Gaussian bounds and parabolic Harnack inequality on locally irregular graphs

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Abstract

A well known theorem of Delmotte is that Gaussian bounds, parabolic Harnack inequality, and the combination of volume doubling and Poincaré inequality are equivalent for graphs. In this paper we consider graphs for which these conditions hold, but only for sufficiently large balls, and prove a similar equivalence.

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1 Introduction

Let $\Gamma = (\mathbb{V}, \mathbb{E})$ be a connected, locally finite graph without double edges or loops. (We allow $\mathbb{V}$ to be either finite or infinite.) Let μ be a weight function on $\mathbb{E}$, such that $\mu_{xy} = \mu_{yx} > 0$ for each $(x, y) \in \mathbb{E}$, while $\mu_{xy} = 0$ for each $(x, y) \notin \mathbb{E}$. Set μ_x to be the weight of a vertex x, i.e.,

$$\mu_x = \sum_{y \in \mathbb{V}} \mu_{xy}.$$

We consider the continuous time random walk $X = \{X_t : t \geq 0\}$ on Γ with generator

$$\mathscr{L}_\mu f(x) = \frac{1}{\mu_x} \sum_{y \in \mathbb{V}} (f(y) - f(x)) \mu_{xy};$$

this is sometimes called the *constant speed random walk* or CSRW on $\mathbb{V}$. Write $\mathbb{P}_x$ for the law of X starting at x. We are interested in the transition density of X, or heat kernel on Γ, given by

$$p_t(x, y) = \frac{\mathbb{P}_x(X_t = y)}{\mu_y}.$$

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For any two vertices x and y denote by $d(x,y)$ the graph distance between x and y. Write $B(x,r) := \{y \in \mathbb{V} : d(y,x) \le r\}$ for balls in Γ. We extend μ to a measure on $\mathbb{V}$ and set

$$V(x,r) = \mu(B(x,r)).$$

In [10] Delmotte, extending the earlier results of [13, 19] on manifolds, proved that the following three properties of Γ are equivalent:

(1) Gaussian bounds on the heat kernel,
(2) A parabolic Harnack inequality,
(3) Volume doubling and a family of Poincaré inequalities (PI).

(See below for the precise definitions.) The hardest implication is that (3) implies (1) and (2). This is also the most useful one, since the conditions on volume growth and Poincaré inequalities can often be verified easily given geometric data on the graph. In addition, it is straightforward to verify that the two conditions in (3) have certain stability properties (for example under rough isometries), while this stability is not evident for the properties (1) or (2).

The ideas in [10], and more generally the methods of Moser and Nash on which [10] is based, have proved very fruitful in the study of random walks in symmetric random environments. For example [3] used Nash's ideas to obtain Gaussian bounds on the heat kernel on supercritical percolation clusters in $\mathbb{Z}^d$, while [1, 2] use Moser's iteration argument to obtain a quenched invariance principle and Harnack inequalities for the random conductance model under quite general conditions.

A general feature of these random environments is that the various kinds of regularity required for heat kernel bounds or Harnack inequalities only hold for sufficiently large balls. In [3], where instead of volume doubling the stronger condition that $V(x,r) \asymp r^d$ is considered, it is shown that, for suitable (non random) constants c_1, c_2, one has with probability one for each $x \in \mathbb{V}(\omega)$ (the unique infinite percolation cluster in the random configuration given by ω) that there exists a (random) $R_x = R_x(\omega) < \infty$ such that

$$c_1 r^d \le V(x,r)(\omega) \le c_2 r^d \text{ for all } r \ge R_x(\omega). \tag{1.1}$$

Thus while the inequality in (1.1) will fail for many small balls, for any x it will hold for all large enough balls with centre x.

In this paper we extend the results of [10] to deterministic graphs such that, for all large enough balls, volume doubling and Poincaré inequality hold. (We do require some additional local regularity.) In the course of our work, we also obtain more precise sufficient conditions on the range of balls for which 'good behaviour' is required in order to obtain heat kernel bounds and Harnack inequalities.

Definition 1.1 Let θ, C_P and λ_P be constants with $C_P > 0$, $\theta \ge 1$ and $\lambda_P \ge 1$. Let $o \in \mathbb{V}$, $h \ge 1$. We say $B(o,h)$ is *good* if $\mathbb{V} - B(o,2h) \ne \emptyset$,

$$V(o,2h) \le 2^\theta V(o,h), \tag{1.2}$$

and for all $f : B(o,\lambda_P h) \to \mathbb{R}$ one has the Poincaré inequality (PI)

$$\min_{a\in\mathbb{R}} \sum_{y\in B(o,h)} (f(y)-a)^2\mu_y \le C_P h^2 \sum_{y,z\in B(o,\lambda_P h)} (f(y)-f(z))^2\mu_{yz}. \tag{1.3}$$

Let $\alpha \in [1,h]$. We say that $B(o,h)$ satisfies the condition $\mathcal{G}_0(\alpha)$ if $B(x,r)$ is good for all $x \in B(o,h)$ and $r \in [\alpha,h]$.

If $B(o,h)$ satisfies $\mathcal{G}_0(\alpha)$ then volume doubling, given by (1.2), and the PI, given by (1.3), hold for balls with centers in $B(o,h)$ over a range of sizes. We call these conditions *restricted volume doubling* (rVD) and *restricted Poincaré inequality* (rPI). Note that in (1.3) the ball on the right side is λ_P times the radius of the one on the left; this is sometimes called a *weak Poincaré inequality*. However, as in [21] we will call this a *Poincaré inequality*, and will use the term *strong PI* for the case $\lambda_P = 1$.

While $\mathcal{G}_0(\alpha)$ controls $V(o,2r)/V(o,r)$ for $\alpha \le r \le h$, to obtain pointwise bounds on the heat kernel we need some additional local regularity of the graph, which controls volume growth for small r. (See Remark 7.6.) We say that $B(o,h)$ satisfies $\mathcal{G}_1(\alpha)$ if it satisfies $\mathcal{G}_0(\alpha)$ and in addition there exists C_0 such that

$$V(x,\alpha) \le C_0\mu_x\alpha^\theta \quad \text{for all } x \in B(o,h). \tag{1.4}$$

Our first theorem gives upper bounds on the heat kernel.

Theorem 1.2 *Let $\delta \in (0,1]$, $1 \le a \le b$ and $x \in \mathbb{V}$. If $B(x,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ for each $h \in [a,b]$, then there exist $c_1 = c_1(\theta, C_P, \lambda_P, C_0, \delta)$ and a universal constant c_2 such that for any $y \in B(x,b/2)$,*

$$p_t(x,y) \le \frac{c_1}{V(x,\sqrt{t})} \exp\left\{-c_2\frac{d(x,y)^2}{t}\right\}, \text{ for } t \in [a^2,b^2] \text{ with } t \ge d(x,y). \tag{1.5}$$

Note that we cannot expect the upper bound (1.5) to hold when $t \ll d(x,y)$. As usual, we need a stronger condition for the heat kernel lower bounds.

Theorem 1.3 *Let $\delta \in (0,1]$ and $o \in \mathbb{V}$. There exist $c_i = c_i(\theta, C_P, \lambda_P, C_0, \delta)$ such that if $1 \le a \le b$, and $(2\lambda_P)^{1/\delta} \le b$, and if $B(z,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ for each $z \in B(o,b)$ and $h \in [a,b]$, then for any $x,y \in B(o,b/8)$,*

$$p_t(x,y) \le \frac{c_1}{V(x,\sqrt{t})} \exp\big(-c_2\frac{d(x,y)^2}{t}\big), \text{ for } t \in [a^2,b^2] \text{ with } t \ge d(x,y), \tag{1.6}$$

$$p_t(x,y) \ge \frac{c_3}{V(x,\sqrt{t})} \exp\big(-c_4\frac{d(x,y)^2}{t}\big), \text{ for } t \in [a^2,b^2] \text{ with } t \ge a d(x,y). \tag{1.7}$$

We can reformulate the results above as follows.

