infty). \tag{3.3}$$

This leads to

Theorem 3.2 *('Domain Markov property'.) Let $G = (V, E)$, $x_0 \in D \subset V$, and let $L^D = \mathrm{LEW}(x_0, D^c)$. Let $\gamma = (w_0, \dots, w_k)$ be a self avoiding path in G with $w_0 = x_0$, and $\gamma' = (w_{k+1}, \dots, w_m)$ such that the concatenation $\gamma \oplus \gamma' = (w_0, \dots, w_m)$ is a self avoiding path from x_0 to D^c. Let Y be SRW on G with $Y_0 = w_k$, conditioned on $\{T_{D^c} < T^+_\gamma\}$. Then*

$$\mathbb{P}\big(L^D = \gamma \oplus \gamma' | (L^D_0, \dots, L^D_k) = \gamma\big) = \mathbb{P}\big(\mathfrak{L}(Y[0, \tau_D]) = \gamma'\big). \tag{3.4}$$

These results mean that study of the law of a loop erased walk is likely to involve studying conditioned random walks.

It is easy to verify that given a deterministic path γ and its time reversal $\mathfrak{R}\gamma$ one has in general $\mathfrak{L}(\gamma) \neq \mathfrak{L}(\mathfrak{R}\gamma)$. However, (see [Law91]) for a Markov chain the loop erasure of X and its time reversal do have the same law.

Lemma 3.3 *([Law99, Section 7.2].) Let $\gamma = (x_0, \dots, x_n)$ be a random walk path. Then $\mathfrak{L}(\gamma)$ and $\mathfrak{R}(\mathfrak{L}(\mathfrak{R}\gamma))$ have the same law.*

We now turn to LEW in $\mathbb{Z}^d$. We can summarise its properties as far as scaling limits are concerned as follows:

For $d \geq 4$ the LEW has standard Brownian motion as its scaling limit. ([Law80, Law86]).

For $d = 2$ LEW has SLE_2 as its scaling limit [LSW04].

For $d = 3$ the scaling limit of LEW exists [Koz].

Length of LEW. For $D \subset \mathbb{Z}^d$ let $M_D = |\mathrm{LEW}(0, D^c)|$ be the length of a LEW run from 0 to D^c. Define

$$L_d(n) = \mathbb{E}^0 M_{B_E(0,n)}.$$

Write

$$f(n) \approx n^\theta \quad \text{to mean} \quad \lim_{n\to\infty} \frac{\log f(n)}{\log n} = \theta.$$

Then Lawler's results give $L_d(n) \approx n^2$ for $d \geq 4$. (In fact it is quite easy to prove that $L_d(n) \asymp n^2$ when $d \geq 5$. Much more delicate is that $L_4(n) \sim n^2(\log n)^{-1/3}$.) For $d = 3$ Shiraishi [Sh] proves there exists θ_3 such that $L_3(n) \approx n^{\theta_3}$. While numerical estimates suggest that $\theta_3 \simeq 1.63$, the best rigorous estimates are $1 < \theta_3 \leq \frac{5}{3}$.

From now on I will concentrate on $\mathbb{Z}^2$.

Theorem 3.4 *([Ken00]).*

$$L_2(n) \approx n^{5/4}. \tag{3.5}$$

Remarks. 1. This was proved by R. Kenyon using dimer coverings, just before the introduction of SLE. Although LEW has SLE_2 as a scaling limit, and SLE_2 has Hausdorff dimension 5/4, I do not know of a quick argument which gives (3.5) just from the SLE theory. (As it is possible to approximate a smooth path by rough ones, one cannot expect to get a lower bound on L_2 just from the convergence to SLE_2.)

2. We will see below that $L_2(n) \asymp n^{5/4}$, i.e. there exists C such that

$$C^{-1} n^{5/4} \leq L_2(n) \leq C n^{5/4}, \qquad n \geq 1. \tag{3.6}$$

3. Given (3.5), one expects that $M_n = M_{B_E(0,n)}$ will be typically be of order $L_2(n)$. But (3.5) is just a first moment result, and does not exclude (say) the possibility that

$$\mathbb{P}(M_n = n) = \mathbb{P}(M_n = n^{5/4}) = \tfrac{1}{2}.$$

Here is a more precise statement of the convergence result in [LSW04].

Theorem 3.5 *Let D be a domain in $\mathbb{C}$ with $0 \in D$. Consider the following measures on simple curves in D. For $\delta > 0$ let μ_δ denote the law of a* $\mathrm{LEW}(0, D^c)$ *in $D \cap \delta\mathbb{Z}^d$. Let ν be the law of* SLE_2 *run from ∂D to 0, with initial position chosen according to harmonic measure. Then $\mu_\delta \Rightarrow \nu$, in the Hausdorff topology.*

Remark. There is no time parametrisation in this result; the problem of obtaining convergence in the 'natural parametrisation' is still open. For a discussion of this problem, and some partial results, see [AKM].

Metric and balls in the UST. Let $\mathcal{U}$ be a USF in $\mathbb{Z}^2$; i.e. $\mathcal{U}$ is a random spanning forest in $\mathbb{Z}^d$, chosen according to the probability measure $\mathbb{P}$ on a space of USFs. Let $d_\mathcal{U}(x,y)$ be the (random) intrinsic graph metric for $\mathcal{U}$: that is the length of the shortest path in $\mathcal{U}$ connecting x and y. (We take $d_\mathcal{U}(x,y) = +\infty$ if there is no such path.) We define balls in $\mathcal{U}$ by

$$B_\mathcal{U}(x,r) = \{y \in \mathbb{Z}^2 : d_\mathcal{U}(x,y) \le r\};$$

and also define Euclidean balls by

$$B_E(x,r) = \{y \in \mathbb{Z}^2 : |x-y| \le r\}.$$

We want to know what $\mathcal{U}$ looks like, and in particular how balls in the intrinsic metric $d_\mathcal{U}(x,y)$ compare with balls in the Euclidean metric $|x-y|$.