Theorem 1.4 *Let $\delta \in (0,1]$, $x, y \in \mathbb{V}$, $r = d(x,y)$ and $t \ge r \vee (2\lambda_P)^{2/\delta}$. There exist constants $c_i = c_i(\theta, C_P, \lambda_P, C_0, \delta)$ such that the following holds.*
(a) If $B(x,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ for $t^{1/2} \le h \le (t^{1/2} \vee 2r)$ then

$$p_t(x,y) \le \frac{c_1}{V(x,\sqrt{t})} \exp\left\{-c_2 \frac{d(x,y)^2}{t}\right\}. \tag{1.8}$$

(b) Let $a = (t/r) \wedge t^{1/2}$ and $b = t^{1/2} \vee 8r$. If $B(z,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ for $z \in B(x,b)$ and $a \le h \le b$ then

$$p_t(x,y) \ge \frac{c_3}{V(x,\sqrt{t})} \exp\left\{-c_4 \frac{d(x,y)^2}{t}\right\}. \tag{1.9}$$

Remark 1.5 (1) See Theorems 4.2 and 6.3 for more precise upper and lower bounds in a fixed ball $B(o,h)$.
(2) Note that while the upper bound (1.6) holds for $t \ge d(x,y)$, the lower bound (1.7) only holds for a more limited range of t. This is because to obtain the Gaussian lower bound when $t \simeq rd(x,y)$, we need to use a chaining argument with a sequence of balls of side order r, and if r is too small then we cannot ensure that we have the necessary lower bounds on $p_{r^2}(x,y)$. For this reason it is helpful to consider a more restricted lower bound condition, such as the condition HKE$(\tau;\varepsilon)$ given on p. 1102 of [7]. The condition (pGB) which is introduced below is similar.
(3) Previous lower bounds, as in [10, 15], require some kind of local regularity of the graph. For example, [15] uses the 'p_0 condition', which is that there exists $p_0 > 0$ such that

$$\mu_{xy}/\mu_x \ge p_0, \qquad \text{for all } x, y \in \mathbb{V}. \tag{1.10}$$

In Lemma 6.1 we see that, provided b is not too small –more precisely provided $b \ge c_0(\lambda_P, \delta)$, then we can use the Poincaré inequality to obtain enough control on small values of μ_{xy} so that we can drop the p_0 condition.

Before we discuss our extensions of Delmotte's theorem, we need some further definitions.

Definition 1.6 Fix $x \in \mathbb{V}$, $R \ge 100$ and $T \ge 100$. Let

$$Q(x,R,T) = (0,T] \times B(x,R-1), \qquad \overline{Q}(x,R,T) = [0,T] \times B(x,R)$$

and

$$Q_-(x,R,T) = [\tfrac{1}{4}T, \tfrac{1}{2}T] \times B(x, \tfrac{1}{2}R), \quad Q_+(x,R,T) = [\tfrac{3}{4}T, T] \times B(x, \tfrac{1}{2}R).$$

Let $u(t,y)$ be a non-negative function on $\overline{Q}$. We say that u is *caloric* on Q, if $u(y,\cdot)$ is continuous on $[0,T]$ and satisfies

$$\frac{\partial}{\partial t} u = \mathscr{L}_\mu u \ \text{ on } \ Q.$$

Set $\mathcal{D}(Q) = \{u : u \ge 0 \text{ on } \overline{Q} \text{ and } u \text{ is caloric on } Q\}$. We say that the *parabolic Harnack inequality* (PHI) holds with constant C_H for $Q(x,R,T)$, if whenever $u \in \mathcal{D}(Q)$ we have

$$\sup_{(t,y)\in Q_-} u(t,y) \le C_H \inf_{(t,y)\in Q_+} u(t,y).$$

Definition 1.7 For $A \subset \mathbb{V}$ write $p_t^A(x,y)$ for the heat kernel of the process X killed on exiting from A. Let $C_1 > 0$, $\eta \ge 1$. We say that *partial Gaussian bounds* (pGB) hold with constants η, C_1 on $B(o,h)$ if

$$p_t(x,x') \le \frac{C_1}{V(o,h)} \Big(\frac{h^2}{t} \vee 1 \Big)^{\eta/2} \exp\Big(- \frac{d(x,x')^2}{C_1 t} \Big), \quad \text{for } x,x' \in B(o,h/2),\ t \ge d(x,x'), \tag{1.11}$$

$$p_t^{B(o,h)}(x,x') \ge \frac{1}{C_1 V(o,h)}, \quad x,x' \in B(o,h/4),\ 10^{-4}h^2 \le t \le h^2. \tag{1.12}$$

Remark 1.8 If there exist C_1, η such that (pGB) hold on enough balls then Gaussian bounds as in (1.6)–(1.7) follow – see Proposition 7.8.

To state our extension of Delmotte's theorem in its simplest form, we will make some overall regularity assumptions on Γ. (For more precise results under weaker hypotheses, see Corollary 6.4 and Theorems 7.2 and 7.5.) So we assume that Γ is an infinite graph such that

$$V(x,r) \le c\mu_x r^\theta \text{ for all } x \in \mathbb{V}, r \ge 1. \tag{1.13}$$

Definition 1.9 Let $x \in \mathbb{V}$, $R_x < \infty$. For $\kappa \in (0,1)$ we introduce the following conditions:
(A_κ) There exist $c_1, \theta, C_P, \lambda_P$ such that $B(y,r)$ is good whenever $r \ge c_1(d(x,y) \vee R_x)^\kappa$.
(B_κ) There exist c_2, C_1, η such that $B(y,r)$ satisfies pGB whenever $r \ge c_2(d(x,y) \vee R_x)^\kappa$.
(C_κ) There exist c_3, C_H such that $Q(y,r,r^2)$ satisfies PHI whenever $r \ge c_3(d(x,y) \vee R_x)^\kappa$.

Theorem 1.10 *Let Γ satisfy* (1.13).
(a) If (A_κ) holds then $(B_{\kappa'})$ holds for any $\kappa' \in (\kappa, 1)$.
(b) If (B_κ) holds then (C_κ) holds.
(c) If (C_κ) holds then (A_κ) holds.

Remark 1.11 In number theory a constant arising in an argument is called *effective* if it could in principle be computed. In this context we say a constant in a result is *effective* if it is a function of the constants in the 'input data'. Thus the constants c_1, c_2 in Theorem 1.2 are effective, since they depend only on the constants $\theta, C_0, \lambda_P, C_P, \delta$ in the hypotheses.

All the constants in this paper and in particular those in the theorem above, are effective. Thus, more precisely, (a) above states that if (A_κ) holds, then $(B_{\kappa'})$ holds, with constants η, C_1 which depend only on κ, κ' and the constants $c_1, \theta, C_P, \lambda_P$ in (A_κ).

The outline of this paper is as follows. Sections 2, 3 and 4 deal with the proof of Theorem 1.2. The first step is to obtain an on-diagonal upper bound, that is an upper bound on $p_t(x,x)$. We use the basic approach of [3], which in turn is based on [17], but extend the argument from the case of regular volume growth of order r^d to the volume doubling case. Our argument uses an auxiliary graph Γ'', which has the same structure as Γ inside a selected ball $B(o,h)$, and a suitably regular global structure. This graph is constructed in Section 2. Then, in Section 3 we obtain heat kernel upper bounds for Γ''. The off-diagonal upper bounds for Γ'' are proved using the 'two-point' method of Grigoryan – see [14, 9, 12, 8]. In Section 4 we bring these bounds back to the original graph Γ.