Since the paths of LEW in $\mathbb{Z}^2$ are quite rough, one expects that $B_\mathcal{U}(x,r)$ should be much smaller than $B_E(x,r)$. In fact, by Wilson's algorithm the $d_\mathcal{U}$ distance from 0 to $B_E(0,r)^c$ should be the length of a LEW run from 0 to $B_E(0,r)^c$ – that is roughly $L_d(r)$. So it is natural to guess that

$$B_E(0,r) \simeq B_\mathcal{U}(0, L_2(r)). \tag{3.7}$$

There is an obvious strategy to study $\mathcal{U}$ via LEW and Wilson's algorithm. We use WA and LEW to construct $\mathcal{U}$, and use the estimates on $L_2(n) = E^x|M_{B_E(x,n)}|$ to study the size of the balls $B_\mathcal{U}(x,r)$. The following difficulties arise:

1. Kenyon's theorem gives a first moment result only.
2. One needs lots of LEW to build $\mathcal{U}$, so to avoid accumulation of errors one

needs very good control of the tail probabilities for LEW: A LEW which is much too short or much too long would have a serious effect on the size and shape of $B_{\mathcal{U}}(0,r)$.
3. Lack of monotonicity of M_D in D. Suppose $D_1 \subset D_2 \subset \mathbb{Z}^2$. Then $\mathrm{LEW}(0, D_2^c)$ stopped when it exits D_1 is not the same as $\mathrm{LEW}(0, D_1^c)$. In fact, if $D_1 \subset D_2 \subset \mathbb{Z}^d$ then one need not have

$$\mathbb{E}M_{D_1} \le \mathbb{E}M_{D_2}.$$

Here is an example, due to O. Angel and G. Kozma. Let $V = \mathbb{Z}$, $D_1 = \{0, 1, \dots, N-1\}$ and $D_2 = \mathbb{Z}_+$. Then $|\mathrm{LEW}(0,-1)| = 1$ a.s., while it is easy to verify that $\mathbb{E}|\mathrm{LEW}(0, \{-1, N\})| = 2N/(N+1)$.

In [Mas09] Robert Masson gave a new, more probabilistic, proof of Theorem 3.4. His argument proceeded as follows. Let $\mathrm{Es}(n)$ be the probability that a SRW and an independent LEW in $B_n = B_E(0,n)$, started at neighbouring points $0 = (0,0)$ and $e_1 = (1,0)$ fail to intersect:

$$\mathrm{Es}(n) = \mathbb{P}\Big(\mathrm{SRW}(0, B_n^c) \cap \mathrm{LEW}(e_1, B_n^c) = \emptyset\Big).$$

Using the connection between LEW and SLE_2, and the known expression for the probability of non-intersection of Brownian motion and an SLE_2 path inside the unit disk, started in a ball of radius $r \in (0,1)$, Masson proved:

Theorem 3.6 *[Mas09]*

$$\mathrm{Es}(n) \approx n^{-3/4}. \tag{3.8}$$

The proof requires a number of estimates on SRW and LEW. Some of these hold for the LEW in more general spaces, but at this point I will just discuss $\mathbb{Z}^2$.

The first is a comparison of LEW in different domains. Let $B_E(0,n) \subset D \subset \mathbb{Z}^2$. Let X be SRW on $\mathbb{Z}^2$, $L^D = \mathfrak{L}(X[0,\tau_D])$, and let $\mu_{n,D}$ be the law of L^D run up to its first exit from $B_E(0,n)$.

Theorem 3.7 *(See [Mas09, Law99]). Let D_1, D_2 satisfy $B_E(0,kn) \subset D_1 \cap D_2$. Let γ be any path from 0 to $\partial B_E(0,n)$. Then*

$$\frac{\mu_{n,D_1}(\gamma)}{\mu_{n,D_2}(\gamma)} \le 1 + \frac{C}{\log k}. \tag{3.9}$$

Corollary 3.8 *For each n the laws $\mu_{n,B_E(0,m)}$ have a limit $\widehat{\mu}_n$ as $m \to \infty$. The laws $\widehat{\mu}_n$ are consistent, and define an infinite self avoiding path in $\mathbb{Z}^2$.*

We call the limit in the corollary above $\mathrm{LEW}(0,\infty)$.

Remark. This gives an alternative construction of the UST in $\mathbb{Z}^2$. Previously the first step in WA was to take $\mathcal{U}_1 = \mathrm{LEW}(0,z)$. We can take the limit as $|z| \to \infty$, and start instead with $\mathcal{U}_1 = \mathrm{LEW}(0,\infty)$.

The following result (which is not as easy as one would like) plays an essential role. Write $d_E(x,A) = \min\{|x-y| : y \in A\}$, and $\tau_r = \tau_{B_E(0,r)}$. To shorten our formulae we use the notation $A \not\cap B$ to mean that $A \cap B = \emptyset$.

Theorem 3.9 *('Separation Lemma', [Mas09]). Let X be a SRW, and L an independent* $\mathrm{LEW}(0,\infty)$. *Set $A_r = \{X[1,\tau_r] \not\cap L[1,\tau_r]\}$, and*

$$W_r = d_E(X_{\tau_r}, L[1,\tau_r]) \wedge d_E(L_{\tau_r}, X[1,\tau_r]).$$

Then there exists $c_i > 0$ such that

$$\mathbb{P}(W_r \geq c_1 r | A_r) \geq c_2.$$

Using this Masson obtained

Lemma 3.10 *[Mas09] The function* $\mathrm{Es}(n)$ *satisfies*

$$c_1 \mathrm{Es}(n) \leq \mathrm{Es}(m) \leq c_2 \mathrm{Es}(n), \quad n \leq m \leq 4n. \tag{3.10}$$

Using these results, Theorem 3.6 follows, using a comparison between SRW, LEW and their continuum limits, i.e. Brownian motion and SLE_2.

Given this control of $\mathrm{Es}(n)$, the next step is to obtain an exact expression for the probability that a point $z \in D$ is in the loop erased walk path. We adopt the convention that when we write $X[0,\tau_A]$ for a path, $\tau_A = \tau_A^X$ is a stopping time for X.

Since we also need a result for conditioned processes, we consider the more general situation of a finite graph $G = (V,E)$ with edge weights μ_{xy}. Let $D \subset V$ be a connected domain in G. We write X^z for the process X started at z, and $Y^{z,w}$ for X^z conditioned on the event $\{T_w < \tau_D\}$.

Proposition 3.11 *Let $x, y \in D$, and let $\sigma_w = \max\{k \le \tau_D : Y_k = w\}$. Then*

$$\mathbb{P}^x(y \in \mathfrak{L}(X[0,\tau_D])) = G_D(x,y)\mathbb{P}(\mathfrak{L}(Y^{x,y}[0,\sigma_y]) \not\cap X^y[1,\tau_D]) \tag{3.11}$$
$$= G_D(x,y)\mathbb{P}(\mathfrak{L}(Y^{y,x}[0,T_x]) \not\cap X^y[1,\tau_D]). \tag{3.12}$$

where X^y, $Y^{y,x}$ and $Y^{x,y}$ are independent versions of these processes.

To obtain the equality between (3.11) and (3.12) one uses Lemma 3.3 and the fact that the time reversal of $Y^{x,y}[0,\sigma_y]$ has the same law as $Y^{y,x}[0,\sigma_x]$).

Proposition 3.12 $L_2(n) = \mathbb{E}M_{B_E(0,n)} \le cn^2\mathrm{Es}(n)$.

Proof. Write $B = B_E(0,n)$. For $z \in B_E(0,n)$ set

$$q(z) = \mathbb{P}\big(\mathfrak{L}(Y^{z,0}[0,T_0]) \not\cap X^z[1,\tau_D]\big). \tag{3.13}$$

For $z \in B$ let $r_z = n - |z|$ be the distance from z to B^c. Set $B' = B(z, r_z/4)$, and write $X = X^{z,B}$. Then

$$q(z) \le \mathbb{P}(\mathfrak{L}(Y^{z,0}[0,\tau_{B'}]) \not\cap X[1,\tau_{B'}]).$$

Inside B' the conditioned process $Y^{z,0}$ has the same distribution, up to constants, as X, and therefore the same holds for $\mathfrak{L}(Y^{z,0})$ and $\mathfrak{L}(X)$. So if X' is a SRW with $X'_0 = z$, independent of X,

$$\mathbb{P}(\mathfrak{L}(Y^{z,0}[0,\tau_{B'}]) \not\cap X[1,\tau_{B'}]) \le c\mathbb{P}(\mathfrak{L}(X'[0,\tau_{B'}]) \not\cap X[1,\tau_{B'}]) = \mathrm{Es}(r_z/4).$$

Hence if X^0 is a SRW started at 0,

$$\mathbb{E}M_{B_E(0,n)} = \sum_{z \in B} \mathbb{P}(z \in \mathfrak{L}(X^{0,B}) \le \sum_{z \in B} G_B(0,z)\mathrm{Es}(r_z/4)$$
$$\asymp \sum_{r=1}^{n} r \log(n/r)\mathrm{Es}(r/4) \asymp n^2\mathrm{Es}(n).$$

The final estimate follows from [BM10, Corollary 3.14]. □

The lower bound is more delicate, since one has to bound $q(z)$ from below, and so has to handle the processes Y^z and $X^{z,D}$ outside the ball B'.

Proposition 3.13 $L_2(n) = \mathbb{E}M_{B_E(0,n)} \ge cn^2\mathrm{Es}(n)$.

Proof. Recall from (3.13) the definition of $q(z)$. We have

$$\begin{aligned}\mathbb{E}M_{B_E(0,n)} &= \sum_{z\in B} q(z)G_D(0,z) \\ &\ge c \sum_{n/4\le|z|\le 3n/4z} q(z) \ge n^2 \min_{n/4\le|z|\le 3n/4} q(z).\end{aligned}$$

Fix z with $|z|\in[n/4,3n/4]$ and let $r=r_z$; we have $r\ge n/16$. Then with A_r as in the separation lemma,

$$\begin{aligned}q(z) &= \mathbb{P}(\mathfrak{L}(Y^z)\not\pitchfork X^z[1,\tau_D]) \\ &\ge \mathbb{P}(\mathfrak{L}(Y^z)\not\pitchfork X^z[1,\tau_D]|A_r)P(A_r) \\ &\ge \mathbb{P}(\{\mathfrak{L}(Y^z)\not\pitchfork X^z[1,\tau_D]\}\cap\{W_r\ge rc_1\}|A_r)\mathrm{Es}(r) \\ &\ge c_2\mathbb{P}(\mathfrak{L}(Y^z)\not\pitchfork X^z[1,\tau_D]|\{W_r\ge rc_1\}\cap A_r)\mathrm{Es}(r).\end{aligned}$$

Standard Harnack type estimates then give

$$\mathbb{P}(L(Y^z)\not\pitchfork X^z[1,\tau_D]|\{W_r\ge rc_1\}\cap A_r)\ge c_3>0,$$

so $q(z)\ge \mathrm{Es}(n/16)$ and the lower bound on $L_2(n)$ then follows. □

In [Law13] Lawler obtained improved results on the probability that a LEW in $\mathbb{Z}^2$ uses a given edge. Let $D=[-(n-1),n]\times[-(n-1),n-1]$, and $e=\{(0,0),(1,0)\}$. Define

$$\begin{aligned}D_L &= \{(-(n-1),j), -(n-1)\le j\le n-1\}, \\ D_R &= \{(n+1,j), -(n-1)\le j\le n-1\}.\end{aligned}$$

Let π be uniform measure on D_L. Let $\widetilde{\mathbb{P}}$ be the law of a SRW X started with measure π and conditioned on the event that X exits D at a point in D_R. We write $\widetilde{X}$ for this conditioned process. Then Lawler proves

Theorem 3.14 *([Law13].) Let $\widetilde{L}=\mathfrak{L}(\widetilde{X}_0,\widetilde{X}_1,\dots,\widetilde{X}_{T_{D_R}})$. Then*

$$\widetilde{\mathbb{P}}(\widetilde{L} \text{ uses the edge } e)\asymp n^{-3/4}. \tag{3.14}$$

Corollary 3.15 *We have*

$$\mathrm{Es}(n)\asymp n^{-3/4}, \tag{3.15}$$

and hence $L_2(n)\asymp n^{5/4}$.