For the lower bounds in Theorem 1.3 we use the general approach of Fabes and Stroock [11]. In this paper, as well as [3] and other works which use these ideas, one considers the heat kernel $p_t(x_0, \cdot)$ in a ball $B(o, h)$, and needs a weighted Poincaré inequality in this ball, where the weight function ψ is of order 1 in $B(o, h/2)$ and is suitably small near the boundary of $B(o, h)$. In the case when the PI holds for all balls, this weighted PI follows from the (weak) PI by an argument of Jerison [16]; see also [20] which gives a somewhat simpler proof. However, both proofs of the weighted PI rely on having the weak PI hold in small balls close to the boundary of $B(o, h)$. In our context this would require that α (the smallest size of ball for which one can be sure that the PI holds) satisfies $\alpha^3 V(o, \alpha) \leq h^2$, i.e. $\alpha \leq ch^{2/(d+3)}$ in the case when all balls satisfy $V(x, r) \asymp r^d$. To improve this to the condition $\alpha \leq h^{1-\delta}$ given in Theorem 6.3, we use the following idea. First, in Theorem 5.1 we prove the weighted PI in a weaker form:

$$\min_{a\in\mathbb{R}} \sum_{x\in\mathcal{B}(o,h-c\alpha)} (f(x) - a)^2 \psi(x) \mu_x \leq c_1 h^2 \sum_{x,y\in B(o,h)} (f(x) - f(y))^2 (\psi(x) \wedge \psi(y)) \mu_{xy}. \quad (1.14)$$

To follow through the argument of [11], we then need to control terms of the form

$$\sum_{y\in\mathcal{B}-\mathcal{B}_e} |\log p_t(x_0, y)|^2 \psi(y) \mu_y,$$

but this can be done using a general lower bound on $p_t(x_0, y)$. With these changes, one can use standard methods (as in [3]) to obtain a near-diagonal lower bound (see Theorem 6.3), and hence also Gaussian lower bounds. Section 6 gives the proof of lower bounds, and Section 7 relates Gaussian bounds, parabolic Harnack inequality, and the conditions rVD and rPI.

For $A \subset \mathbb{V}$ we define the exterior boundary of A by

$$\partial A = \{y \in \mathbb{V} - A : \text{ there exists } x \in A \text{ such that } y \sim x\},$$

and set $\partial_i(A) = \partial(\mathbb{V} - A)$. We use the notation $C_\cdot$ to denote fixed positive constants. c, c' etc denote positive constants which may vary on each appearance, and c_i denote positive constants which are fixed in each argument. If we need to refer to constant c_1 of Lemma 2.1 elsewhere we will use the notation $c_{2.1.1}$.

2 Graph Modifications

We begin by stating some results on ball covers. We write $C_V = 2^\theta$.

Lemma 2.1 *Let $o \in \mathbb{V}$ and $\alpha \leq h$.*
(a) Suppose $B(o, h)$ satisfies $\mathcal{G}_0(\alpha)$. Then, for any $x, y \in B(o, h)$, $r \in [\alpha, h]$ and $R \in [0, h]$, if $R + d(x, y) \geq r$ then

$$\frac{V(y, R)}{V(x, r)} \leq C_V^2 \left(\frac{R + d(x, y)}{r}\right)^\theta.$$

(b) Suppose $B(o, h)$ satisfies $\mathcal{G}_1(\alpha)$. Then

$$V(o, h) \leq c_1 h^\theta \mu_x \text{ for all } x \in B(o, h). \quad (2.1)$$

Proof. (a) This is well known when Γ satisfies VD; the proofs are the same in the rVD case.
(b) By (1.4) and the first result,

$$\frac{V(o,h)}{\mu_x} = \frac{V(x,\alpha)}{\mu_x}\frac{V(o,h)}{V(x,\alpha)} \le C_0\alpha^\theta \cdot C_V^2 \left(\frac{h+d(o,x)}{\alpha}\right)^\theta \le C_0 C_V^2 2^\theta h^\theta.$$

□

Standard arguments using the volume doubling condition give

Lemma 2.2 *Let $\kappa \ge 1$. Let $B(o,h)$ satisfy $\mathcal{G}_0(\alpha)$. Let $\alpha \le R_1 \le r_i \le R_2$ for all $i \in I$. If $B(x_i,r_i), i \in I$ are pairwise disjoint balls with $x_i \in B(o,h)$, then there exists a constant C, depending only on C_V, R_2/R_1 and κ, such that for any $x \in \mathbb{V}$,*

$$|\{i : x \in B(x_i, \kappa r_i)\}| \le C.$$

We now fix $h \in \mathbb{N}$ and for the remainder of this section we will assume that $\mathcal{B} = B(o,h)$ is a fixed ball which satisfies $\mathcal{G}_0(\alpha)$. We shall construct modifications Γ' and Γ'' of Γ which keep the same structure inside $\mathcal{B}$ but modify the remainder of the graph. Set

$$\mu_{\text{avg}} = \frac{V(o,h)}{h}; \tag{2.2}$$

this choice is made so that the estimates in Lemma 2.5 will hold. We wish to modify Γ so that μ_x is not too large outside $\mathcal{B}$. Let

$$D = \{x \in \mathbb{V} - \mathcal{B} : \mu_x > \mu_{\text{avg}}\}. \tag{2.3}$$

For each $x \in D$ let $n_x = \lceil \mu_x/\mu_{\text{avg}} \rceil$. Choose $x \in D$ which minimizes $d(o,x)$, and replace x by n_x vertices (x,i), $i = 1,\dots,n_x$. We define the weights

$$\mu'_{(x,i),y} = n_x^{-1}\mu_{xy}, \tag{2.4}$$

so that $\mu'_{(x,i)} = \sum_y \mu'_{(x,i),y} = n_x^{-1}\mu_x$. Thus $x \in \mathbb{V}$ is replaced by n_x vertices in $\mathbb{V}'$, with the same neighbors as x. We then continue this construction, and denote the resulting graph $\Gamma' = (\mathbb{V}', \mathbb{E}')$. Write $\pi : \mathbb{V}' \to \mathbb{V}$ for the function which maps $x \in \mathbb{V} - D$ to x and (x,i) to x for $x \in D$, and write $B'(x,r)$ for balls in $\mathbb{V}'$.

Proposition 2.3 *(a) For every vertex $x \in \mathbb{V}' - \mathcal{B}$ we have $\mu'_x \le \mu_{\text{avg}}$.*
(b) The ball $B'(o,h)$ in the weighted graph (Γ',μ') satisfies the condition rPI with constants $\lambda'_P = \lambda_P \vee 2$ and $C'_P = 1 \vee C_P$.

Proof. (a) This is immediate from the construction.
(b) By induction, it is enough to consider the case when only one vertex $z \in \mathbb{V}$ is split into $n = n_z$ new vertices. Fix $x_0 \in B(o,h)$ and $r \in [\alpha, h]$. Let $A_0 = B(x_0, r)$ and $A_1 = B(x_0, \lambda'_P r)$ be balls in Γ, and $A'_i = \pi^{-1}(A_i)$ be the related balls in Γ'. We need to establish (1.3) for $A'_0 \subseteq A'_1$.

If $z \notin A_1$ then $A'_1 = A_1$ and the inequality is immediate since $B(o,h)$ satisfies $\mathcal{G}_0(\alpha)$.