Proof. Let $A_1 = [-n/2, n/2]^2$, $A_2 = [-3n/4, 3n/4]^2$. Then for each $x \in D_L$ we have $\mathbb{P}^x(\widetilde{X} \text{ hits } A_1) \asymp 1$. Write $T_1 = \min\{k \geq 0 : \widetilde{X}_k \in A_1\}$, $\sigma_2 = \sup\{k \leq \tau_D : \widetilde{X}_k \in A_2\}$. If we write $\widetilde{X}^{(i)}$ the rotation of $\widetilde{X}$ by $i\pi/4$ for $i = 0, 1, 2, 3$, then the laws of the path $\widetilde{X}^{(i)}[T_1, \sigma_2]$ are all absolutely continuous. It follows from this and (3.14) that we also have

$$\mathbb{P}(0 \in \widetilde{L}) \asymp n^{-3/4}. \tag{3.16}$$

Applying Proposition 3.11 to the conditioned random walk $\widetilde{X}$, and using the notation of that Proposition, we obtain from (3.16) that

$$\sum_{x \in D_L} \pi_x \widetilde{G}_D(x, 0) \mathbb{P}(\mathfrak{L}(\widetilde{Y}^{0,x}[0, T_x]) \not\cap \widetilde{X}^0[1, \tau_D]) \asymp n^{-3/4}. \tag{3.17}$$

For each $x \in D_L$ we have $\widetilde{G}_D(x, 0) \asymp 1$, and since the laws of X, $\widetilde{X}$ and $Y^{0,x}$ are comparable inside $[-n/2, n/2]^2$, we have

$$\mathbb{P}(\mathfrak{L}(\widetilde{Y}^{0,x}[0, T_x]) \not\cap \widetilde{X}^0[1, \tau_D]) \asymp \mathrm{Es}(n/2).$$

So the right side of (3.17) is comparable to $\mathrm{Es}(n)$, and thus $\mathrm{Es}(n) \asymp n^{-3/4}$. □

We will (mainly) continue to use the notation $L_2(n)$ rather than $n^{5/4}$ in what follows, since it will make the structure of many expressions clearer.

Theorem 3.16 *([BM10].) Let $D \subset \mathbb{Z}^2$ be simply connected and suppose each point $z \in D$ is within n of ∂D.*
(a) Let $z_0, z_1, \dots, z_k \in D$, and write $r_i = |z_i - z_{i-1}| \wedge |z_i - z_{i+1}| \wedge r_{z_i}$, where $r_z = dist(z, \partial D)$. Then

$$\mathbb{P}^{z_0}(z_1, \dots, z_k \in \mathrm{LEW}(0, D^c) \textit{ in order }) \leq C^k \prod_{i=1}^{k} G_D(z_{i-1}, z_i) \mathrm{Es}(r_i).$$

(b) For $k \geq 0$, $\mathbb{E}^0(M_D)^k \leq C^k k! L_2(n)^k$.
(c) For $\lambda \geq 1$, $\mathbb{P}(M_D \geq \lambda L_2(n)) \leq 2\exp(-\lambda/2C)$.

Proofs. Given (b), it is easy to prove (c):

$$\mathbb{P}(M_D \geq \lambda L_2(n)) = \mathbb{P}(e^{M_D/2CL_2(n)} \geq e^{\lambda/2C}) \leq \frac{\mathbb{E}(e^{M_D/2CL_2(n)})}{e^{\lambda/2C}}.$$

Then

$$\mathbb{E}(e^{M_D/2CL_2(n)}) \le \sum_{k=0}^{\infty} E\Big(\frac{M_D^k}{k!(2CL_2(n))^k}\Big) \le \sum_{k=0}^{\infty} 2^{-k} = 2.$$

Proving (b) from (a) is a bit more complicated, but is straightforward. Given (a), we have writing $L_D = \mathrm{LEW}(0, D)$,

$$\begin{aligned}
\mathbb{E}M_D^k = E\Big(\sum_z 1_{(z\in L_D)}\Big)^k &= \mathbb{E}\sum_{z_1}\cdots\sum_{z_k} 1_{(z_1,z_2,\dots,z_k\in L_D)} \\
&= \sum_{z_1}\cdots\sum_{z_k} \mathbb{P}(z_1, z_2, \dots, z_k \in L_D) \\
&= k!\sum_{z_1}\cdots\sum_{z_k} \mathbb{P}(z_1, z_2, \dots, z_k \in L_D \text{ in order}) \\
&\le k!\sum_{z_1}\cdots\sum_{z_k} C^k \prod_{i=1}^{k} G_D(z_{i-1}, z_i)\mathrm{Es}(r_i)
\end{aligned}$$

This sum can be handled inductively:

$$\begin{aligned}
\frac{\mathbb{E}M_D^k}{C^k k!} &\le \sum_{z_1}\cdots\sum_{z_{k-1}}\prod_{i=1}^{k-1} G_D(z_{i-1}, z_i)\mathrm{Es}(r_i)\sum_{z_k} G_D(z_{k-1}, z_k)\mathrm{Es}(r_k) \\
&\le \sum_{z_1}\cdots\sum_{z_{k-1}}\prod_{i=1}^{k-1} G_D(z_{i-1}, z_i)\mathrm{Es}(r_i)C' L_2(n),
\end{aligned}$$

and continuing one gets (b).

The hard work is in proving (a), and this uses a generalisation of Proposition 3.11 to k points.

Definition 3.17 *Suppose that $z_0, z_1, \dots, z_k$ are any distinct points in a domain $D \subset \mathbb{Z}^2$, and X is a Markov chain on $\mathbb{Z}^2$ with $\mathbb{P}^{z_0}(\sigma_D^X < \infty) = 1$. Let $V_{z_0,\dots,z_k}$ be the event that $X_0 = z_0$ and $z_1, z_2, \dots, z_k$ are all visited by the path $LX[0, \sigma_D]$ in order.*

Proposition 3.18 *Suppose that $z_0, z_1, \dots, z_k$ are distinct points in a domain $D \subset \mathbb{Z}^2$, and X is a Markov chain on $\mathbb{Z}^2$ with $\mathbb{P}^{z_0}(\sigma_D^X < \infty) = 1$. Define z_{k+1} to be ∂D and for $i = 0, \dots, k$, let X^i be independent versions of X started at z_i and Y^i be X^i conditioned on the event $\{T_{z_{i+1}} \le \tau_D\}$. Let $\sigma_i = \max\{k \le$*

$\tau_D^{Y^i} : Y_k^i = z_{i+1}\}$, *and*

$$F_i = \big\{\mathfrak{L}(Y^{i-1}[0,\sigma_{i-1}]) \not\cap \bigcup_{j=i}^{k} Y^j[1,\sigma_j]\big\},\ i = 1,\dots,k.$$

Then,

$$\mathbb{P}^{z_0}(V_{z_0,\dots,z_k}) = \mathbb{P}\Big(\bigcap_{i=1}^{k} F_i\Big)\prod_{i=1}^{k} G_D^X(z_{i-1},z_i).$$

To see why this is true, look at the case $k = 2$:

$$\mathbb{P}^{z_0}(z_1 \text{ and then } z_2 \in \mathrm{LEW}(0,D^c)) \le cG_D(z_0,z_1)\mathrm{Es}(r_1)G_D(z_1,z_2)\mathrm{Es}(r_2).$$

For this to happen we need (at least):
(1) X hits z_1,
(2) After hitting z_1 the process X avoids the previous path, i.e. $\mathrm{LEW}(0,z_1)$,
(3) X then hits z_2,
(4) After hitting z_2 the process X avoids its previous path up to its exit from $B(z_2,r_2)$.

The Green's function terms relate to the hitting probabilities, weighted by the fact that one can have more than one attempt to include the points z_i. The $\mathrm{Es}(r_i)$ terms then bound from above the collision probabilities.

We also have a lower bound:

Theorem 3.19 *([BM10].) Let $B_E(0,m) \subset D \subset \mathbb{Z}^2$. Then for each $\varepsilon > 0$*

$$\mathbb{P}(M_D \le \lambda^{-1}L_2(m)) \le c(\varepsilon)\exp(-c'(\varepsilon)\lambda^{4/5-\varepsilon}). \tag{3.18}$$

It is clear that one cannot obtain this from moment bounds, but will need to use some independence.

Here is an outline of the argument. It is easier to use boxes than balls so we will write $B_\infty(x,r)$ for the square in $\mathbb{Z}^2$ with centre x and side $2r$ – i.e. balls in the ℓ_∞ norm in $\mathbb{Z}^2$.

1. First, one can restrict to proving the result for $B_n = [-n,n]^2$. More precisely, we use Theorem 3.7 to compare $\mathrm{LEW}(0,D^c)$ with $\mathrm{LEW}(0,B_n^c)$ inside $B_\infty(0,n/4)$.

2. Let $k \geq 1$, to be chosen later, and set $m = n/4k$. Define squares $Q_j = B_\infty(0, jm)$, $j = 1, \dots, k$. Let $L = \mathrm{LEW}(0, B_n^c)$, let

$$T_j = \min\{i : L_i \in \partial Q_j\},$$

and write $x_j = L_{T_j}$. Let $e_1 = (1,0)$, $e_2 = (0,1)$ and $D_j = \{x_j \pm e_i, i = 1,2\}$ be the set of four points in $\mathbb{Z}^2$ which are either a horizontal or vertical distance $m/2$ from x_j. Let x'_j be a point in D_j chosen so that $A_j = B_\infty(x'_j, m/4)$ lies outside Q_j, and where we use some procedure to choose x'_j if there exist two such points. Let $Q_j^* = B_\infty(x_j, m)$. Let $\mathcal{F}_j = \sigma(L_i, i \leq T_j)$, and α_j be the path L between T_j and T'_j, where

$$T'_j = \min\{i \geq T_j : L_i \notin Q_j^*\}.$$

Let ξ_j be the number of hits by α_j on A_j.

Lemma 3.20 *For $z \in A_j$,*

$$\mathbb{P}(z \in \alpha_j | \mathcal{F}_j) \geq c(\log k)^{-3} \mathrm{Es}(m). \tag{3.19}$$

For Lemma 3.20, if one had the unconditioned random walk, then by Propositions 3.12 and 3.13 one would have $\mathbb{P}(z \in \alpha_j) \asymp G_{B_\infty(x_j,m)}(x_j, z)\mathrm{Es}(r_z) \asymp \mathrm{Es}(m)$. Let $\gamma_j = L[0, T_j]$, and condition on a fixed path γ_j. Let Y be SRW in B_n conditioned on $\{\tau_{B_n} < T^+_{\gamma_j}\}$. Then to prove (3.19) we need to calculate with the process Y rather than the SRW X. One needs the following rather delicate estimate, which is a kind of boundary Harnack principle which is uniform in the set γ_j.