Suppose $z \in A_0$, and write $z_i = (z, i)$ for each $1 \le i \le n$. Set $V_0 = A_0 - \{z\}$ and $V_1 = A_1 - \{z\}$. Then

$$A'_0 = V_0 \cup \{z_i : 1 \le i \le n\} \ \text{ and } \ A'_1 = V_1 \cup \{z_i : 1 \le i \le n\}.$$

Let $f : A'_1 \to \mathbb{R}$, and define $\tilde{f} : A_1 \to \mathbb{R}$ by:

$$\tilde{f}(x) = f(x),\ x \in V_1, \quad \tilde{f}(z) = n^{-1} \sum_{i=1}^{n} f(z_i).$$

We can assume that

$$\overline{f} = \sum_{x \in A'_0} f(x)\mu'_x = 0,$$

so we also have $\sum_{x \in A_0} \tilde{f}(x)\mu_x = 0$. Set

$$V_z(f) = n^{-1}\mu_z \sum_{i=1}^{n} (f(z_i) - \tilde{f}(z))^2.$$

Note that $B(z, 1) \subseteq B(x_0, r + 1) \subseteq A_1$. Therefore

$$\begin{aligned} &\tfrac{1}{2} \sum_{x,y \in A'_1} (f(x) - f(y))^2 \mu'_{xy} - \tfrac{1}{2} \sum_{x,y \in A_1} (\tilde{f}(x) - \tilde{f}(y))^2 \mu_{xy} \\ &\qquad = \sum_{y \in V_1} \sum_{i=1}^{n} (f(z_i) - f(y))^2 n^{-1} \mu_{zy} - \sum_{y \in V_1} (\tilde{f}(z) - f(y))^2 \mu_{zy} = V_z(f). \end{aligned} \tag{2.5}$$

A direct calculation gives

$$\sum_{x \in A'_0} f(x)^2 \mu'_x - \sum_{x \in A_0} \tilde{f}(x)^2 \mu_x = n^{-1} \sum_{i=1}^{n} f(z_i)^2 \mu_z - \tilde{f}(z)^2 \mu_z = V_z(f). \tag{2.6}$$

Using (2.6), (2.5) and the PI (1.3) for the original graph Γ we obtain (1.3) for $A'_0 \subseteq A'_1$.

If $z \in A_1 - A_0$, then $A'_0 = A_0$ and (1.3) for $A'_0 \subseteq A'_1$ is proved in a similar fashion as the second case. □

If X' is the random walk on Γ' then we can couple X and X' so that $\pi(X'_t) = X_t$ for all t. Hence, writing $p'_t(x, y)$ for the heat kernel on Γ', we have

$$p_t(x, \pi(y)) = p'_t(x, y) \ \text{ for all } x \in \mathcal{B},\ y \in \mathbb{V}'. \tag{2.7}$$

In particular, the heat kernels agree on $B(o, h)$. So, as long as we are just concerned with $p_t(x, y)$ for $x, y \in B(o, h)$, we can replace our original graph Γ by Γ', or equivalently (which will simplify our notation) we can assume that the set D given by (2.3) is empty, so that $\Gamma = \Gamma'$.

We now construct another weighted graph Γ'', which also keeps the same structure inside $\mathcal{B}$. Without loss of generality, we can assume that $\mathbb{V} \cap \mathbb{N} = \emptyset$. The new graph has vertex set $B(o,h) \cup (\mathbb{N} \cap [h+1, M''])$, where $M'' \in \mathbb{N}$ is defined below, and is obtained by mapping points in the shells $\partial B(o,r)$ into $\mathbb{N} \cap [h+1, M'']$ in such a way that the total weights of points mapped onto a point j is, as far as possible, close to μ_{avg}.

Set $H = 2h$. Then $\partial B(o, H-1) \neq \emptyset$ as $B(o,h)$ satisfies $\mathcal{G}_0(\alpha)$. We define a function χ from $B(o,H)$ to $\mathcal{B} \cup \{r \in \mathbb{N} : r \geq h+1\}$ and a strictly increasing function ξ from $\{0, 1, 2, \cdots, H+1\}$ to $\mathbb{Z}_+$ such that:

(1) $\chi(x) = x$ for each $x \in \mathcal{B}$,
(2) $\xi_r = r$ for each $0 \leq r \leq h+1$,
(3) $\chi^{-1}(\{l \in \mathbb{N} : \xi_r \leq l < \xi_{r+1}\}) \subseteq \partial B(o, r-1)$ for each $h+1 \leq r \leq H$,
(4) $0 < \mu(\chi^{-1}(\xi_r)) < 2\mu_{\text{avg}}$, for each $h+1 \leq r \leq H$,
(5) $\mu_{\text{avg}} \leq \mu(\chi^{-1}(l)) < 2\mu_{\text{avg}}$ for each $h+1 \leq r \leq H$, $\xi_r < l < \xi_{r+1}$.

The existence of such functions χ, ξ is clear from Proposition 2.3. We set $M'' = \xi_{H+1} - 1$. Then $M'' \geq H = 2h$. Now we define a weighted graph $\Gamma'' = (\mathbb{V}'', \mathbb{E}'', \nu)$ as follows. Let

$$\begin{aligned} \mathbb{V}'' &= \mathcal{B} \cup (\mathbb{N} \cap [h+1, M'']); \\ \mathbb{E}'' &= \{(\chi(x), \chi(y)) : x, y \in B(o,H), \chi(x) \neq \chi(y), (x,y) \in \mathbb{E}\} \\ &\qquad \cup \{(r, r+1) : r \in \mathbb{N} \cap [h+1, M''-1]\}. \end{aligned}$$

For any $x, y \in \mathbb{V}''$ with $x \neq y$, we set

$$\nu_{xy} = \sum_{u \in \chi^{-1}(x)} \sum_{v \in \chi^{-1}(y)} \mu_{uv} + 2\mu_{\text{avg}} 1_{\{x > h, y > h, |x-y|=1\}}. \tag{2.8}$$

Thus $\nu_{xy} = \nu_{yx}$ and

$$\nu_{xy} = \begin{cases} \mu_{xy} & \text{if } x, y \in \mathcal{B}; \\ \sum_{u \in \chi^{-1}(x)} \mu_{uy} & \text{if } x \geq h+1, y \in \mathcal{B} \setminus B(o, h-1); \\ 2\mu_{\text{avg}} + \sum_{u \in \chi^{-1}(x), v \in \chi^{-1}(y)} \mu_{uv} & \text{if } x - 1 = y > h; \\ \sum_{u \in \chi^{-1}(x), v \in \chi^{-1}(y)} \mu_{uv} & \text{if } x - 1 > y > h. \end{cases}$$

We set

$$\nu_x = \sum_y \nu_{xy}.$$

Lemma 2.4 *If $x \in \mathcal{B}$ then $\nu_x = \mu_x$. If $x \in \mathbb{N} \cap [h+1, M'']$, then*

$$2\mu_{\text{avg}} \leq \nu_x \leq 6\mu_{\text{avg}}.$$

Proof. The first result is obvious. For the second, the lower bound on ν_x is immediate from the definition (2.8). Also,

$$\nu_x \leq 4\mu_{\text{avg}} + \sum_{u \in \chi^{-1}(x)} \sum_{y \in \mathbb{V}''} \sum_{v \in \chi^{-1}(y)} \mu_{uv}$$

$$
\begin{aligned}
&= 4\mu_{\rm avg} + \sum_{u\in\chi^{-1}(x)} \sum_{v\in B(o,H)} \mu_{uv} \\
&\le 4\mu_{\rm avg} + \sum_{u\in\chi^{-1}(x)} \mu_u = 4\mu_{\rm avg} + \mu(\chi^{-1}(x)) \le 6\mu_{\rm avg}.
\end{aligned}
$$

Here the last inequality follows from the conditions (4) and (5) of the construction. □

Write $d_{\Gamma''}$ for graph distance in Γ''.

Lemma 2.5 *(a) There exists* c_1 *so that if* $x, y \in \mathbb{V}''$ *then* $d_{\Gamma''}(x,y) \le c_1 h$.
(b) There exists c_2 *so that* $\nu(\mathbb{V}'') \le c_2 V(o,h)$*, where* $\nu(\mathbb{V}'') = \sum_{y\in\mathbb{V}''} \nu_y$.