Proposition 3.21 *([Mas09, Proposition 3.5].) Let γ_j be any path in Q_j ending at x_j, and Y be as above. Write $\tau = \tau_{B_\infty(x_j, m/4)}(Y)$. Then*

$$\mathbb{P}(Y_\tau \in A_j) \geq p_0 > 0. \tag{3.20}$$

Using Lemma 3.20 and similar estimates, one obtains by summing over $z \in A_j$,

Lemma 3.22

$$c_1(\log k)^{-3} L_2(n) \leq \mathbb{E}(\xi_j | \mathcal{F}_j) \leq c_2(\log k) L_2(n), \tag{3.21}$$

$$\mathbb{E}(\xi_j^2 | \mathcal{F}_j) \leq c_2(\log k)^2 L_2(n)^2. \tag{3.22}$$

Hence

$$\mathbb{P}\big(\xi_j < L_2(m)/(\log k)^3\big) \le 1 - \frac{c}{(\log k)^8}. \tag{3.23}$$

The lower tail estimate (3.23) follows from (3.21) and (3.22) by the second moment method. We remark that the bounds (3.21) and (3.22) also hold for the unconditioned walk.

Proof of Theorem 3.19. Since $M_{B_n} \ge \sum_j \xi_j$, we have

$$\begin{aligned}\mathbb{P}(M_{B_n} < b^{-3}L_2(m)) &\le \mathbb{P}(\xi_j < (\log k)^{-3}L_2(m) \text{ for } j = 1, \dots, k)\\ &\le (1 - c(\log k)^{-8})^k \le c\exp(-c'k(\log k)^{-8}).\end{aligned}$$

We have $L_2(m) = L_2(n/k) \asymp k^{-5/4}L_2(n)$, so if $\lambda^{-1} = L_2(m)/L_2(n)(\log k)^3$ then $k(\log k)^{-8} \le \lambda^{4/5-\varepsilon}$, and we obtain (3.18). □

4 Geometry of the UST in two dimensions

Using these bounds, we can compare the sizes of balls in the Euclidean and $d_{\mathcal{U}}$ metrics. Recall that $\mathbb{P}$ is the law of the UST, B_E denotes Euclidean balls, and $B_{\mathcal{U}}$ the (random) balls in the intrinsic $d_{\mathcal{U}}$ metric. Earlier we guessed that $B_E(0,r) \simeq B_{\mathcal{U}}(0, r^{5/4})$.

Theorem 4.1 *For* $\lambda \ge 1$

$$\mathbb{P}\big(B_{\mathcal{U}}(0, \lambda^{-1}L_2(r)) \not\subset B_E(0,r)\big) \le Ce^{-\lambda^{2/3}} \tag{4.1}$$

$$\mathbb{P}\big(B_E(0,r) \not\subset B_{\mathcal{U}}(0, \lambda L_2(r))\big) \le C(\varepsilon)\lambda^{-4/15-\varepsilon}. \tag{4.2}$$

Remark. It is not hard to see that one cannot expect to do better than the polynomial bound in (4.2). Consider the first stage of the construction of $\mathcal{U}$, by making an infinite LEW started from 0. This walk has probability at least $k^{-\delta}$ of returning inside $B_E(0,r)$ after leaving $B_E(0,kr)$ – see [BM11, Remark 2.5]. If this event occurs, then one would expect $B_E(0,r)$ to contain points z with $d_{\mathcal{U}}(0,z) \simeq (kr)^{5/4}$.

Outline of proof of (a). Our estimates on LEW from Theorem 3.16 give that

$$\mathbb{P}\big(d_{\mathcal{U}}(0,w) < \lambda^{-1}L_2(|w|)\big) \le ce^{-c\lambda^{3/4}}.$$

Consider first the following crude argument. Set $R = \lambda^{-1} L_2(r)$. Suppose there exists $x \in B_{\mathcal{U}}(0,R)$ with $|x| > r$. Then following the unique geodesic in $\mathcal{U}$ from x to 0, there exists $y \in \partial B_E(0,r)$ with $d_{\mathcal{U}}(0,y) \le d_{\mathcal{U}}(0,x) \le R$. So

$$\begin{aligned} \mathbb{P}\big(B_{\mathcal{U}}(0,R) \not\subset B_E(0,r)\big) &\le \sum_{y \in \partial B_E(0,r)} \mathbb{P}(d_{\mathcal{U}}(0,y) \le \lambda^{-1} L_2(r)) \\ &\le cr \exp(-c\lambda^{3/4}) = c \exp(-c\lambda^{3/4} + \log r). \end{aligned}$$

So we obtain a good bound only when $\lambda > c(\log r)^{4/3}$, while we want a bound which holds for fixed λ and for all r.

To improve this estimate, we use the same idea – that the probability that $d_{\mathcal{U}}(0,x) \ll L_2(|x|)$ is small, but use a more careful construction.

Step 1. Choose small but fixed $\delta > 0$ and cover $B_E(0,r) - B_E(0,r/2)$ by a set D_1 with roughly δ^{-2} points, with separation $r\delta$.

Step 2. Start the Wilson construction of $\mathcal{U}$ by running LEW from the points in D_1 to 0. Call this tree $\mathcal{V}_1$. With high probability the parts of these paths inside $B(0,r/4)$ will all have length greater than $\lambda^{-1} L_2(r/4)$.

Step 3. Fill in the remainder of the UST. With high probability the LEW started at any point $y \in B_E(0,3r/4)^c$ will hit $\mathcal{V}_1$ before it reaches $B_E(0,r/4)$, and therefore

$$d_{\mathcal{U}}(y,0) \ge \min_{x \in \mathcal{V}_1 \cap \partial B_E(0,r/4)} d_{\mathcal{U}}(x,0) \ge \lambda^{-1} L_2(r/4).$$

(Actually, again to avoid $\log r$ error terms, one needs to choose a sequence of finite sets D_k with separation $2^{-k}\delta r$ and build trees $\mathcal{V}_k$.) □

While usually $B_{\mathcal{U}}(x,r)$ will contain a small Euclidean ball, there have to be neighbouring points in $\mathbb{Z}^2$ which are distant in $\mathcal{U}$.