Proof. By the conditions (3),(4) and (5) of the construction,

$$
\xi_{r+1} - 1 - \xi_r \le \sum_{j=\xi_r+1}^{\xi_{r+1}-1} \frac{\mu(\chi^{-1}(j))}{\mu_{\rm avg}} \le \frac{V(o,r) - V(o,r-1)}{\mu_{\rm avg}}.
$$

So, using rVD,

$$
\begin{aligned}
\xi_{H+1} - \xi_h &= \sum_{r=h}^{H} (\xi_{r+1} - \xi_r) \le \sum_{r=h}^{H} \left(1 + \frac{V(o,r) - V(o,r-1)}{\mu_{\rm avg}}\right) \\
&\le 1 + h + \mu_{\rm avg}^{-1} V(o,2h) \\
&\le 1 + h + \mu_{\rm avg}^{-1} C_V V(o,h) = 1 + (1 + C_V)h.
\end{aligned}
$$

Consequently

$$
M'' = \xi_{H+1} - 1 \le h + 1 + (1 + C_V)h - 1 = (2 + C_V)h;
$$

and for all $z \in \mathbb{V}''$, $d_{\Gamma''}(o,z) \le M''$, proving (a). For (b),

$$
\begin{aligned}
\nu(\mathbb{V}'') =& \sum_{y\in\mathcal{B}} \nu_y + \sum_{y\in\mathbb{N}\cap[h+1,M'']} \nu_y \le \sum_{y\in\mathcal{B}} \mu_y + 6 \sum_{y\in\mathbb{N}\cap[h+1,M'']} \mu_{\rm avg} \\
\le& V(o,h) + 6M''\mu_{\rm avg} \le V(o,h) + 6(2 + C_V)h\mu_{\rm avg} \\
=& (13 + 6C_V)V(o,h).
\end{aligned}
$$

□

Our heat kernel estimate will use Poincaré inequalities in the two different parts of the graph $\mathbb{V}''$. The first is already given by Proposition 2.3, and controls balls with centers in $B(o,h)$. The second is a strong Poincaré inequality on $\mathbb{N}$, which it is straightforward to prove by elementary arguments.

Lemma 2.6 *Let* $a, b \in \mathbb{N}$ *with* $b > a$*. Then for any function* f *on* $\mathbb{N}$,

$$
\min_{z\in\mathbb{R}} \sum_{i=a}^{b} (f(i) - z)^2 \le (b-a)^2 \sum_{i=a}^{b-1} (f(i+1) - f(i))^2.
$$

The following bound will play an essential role in the heat kernel upper bound in Section 3.

Lemma 2.7 *Suppose $2\alpha \le r \le h/\lambda_P$. Let $g : \mathbb{V}'' \to \mathbb{R}_+$ with $\sum_x g(x)\nu_x = 1$. Then*

$$\sum_{x,y\in\mathbb{V}''} (g(x)-g(y))^2 \nu_{xy} \ge c_1 r^{-2}\Big(\sum_x g(x)^2\nu_x - c_2 \frac{h^\theta}{V(o,h)r^\theta}\Big). \tag{2.9}$$

Proof. Define $\tilde g : \mathbb{V} \to \mathbb{R}_+$ by $\tilde g(x) = g(\chi(x))$ if $x \in B(o,H)$, and $\tilde g(x) = 0$, $x \in \mathbb{V} - B(o,H)$. Then

$$\sum_{u\in\mathbb{V}} \tilde g(u)\mu_u = \sum_{x\in\chi(B(o,H))}\ \sum_{u\in\chi^{-1}(x)} g(x)\mu_u \le \sum_{x\in\mathbb{V}''} g(x)\nu_x = 1.$$

By the definition of ν_{xy},

$$\begin{aligned}
&\sum_{x,y\in\mathbb{V}''} [g(x)-g(y)]^2\nu_{xy}\\
&\qquad = \sum_{x,y\in\mathbb{V}''} [g(x)-g(y)]^2 \sum_{u\in\chi^{-1}(x),v\in\chi^{-1}(y)} \mu_{uv} + 4\mu_{\mathrm{avg}} \sum_{x=h+1}^{M''-1} [g(x+1)-g(x)]^2.
\end{aligned}$$

We estimate the two sums separately, beginning with the first. We have

$$\begin{aligned}
\sum_{x,y\in\mathbb{V}''} [g(x)-g(y)]^2 \sum_{u\in\chi^{-1}(x),v\in\chi^{-1}(y)} \mu_{uv} &= \sum_{x,y\in\mathbb{V}''}\ \sum_{u\in\chi^{-1}(x),v\in\chi^{-1}(y)} [\tilde g(u)-\tilde g(v)]^2\mu_{uv}\\
&= \sum_{u,v\in B(o,H)} [\tilde g(u)-\tilde g(v)]^2\mu_{uv}.
\end{aligned}$$

Let $\Theta_r \subset \mathcal{B}$ be such that $\{B(x,r/2), x\in\Theta_r\}$ are disjoint, and $\mathcal{B} \subset \cup_{x\in\Theta_r} B(x,r)$. Since $r \le h/\lambda_P$, for any $x \in \Theta_r$ we have

$$B(x,\lambda_P r) \subseteq B(x,h) \subseteq B(o,H).$$

By Lemma 2.2 there exists a constant K so that any $y \in B(o,H)$ is contained in at most K of the balls $B(x,\lambda_P r), x \in \Theta_r$. Let $\overline{g}_{x,r}$ be the mean of $\tilde g$ on $B(x,r)$. Using the rPI (1.3) for the balls $B(x,r), x\in\Theta_r$,

$$\begin{aligned}
\sum_{u,v\in B(o,H)} [\tilde g(u)-\tilde g(v)]^2\mu_{uv} &\ge K^{-1}\sum_{x\in\Theta_r}\ \sum_{u,v\in B(x,\lambda_P r)} [\tilde g(u)-\tilde g(v)]^2\mu_{uv}\\
&\ge K^{-1}C_P^{-1}r^{-2}\sum_{x\in\Theta_r}\ \sum_{y\in B(x,r)} (\tilde g(y)-\overline{g}_{x,r})^2\mu_y\\
&= (KC_P r^2)^{-1}\sum_{x\in\Theta_r}\left(\sum_{y\in B(x,r)} \tilde g(y)^2\mu_y - \frac{[\sum_{y\in B(x,r)}\tilde g(y)\mu_y]^2}{V(x,r)}\right)
\end{aligned}$$

$$\geq (KC_P r^2)^{-1}\left(\sum_{y\in\mathcal{B}} g(y)^2\nu_y - C_V^2 2^\theta \frac{h^\theta}{V(o,h)r^\theta}\sum_{x\in\Theta_r}[\sum_{y\in B(x,r)}\tilde{g}(y)\mu_y]^2\right).$$

In the final inequality we used Lemma 2.1 to bound $V(x,r)$ from below, and also the first statement in Lemma 2.4.

We bound the second sum in a similar fashion. We divide $\mathbb{N}\cap[h+1,M'']$ into disjoint intervals A_l, $l=1,\dots,k$, chosen so that $r\leq|A_l|\leq 2r$ for each l. Using Lemma 2.6, and writing $\overline{g}_l$ for the mean of g on A_l,

$$\begin{aligned}\mu_{\mathrm{avg}}\sum_{x\in A_l,x+1\in A_l}(g(x+1)-g(x))^2 &\geq 2^{-2}r^{-2}\mu_{\mathrm{avg}}\sum_{x\in A_l}[g(x)-\overline{g}_l]^2\\ &= 2^{-2}r^{-2}\mu_{\mathrm{avg}}\left(\sum_{x\in A_l}g(x)^2-\overline{g}_l^2|A_l|\right)\\ &\geq cr^{-2}\left(\sum_{x\in A_l}g(x)^2\nu_x - \frac{c'\mu_{\mathrm{avg}}\big(\sum_{x\in A_l}g(x)\big)^2}{|A_l|}\right).\end{aligned}$$

Here we used the bounds on ν_x from Lemma 2.4. Since $\theta\geq 1$ and $|A_l|\geq r$,

$$\begin{aligned}\frac{\mu_{\mathrm{avg}}\big(\sum_{x\in A_l}g(x)\big)^2}{|A_l|} &\leq \frac{1}{\mu_{\mathrm{avg}}r}\Big(\sum_{x\in A_l}g(x)\mu_{\mathrm{avg}}\Big)^2\\ &= \frac{h}{V(o,h)r}\Big(\sum_{x\in A_l}g(x)\mu_{\mathrm{avg}}\Big)^2 \leq \frac{h^\theta}{V(o,h)r^\theta}\Big(\sum_{x\in A_l}g(x)\mu_{\mathrm{avg}}\Big)^2.\end{aligned}$$