Lemma 4.2 *([BLPS].) The box $[-n,n]^2$ contains with probability 1 neighbouring points x, y in $\mathbb{Z}^2$ with $d_{\mathcal{U}}(x,y) \ge n$.*

Proof. Look at the path (in $\mathbb{Z}^2$) of length $4n$ around the box $[-n,n]^2$: call this $z_0, z_1, \dots, z_{4n}$. If each pair z_j, z_{j+1} were connected by a path in $\mathcal{U}$ of length less than n then this path would not contain 0. Hence we would obtain a loop around 0 – which is impossible since $\mathcal{U}$ is a tree. □

Volume Bounds

Let $\ell_2(R)$ be the inverse of $L_2(r)$, so that

$$\ell_2(R) \asymp R^{4/5}.$$

Theorem 4.3 *For $\lambda \geq 1$,*

$$\mathbb{P}(|B_{\mathcal{U}}(0,R)| \geq \lambda \ell_2(R)^2) \leq c\exp(-c\lambda^{1/3}) \tag{4.3}$$

$$\mathbb{P}(|B_{\mathcal{U}}(0,R)| \leq \lambda^{-1} \ell_2(R)^2) \leq c\exp(-c\lambda^{1/9}). \tag{4.4}$$

Outline of proof. The upper bound (4.3) follows from Theorem 4.1, which gives that with high probability $B_{\mathcal{U}}(0,R) \subset B_E(0, \lambda^{1/2}\ell_2(R))$.

For the lower bound (4.4) one shows it is very likely that the first few LEW paths are of about the right length, and nearly enclose some small ball $B_E(z, \varepsilon^2 r)$, where $r = \ell_2(R)$. More precisely we start with a list of k candidate balls $B_E(z_i, \varepsilon r)$ on the part of the first path $L = \mathrm{LEW}(0,\infty)$ until it hits $\partial B_E(0,r)$.

1. If L returns to $B_E(0,r)$ after its first hit on $\partial B_E(0,2r)$ we remove any ball B_i hit by L on its return. (One can show that the probability that more than $k/2$ balls are removed is less than $c\exp(-ck^{1/3})$.)

2. We then take a ball remaining on the list, and run Wilson's algorithm in $B_E(z_i, \varepsilon^2 r)$. 'Success' means all paths are inside $B_E(z_i, \varepsilon r)$ and are not too long. We can show $\mathbb{P}(\text{'success'}) \geq \frac{1}{4}$. If a path escapes it may 'compromise' other balls by entering them – but it is very unlikely to compromise more than $k^{1/4}$ balls. Any compromised ball is removed from the list.

3. If we do not succeed we try again, until we run out of balls.

If a 'success' occurs then we have obtained a Euclidean ball $B_E(z_i, \varepsilon^2 r)$ which with high probability is contained in $B_{\mathcal{U}}(0, \lambda L_2(r))$, so that

$$|B_{\mathcal{U}}(0, \lambda L_2(r))| \geq \varepsilon^4 r^2.$$

Effective Resistance. Recall that $R_{\mathrm{eff}}(0, B_{\mathcal{U}}(0,R)^c)$ is the effective resistance between 0 and the boundary of $B_{\mathcal{U}}(0,R)$. Since $\mathcal{U}$ contains a path length R from 0 to the boundary of the ball, we have $R_{\mathrm{eff}}(0, B_{\mathcal{U}}(0,R)^c) \leq R+1$; we want a lower bound of the same order.

We can do this by counting 'cut sets'. For $0 < s < R$ let N_s be the number of points z with $d_{\mathcal{U}}(0,z) = s$ such that z is connected to $B_{\mathcal{U}}(0,R)^c$ by a path which, except for its first step, lies outside $B_{\mathcal{U}}(0,s)$. Since $\mathcal{U}_2$ is a tree, we have that N_s is increasing in s. So

$$R_{\rm eff}(0, B_{\mathcal{U}}(0,R)^c) \ge \sum_{s=1}^{R/2} \frac{1}{N_s} \ge \frac{\frac{1}{2}R}{N_{R/2}}. \tag{4.5}$$

Thus if $N_R \le c_1$ we obtain $R_{\rm eff}(0, B_{\mathcal{U}}(0,R)^c) \ge R/(2c_1)$.

If the construction of $\mathcal{U}$ in the proof of Theorem 4.1 works, then any path γ in $\mathcal{U}$ from 0 to $B_E(0,r)^c$ will lie in the tree $\mathcal{V}_1$ when it is inside $B_E(0,r/8)$. This tree has at most δ^{-2} paths, so we obtain

$$R_{\rm eff}(0, B_E(0,r)^c) \ge c\delta^{-2} L_2(r/4).$$

It then follows (set $R = L_2(r)$) that

$$R_{\rm eff}(0, B_{\mathcal{U}}(0,R)^c) \ge c'R.$$

Theorem 4.4 *([BM11].) For $\lambda \ge 1$,*

$$\mathbb{P}(R_{\rm eff}(0, B_{\mathcal{U}}(0,r)^c) < \lambda^{-1} r) \le c \exp(-c\lambda^{2/11}).$$

5 Random walks on $\mathcal{U}_2$.

Definition. A graph $G = (V,E)$ satisfies the condition $V(\alpha)$ if

$$|B(x,r)| \asymp r^\alpha, \ \text{ for all } x \in V, r \ge 1.$$

G satisfies RES(β) if:

$$R_{\rm eff}(x,y) \asymp \frac{d(x,y)^\beta}{V(x,d(x,y))} \quad \Big(\asymp d(x,y)^{\beta-\alpha} \quad \text{if } V(\alpha) \text{ holds}\Big).$$

The case of interest is when $\alpha < \beta$, and so $R_{\rm eff}(x,y) \to \infty$ as $d(x,y) \to \infty$.

Example. For $\mathbb{Z}^1$ one has $R_{\rm eff}(x,y) = |y-x| = |y-x|^{2-1}$ so RES(2) holds. The condition does not hold for $\mathbb{Z}^d$ for any $d \ge 2$, since for $x \ne y$,

$$R_{\rm eff}(x,y) \asymp \log|y-x|, \ \text{ if } d = 2,$$
$$0 < c_0 \le R_{\rm eff}(x,y) \le c_1 \text{ for all } x \ne y \text{ if } d \ge 3.$$

(For a transient graph the resistance from a point 'to infinity' is finite.)