Combining the inequalities above we obtain

$$\sum_{x,y\in\mathbb{V}''}(g(x)-g(y))^2\nu_{xy}\geq cr^{-2}\Big(\sum_{x\in\mathbb{V}''}g(x)^2\nu_x - c'\frac{h^\theta}{V(o,h)r^\theta}\,J^2\Big), \tag{2.10}$$

where

$$J^2=\sum_{x\in\Theta_r}\Big(\sum_{y\in B(x,r)}\tilde{g}(y)\mu_y\Big)^2+\sum_{l=1}^k\Big(\sum_{y\in A_l}g(y)\mu_{\mathrm{avg}}\Big)^2. \tag{2.11}$$

Since $\sum x_i^2+\sum y_j^2\leq(\sum|x_i|+\sum|y_j|)^2$, and each point in $\mathbb{V}$ is in at most K of the balls $B(x,r)$,

$$\begin{aligned}J &\leq \sum_{x\in\Theta_r}\sum_{y\in B(x,r)}\tilde{g}(y)\mu_y+\sum_{l=1}^k\sum_{x\in A_l}g(x)\mu_{\mathrm{avg}}\\ &\leq K\sum_{y\in\mathbb{V}}\tilde{g}(y)\mu_y+\sum_{x=h+1}^{M''}g(x)\nu_x\leq K+1.\end{aligned}$$

Substituting this bound on J into (2.10) completes the proof. □

3 Heat kernel estimates for the extended graph

Throughout this section we fix a ball $\mathcal{B} = B(o,h)$ and assume that for some $\delta \in (0,1]$ the ball $\mathcal{B}$ satisfies $\mathcal{G}_1(h^{1-\delta})$. Let Γ'' be the graph constructed in the previous section, and let Y be the continuous time random walk on Γ'', with generator $\mathscr{L}_\nu$ and heat kernel $q_t(x,y)$ given by

$$\mathscr{L}_\nu f(x) = \frac{1}{\nu_x}\sum_y (f(y)-f(x))\nu_{xy}, \quad q_t(x,y) = \frac{\mathbb{P}_x(Y_t = y)}{\nu_y}.$$

Let $\mu_{\min} = \min_{x\in\mathcal{B}} \mu_x$, and set

$$\eta = \theta/\delta; \tag{3.1}$$

note that $\eta \ge \theta \ge 1$.

Lemma 3.1 *We have $\nu_x \ge \mu_{\min}$ for all $x \in \mathbb{V}''$.*

Proof. By the construction, $\nu_x = \mu_x \ge \mu_{\min}$ for each $x \in \mathcal{B}$. As $\mathcal{B}$ satisfies $\mathcal{G}_0(\alpha)$ we have $\mathbb{V} - B(o,2h) \neq \emptyset$ and $V(o,h) \ge h\mu_{\min}$. So, using Lemma 2.4, for each $x \in \mathbb{N} \cap [h+1, M'']$, $\nu_x \ge \mu_{\text{avg}} = V(o,h)/h \ge \mu_{\min}$. □

Proposition 3.2 *There exists c_1 such that for any $w \in \mathbb{V}''$ and $t \ge 0$,*

$$q_t(w,w) \le c_1 \left(\frac{h^2}{t} \vee 1\right)^{\eta/2} \frac{1}{V(o,h)}. \tag{3.2}$$

Proof. If $2h^{1-\delta} \ge h/\lambda_P$ then $h \le (2\lambda_P)^{1/\delta}$, and for any $t \ge 0$,

$$q_t(w,w) \le \frac{1}{\nu_w} \le \frac{1}{\mu_{\min}} \le \frac{c_{2.1.1}h^\theta}{V(o,h)} \le \frac{c_{2.1.1}(2\lambda_P)^{\theta/\delta}}{V(o,h)}. \tag{3.3}$$

Thus (3.2) holds in this case.

Now let $2h^{1-\delta} < h/\lambda_P$. Fix $w \in \mathbb{V}''$. For $t \ge 0$ and $x \in \mathbb{V}''$, set

$$\varphi(t) = q_{2t}(w,w), \text{ and } f_t(x) = q_t(w,x).$$

Then $\varphi(t) = \sum_x f_t(x)^2 \nu_x$ and

$$-\varphi'(t) = -\frac{d}{dt}\langle f_t, f_t\rangle = -2\Big\langle \frac{\partial f_t}{\partial t}, f_t\Big\rangle = -2\langle \mathscr{L}_\nu f_t, f_t\rangle = \sum_{x,y\in\mathbb{V}''} [f_t(x)-f_t(y)]^2 \nu_{xy}.$$

Therefore by Lemma 2.7, for any $r \in [2h^{1-\delta}, h/\lambda_P]$,

$$-\varphi'(t) \ge c_2 r^{-2}\left(\varphi - \frac{c_3 h^\theta}{2V(o,h)r^\theta}\right). \tag{3.4}$$

Set $c_4 = c_3 + 2^\eta c_{2.1.1}$ and

$$g(r) = \frac{c_4 h^\eta}{r^\eta V(o,h)}.$$

By (2.1),

$$g(2h^{1-\delta}) = \frac{c_4 h^\eta}{(2h^{1-\delta})^\eta V(o,h)} \geq \frac{c_{2.1.1} h^\theta}{V(o,h)} \geq \mu_{\min}^{-1} \geq \nu_w^{-1} = q_0(w,w) = \varphi(0).$$

Define $r(t)$ by

$$r(t) = g^{-1}(\varphi(t)).$$

Then $r(0) \geq 2h^{1-\delta}$, so $r(t) \geq 2h^{1-\delta}$ for all $t \geq 0$. Now let t_1 be such that $r(t_1) = h/\lambda_P$. We obtain from (3.4),

$$\varphi'(t) \leq -\tfrac{1}{2} c_2 r(t)^{-2} \varphi(t), \text{ for } t \in [0, t_1]. \tag{3.5}$$

So for $t \in [0, t_1]$,

$$-\varphi'(t) \geq \tfrac{1}{2} c_2 \varphi(t) (g^{-1}(\varphi(t)))^{-2} = \tfrac{1}{2} c_2 \varphi^{1+2/\eta} (V(o,h)/c_4 h^\eta)^{2/\eta}.$$

Set $\psi(t) = \varphi(t)^{-2/\eta}$; then for $t \in [0, t_1]$,

$$\psi'(t) = -(2/\eta)\varphi(t)^{-1-2/\eta}\varphi'(t) \geq c_5^{2/\eta} \Big(\frac{V(o,h)}{h^\eta}\Big)^{2/\eta}.$$

Hence

$$\psi(t) \geq \psi(0) + t c_5^{2/\eta} \Big(\frac{V(o,h)}{h^\eta}\Big)^{2/\eta} \geq t c_5^{2/\eta} \Big(\frac{V(o,h)}{h^\eta}\Big)^{2/\eta},$$

which implies that

$$\varphi(t) \leq \frac{h^\eta}{c_5 t^{\eta/2} V(o,h)}, \text{ for } 0 \leq t \leq t_1. \tag{3.6}$$

This proves (3.2) for $0 \leq t \leq t_1$. If $t \geq t_1$ then

$$\varphi(t) \leq \varphi(t_1) = g(h/\lambda_P) = \frac{c_4 \lambda_P^\eta}{V(o,h)},$$

so again (3.2) holds. □

We now obtain general Gaussian upper bounds on Γ''.