Given a graph G write $p_n(x,y)$ for the (discrete time) heat kernel on G:

$$p_n(x,y) = \frac{P^x(X_n = y)}{\mu_y}.$$

Define the *spectral dimension* of G by

$$d_s(G) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log(2n)}, \text{ if this limit exists.} \tag{5.1}$$

Theorem 5.1 *[BCK05]. Let G be a locally finite infinite connected graph, and $\beta > \alpha \geq 1$. Suppose that G satisfies $V(\alpha)$ and $RES(\beta)$. Then*

$$R_{\text{eff}}(x, B_d(x,R)^c) \asymp R^{\beta-\alpha}$$

and

$$E^x \tau(x,R) \asymp r^\beta, \quad x \in V, r \geq 1, \tag{5.2}$$

$$E^x d(x, X_n) \asymp n^{1/\beta}, \tag{5.3}$$

$$p_{2n}(x,x) \asymp n^{-\alpha/\beta}. \tag{5.4}$$

In particular one has

$$d_s(G) = \frac{2\alpha}{\beta}.$$

In fact one also obtains, under the same hypotheses, sub-Gaussian heat kernel bounds for $p_n(x,y)$.

Estimates from one point. Theorem 5.1 assumes we have control of $B_d(x,r)$ for all $x \in V$, $r \geq 1$. Suppose instead that we just have estimates for the balls $B_d(o,r)$, where $o \in V$ is some marked point.

Theorem 5.2 *(Implicit in [BJKS, KM08].) Let $o \in V$, $\alpha > 0$. Suppose:*

$$V(o,r) \asymp r^\alpha,$$
$$R_{\text{eff}}(o,x) \asymp d(0,x).$$

Then writing $\beta = 1+\alpha$,

$$p_{2n}(o,o) \asymp n^{-\alpha/\beta}, \qquad E^o \tau(o,r) \asymp r^\beta,$$

and

$$d_s(G) = \frac{2\alpha}{\beta}.$$

This is useful for random graphs, where control of balls $B_d(o,r)$ may be possible, while control of all balls $B_d(x,r)$ could be hopeless.

Application to random graphs. Let $(\Omega, \mathbb{P})$ be a probability space, carrying a random graphs $\mathcal{G}(\omega)$, $\omega \in \Omega$. Assume each $\mathcal{G}(\omega)$ has a marked point o. Write P^x_ω for the law of the SRW X on $\mathcal{G}(\omega)$ with $X_0 = x \in \mathcal{G}(\omega)$.

Definition. Let $\alpha \geq 1$. For $\lambda \geq 1$ say that $B_d(o,r) \in \mathcal{G}(\omega)$ is λ-good if:

$$\lambda^{-1} r^\alpha \leq |B_d(o,r)| \leq \lambda r^\alpha, \tag{5.5}$$

$$r/\lambda \leq R_{\text{eff}}(o, B_d(o,r)^c) \leq r+1. \tag{5.6}$$

(Note that the right hand inequality in (5.6) always holds.)

We say the family $\mathcal{G}(\omega)$ satisfies *Condition A* if there exists $c_1 \geq 1$ and $\theta > 0$ such that

$$\mathbb{P}(\ B_d(o,r) \text{ is } \lambda\text{-good }) \geq 1 - \lambda^{-\theta}, \qquad \text{for all } r \geq c_1. \tag{A}$$

λ-good means that (within a factor of λ) the volume and resistance properties of $B_d(o,r)$ are what we want them to be.

Theorem 5.3 *([BJKS, KM08].) Suppose $\mathcal{G}(\omega)$ satisfies Condition A. Then there exists $\gamma > 0$ such that for all large t and r, writing $\beta = 1 + \alpha$,*

$$(\log r)^{-\gamma} r^\beta \leq E^o_\omega \tau(o,r) \leq (\log r)^\gamma r^\beta,$$

$$(\log t)^{-\gamma} n^{-\alpha/\beta} \leq P_{2n}(o,o)(\omega) \leq (\log n)^\gamma n^{-\alpha/\beta}.$$

In particular, $\mathbb{P}$–a.s.,

$$d_s(G) = \frac{2\alpha}{\beta} = \frac{2\alpha}{1+\alpha},$$

$$d_w(G) = \lim_{R \to \infty} \frac{E^o_\omega \tau(o,R)}{\log R} = 1 + \alpha.$$

Several families of random graphs are now known to satisfy Condition (A):

(1) The incipient infinite cluster for spread out oriented percolation in $\mathbb{Z}_+ \times \mathbb{Z}^d$ with $d \geq 6$. [BJKS].

(2) Invasion percolation on a regular tree. [AGHS].

(3) The incipient infinite cluster for percolation on $\mathbb{Z}^d$ with $d \geq 19$ [KN].

(4) Critical finite variance Galton-Watson trees conditioned to survive forever [FK].

In all these cases the trees are close to a critical GW tree conditioned on non-extinction, and $\alpha = 2$ and $d_s = 4/3$. In addition we have the following examples with $\alpha \neq 2$.

(5) θ stable Galton-Watson trees conditioned to survive forever [CK]. Here $\alpha = \theta/(\theta - 1)$, and

$$d_s = \frac{2\alpha}{1+\alpha} = \frac{2\theta}{2\theta - 1}.$$

(6) The uniform spanning tree in $d = 2$. Here, as we have seen from the results above, $\alpha = 8/5$, $\beta = 1 + \alpha = 13/5$ and so $d_s = 2\alpha/\beta = 16/13$.

Remark. Theorem 5.3 was particularly useful for studying the IIC for oriented percolation, since estimates for ball volumes were only available from the base point $(0, 0)$. However, the UST is translation invariant, and so one can obtain better estimates of the off-diagonal terms $p_n(x, y)$ – see [BM11] for details.

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