Proposition 3.3 *There exist $c_1 = c_1(C_P, \lambda_P, \theta, C_0, \delta)$ and a universal constant c_2 such that for $x, y \in \mathbb{V}''$ with $t \geq 1 \vee d_{\Gamma''}(x,y)$,*

$$q_t(y,x) \leq c_1 \left(\frac{h^2}{t} \vee 1\right)^{\eta/2} \frac{1}{V(o,h)} \exp\left(-c_2 \frac{d_{\Gamma''}^2(x,y)}{t}\right). \tag{3.7}$$

Proof. For $x \in \mathbb{V}''$ set

$$f_x(t) = (c_{3.2.1})^{-1} \left(\frac{h^2}{t} \vee 1\right)^{-\eta/2} \frac{V(o,h)}{\nu_x}.$$

By Proposition 3.2,

$$\mathbb{P}_x(Y_t = x) = q_t(x,x)\nu_x \leq \frac{1}{f_x(t)} \text{ for } x \in \mathbb{V}''.$$

Following Grigoryan [14] (in the manifold case), and [9, 12] (for random walks on graphs), [8] obtains Gaussian upper bounds for a random walk on a graph given only on-diagonal upper bounds as in (3.2). The metric $d_\nu(x,y)$ in [8] is just $d_{\Gamma''}(x,y)$, and one can easily check that $f_x(t)$ is $(2^{\eta/2},2)$–regular: see [14, 8] for the definition. Therefore, by [8, Theorem 1.1 and Remark 1.1] there exist universal constants c_3, c_4 such that for $t \ge 1 \vee d_{\Gamma''}(x,y)$,

$$\begin{aligned}\mathbb{P}_y(Y_t = x) \le & \frac{c_3 2^{\eta/2}(\nu_x/\nu_y)^{1/2}}{\sqrt{f_x(t/64)f_y(t/64)}}\exp\{-c_4\frac{d^2_{\Gamma''}(x,y)}{t}\} \\ =& c_3 2^{\eta/2} c_{3.2.1}\left(\frac{64h^2}{t}\vee 1\right)^{\eta/2}\frac{\nu_x}{V(o,h)}\exp\{-c_4\frac{d^2_{\Gamma''}(x,y)}{t}\}.\end{aligned}$$

The bound (3.7) follows immediately. □

By [8, Corollary 2.8] the following 'long range' bounds hold for $q_t(x,y)$. There exist $c_1, \delta_0 > 0$ such that if $x, y \in \mathbb{V}''$, $r = d_{\Gamma''}(x,y)$, $t > 0$, then

$$q_t(x,y) \le (\nu_x\nu_y)^{-1/2}\exp(-r^2/(16t)), \quad \text{if } t \ge r, \tag{3.8}$$

$$q_t(x,y) \le c_1(\nu_x\nu_y)^{-1/2}\exp\big(-\tfrac12 r\log((1+\delta_0)r/t)\big) \quad \text{if } t \le r. \tag{3.9}$$

Corollary 3.4 *Let $x, y \in \mathbb{V}''$ with $d_{\Gamma''}(x,y) \ge h/16$. There exists c_i so that if $h \le t \le h^2$ then*

$$\sup_{0\le s\le t} q_s(x,y) \le \frac{c_1}{V(o,h)}\left(\frac{h^2}{t}\right)^{\eta/2} e^{-c_2h^2/t}. \tag{3.10}$$

Proof. We bound separately the supremum on the intervals $[0, C''h]$ and $[C''h, t]$. By (3.8) and (3.9) we have for $0 \le s \le C''h$,

$$q_s(x,y) \le c(\nu_x\nu_y)^{-1/2}e^{-c_3h}.$$

So, by Lemma 2.1,

$$\begin{aligned}\sup_{0\le s\le C''h} q_s(x,y) \le & \frac{c}{V(o,h)}h^\theta e^{-c_3h} \\ \le & \frac{c}{V(o,h)}\left(\frac{h^2}{t}\right)^{\eta/2} e^{-\frac12 c_3h}\Big(\sup_h h^\theta e^{-\frac12 c_3h}\Big) \\ \le & \frac{c'}{V(o,h)}\left(\frac{h^2}{t}\right)^{\eta/2} e^{-\frac12 c_3h^2/t}.\end{aligned}$$

By Proposition 3.3 and Lemma 2.5(a),

$$\sup_{C''h\le s\le t} q_s(x,y) \le \frac{c}{V(o,h)}\sup_{C''h\le s\le t}\left(\frac{h^2}{s}\right)^{\eta/2}\exp(-c'h^2/s),$$

and it is straightforward to verify that this supremum is bounded by (3.10). □

Lemma 3.5 *There exist constants c_i such that if $h \geq c_1$, then for $x \in B(o, 7h/8)$ and $c_2 h \leq t \leq c_3 h^2$,*

$$\mathbb{P}_x\big(Y_t \notin B(x, h/16)\big) \leq \frac{1}{4}.$$

Proof. Since Γ'' keeps the same structure with Γ inside $\mathcal{B}$, we have $d_{\Gamma''}(x,y) \geq h/16$ for $y \in \mathbb{V}'' - B(x, h/16)$. Then by Proposition 3.3 and Lemma 2.5, if $c_{2.5.1} h \leq t \leq h^2$ then

$$\begin{aligned}\mathbb{P}_x\big(Y_t \notin B(x, h/16)\big) &\leq \sum_{y \in \mathbb{V}'' - B(x,h/16)} \frac{c_1}{V(o,h)} \left(\frac{h^2}{t}\right)^{\eta/2} \exp(-c_2 \frac{d^2_{\Gamma''}(x,y)}{t}) \nu_y \\ &\leq \frac{c_1 \nu(\mathbb{V}'')}{V(o,h)} \left(\frac{h^2}{t}\right)^{\eta/2} \exp(-c_2 \frac{(h/16)^2}{t}) \\ &\leq c_1 c_{2.5.2} \left(\frac{h^2}{t}\right)^{\eta/2} \exp(-c \frac{h^2}{t}). \end{aligned} \tag{3.11}$$

So taking c_3 small enough we can make the right side of (3.11) less than $\frac{1}{4}$. □

4 Upper bound of heat kernel on the original graph

We now fix a ball $\mathcal{B} = B(o,h)$ satisfying $\mathcal{G}_1(h^{1-\delta})$ and define Γ'' and Y as in the previous two sections. Recall from Section 1 the definition of X. We set

$$\tau_A^X = \inf\{t \geq 0 : X_t \notin A\},$$

and write

$$\tau_{x,r}^X = \tau_{B(x,r)}^X.$$

We define the stopping times $\tau_\cdot^Y$ in the same way. If X and Y start at the same vertex in $\mathcal{B}$, then since Γ and Γ'' have the same structure in $\mathcal{B}$, we can couple X and Y on the same probability space such that

$$X_s = Y_s \text{ for all } 0 \leq s < \tau_{\mathcal{B}}^X = \tau_{\mathcal{B}}^Y. \tag{4.1}$$

We use the same probability measure $\mathbb{P}_x$ for both X and Y.

Lemma 4.1 *Let $B(o,h)$ satisfy $\mathcal{G}_1(h^{1-\delta})$. If $h \geq c_1$ then for any $x \in B(o, 5h/8)$,*

$$\mathbb{P}_x(\tau_{x,h/8}^X < c_2 h^2) \leq \tfrac{1}{2}.$$

Proof. Fix $x \in B(o, 5h/8)$. Since X and Y agree until time $\tau_{\mathcal{B}}^X$, we have $\tau_{x,h/8}^X = \tau_{x,h/8}^Y$ $\mathbb{P}_x$–a.s. Write $\sigma = \tau_{x,h/8}^Y$. Then $d(Y_\sigma, x) > \frac{h}{8}$, $\mathbb{P}_x$–a.s. Let $t_2 = c_{3.5.3} h^2$ and $t_1 = t_2 - c_{3.5.2} h$. Let $B_1 = B(x, h/16)$. Then if c_1 is large enough so that $t_1 \geq c_{3.5.2} h$ and $t_1 \geq \frac{1}{2} t_2$,

$$\begin{aligned}\mathbb{P}_x(\sigma < \tfrac{1}{2} t_2) &\leq \mathbb{P}_x(\sigma < t_1, Y_{t_2} \notin B_1) + \mathbb{P}_x(\sigma < t_1, Y_{t_2} \in B_1) \\ &\leq \mathbb{P}_x(Y_{t_2} \notin B_1) + \mathbb{E}_x(1_{(\sigma < t_1)} \mathbb{P}_{Y_\sigma}(Y_{t_2 - \sigma} \in B_1)) \\ &\leq \tfrac{1}{4} + \max_{z \in B(o, 7h/8)} \sup_{c_{3.5.2} h \leq s \leq t_2} \mathbb{P}_z(Y_s \notin B(z, h/16)) \leq \tfrac{1}{2}.\end{aligned}$$

□

Theorem 4.2 *Let $B(o,h)$ satisfy $\mathcal{G}_1(h^{1-\delta})$. Then there exist $c_1 = c_1(C_P, \lambda_P, \theta, C_0, \delta)$ and a universal constant c_2 such that for any $x, x' \in B(o, h/2)$ and $t \geq d(x, x')$,*

$$p_t(x,x') \leq \frac{c_1}{V(o,h)} \Big(\frac{h^2}{t} \vee 1 \Big)^{\theta/2\delta} \exp\Big(- c_2 \frac{d(x,x')^2}{t} \Big). \tag{4.2}$$

In particular, the upper bound (1.11) *of (pGB) holds with $\eta = \theta/\delta$.*

Proof. If $h \leq c_1$, then the result is trivial by (3.3). So, let $h > c_1$. Define the heat kernel for the process X killed on exiting $\mathcal{B}$ by

$$p_t^{\mathcal{B}}(x,y) = \frac{\mathbb{P}_x(X_t = y, \tau_{\mathcal{B}}^X > t)}{\mu_y};$$

by (4.1) this is also the heat kernel for Y killed on exiting $\mathcal{B}$. Write $\tau = \tau_{\mathcal{B}}^X$. It is clear that

$$p_t^{\mathcal{B}}(x,y) \leq p_t(x,y) \wedge q_t(x,y), \; x, y \in \mathcal{B}.$$

Set

$$A = \partial_i B(o, 5h/8).$$

Let $T = \inf\{t \geq 0 : X_t \in A\}$. Applying the strong Markov property at T we have for $x \in B(o, h/2)$, and $w \in \partial B(o,h)$,

$$\mathbb{P}_w(X_t = x) = \mathbb{E}_w \mathbb{P}_{X_T}(X_{t-T} = x) \leq \sup_{0<s\leq t} \max_{z \in A} \mathbb{P}_z(X_s = x). \tag{4.3}$$

Fix $x \in B(o, h/2)$ and set

$$p^*(t) = \max_{y \in A} p_t(y,x), \quad q^*(t) = \max_{y\in A} q_t(y,x).$$

Then using (4.3) and the strong Markov property, for $y \in \mathcal{B}$,

$$\begin{aligned} p_t(y,x) &= p_t^{\mathcal{B}}(y,x) + \mathbb{E}_y\big(1_{(\tau<t)} p_{t-\tau}(X_\tau, x)\big) \\ &\leq q_t(y,x) + \mathbb{P}_y(\tau < t) \sup_{0<s\leq t} \max_{z\in A} p_s(z,x) \\ &\leq q_t(y,x) + \mathbb{P}_y(\tau<t) \sup_{0<s\leq t} p^*(s). \end{aligned} \tag{4.4}$$

Now let $t \leq (1 \wedge c_{4.1.2})h^2$. By Lemma 4.1 we have $\mathbb{P}_y(\tau < t) \leq \frac{1}{2}$ for $y \in A$, so by (4.4)

$$p^*(t) \leq q^*(t) + \tfrac{1}{2} \sup_{0<s\leq t} p^*(s),$$

and therefore

$$\sup_{0<s\leq t} p^*(s) \leq 2 \sup_{0<s\leq t} q^*(s). \tag{4.5}$$

Let $x, x' \in B(o,h/2)$. By (4.4) and (4.5)

$$p_t(x',x) \leq q_t(x,x') + \tfrac{1}{2} \sup_{0<s\leq t} p^*(s) \leq q_t(x,x') + \sup_{0<s\leq t} q^*(s). \tag{4.6}$$

If $h \le t \le (1 \wedge c_{4.1.2})h^2$ then using (4.6), Proposition 3.3 and Corollary 3.4 gives (4.2). Next suppose $h \ge t \ge d(x,x')$. Then by (4.6) and Corollary 3.4, writing $r = d(x,x')$,

$$p_t(x,x') \le q_t(x,x') + \sup_{0<s\le h} q^*(s) \le \frac{c_1}{V(o,h)}\Big[\Big(\frac{h^2}{t}\Big)^{\eta/2} e^{-c_2 r^2/t} + h^{\eta/2} e^{-c_2 h^2/t}\Big]. \tag{4.7}$$

Now $(t/h)^{\eta/2} \le 1 \le \exp(c_2(h^2 - r^2)/t)$, and rearranging we see that the second term in (4.7) is dominated by the first, so that we obtain (4.2).

Finally, if $t \ge ch^2$ then

$$p_t(x,x) \le p_{ch^2}(x,x) \le \frac{c'}{V(o,h)},$$

and since $p_t(x,x') \le p_t(x,x)^{1/2} p_t(x',x')^{1/2}$, we obtain (4.2) for this range of t also.

The argument above proves (4.2) with constants $c_i = c_i(\theta, C_P, \lambda_P, C_0, \delta)$ for $i = 1, 2$. We can now use [8] again, as in the proof of Proposition 3.3, to obtain (4.2) with a universal constant c_2. □

Proof of Theorem 1.2. Let $y \in B(x, b/2)$ and $t \in [a^2, b^2]$ with $t \ge d(x,y)$. If $h \in [a,b]$ and $d(x,y) \le h/2$ then by Theorem 4.2,

$$p_t(x,y) \le \frac{c_1}{V(x,h)}\Big(\frac{h^2}{t} \vee 1\Big)^{\eta/2} \exp\Big(-c_2 \frac{d(x,y)^2}{t}\Big). \tag{4.8}$$

We now consider two cases.

Case 1. $t \ge 4d(x,y)^2$. Let $h = t^{1/2}$; then $h \in [a,b]$ and $d(x,y) \le h/2$, so the bound (1.5) follows immediately from (4.8). (Note that in this case the exponential term is of order 1.)

Case 2. $t < 4d(x,y)^2$. Let $h = 2d(x,y)$, and write $r = d(x,y)$, $u = r^2/t$. Then (4.8) gives

$$p_t(x,y) \le \frac{c_1}{V(x,2r)}\Big(\frac{4r^2}{t}\Big)^{\eta/2} e^{-c_2 u} \le \frac{c_1'}{V(x,\sqrt{t})} e^{-\frac{1}{2}c_2 u} \sup_{u>0} u^{\eta/2} e^{-\frac{1}{2}c_2 u},$$

from which again (1.5) follows. □

5 Weighted Poincaré Inequality

Let $\beta \ge 2$, $h \in \mathbb{N}$, $1 \le \alpha \le h$, and assume that $\mathcal{B} = B(o,h)$ satisfies $\mathcal{G}_1(\alpha)$. Let

$$\rho(x) = h - d(o,x) \quad \text{for each} \quad x \in \mathcal{B}, \tag{5.1}$$

so that $\rho(x) \le d(x, \partial_i \mathcal{B})$ and let

$$\psi(x) = \frac{1 \vee (h - d(o,x))^\beta}{h^\beta} = \frac{(1 \vee \rho(x))^\beta}{h^\beta}. \tag{5.2}$$

Let $\lambda_1 = \lceil 10 \vee 4\lambda_P \rceil$, $R_0 = \lceil 10 \vee 4\alpha \rceil$ and $\lambda_2 = \lambda_1^2/(\lambda_1 - 1)$. In this section we prove that $\mathcal{B}$ satisfies the following weighted Poincaré inequality.

Theorem 5.1 *Let $\mathcal{B} = B(o,h)$ satisfy $\mathcal{G}_1(\alpha)$, and let $\mathcal{B}_{\rm e} = B(o, h - \lambda_2 R_0)$. Then*

$$\min_{a\in\mathbb{R}} \sum_{x\in\mathcal{B}_{\rm e}} (f(x)-a)^2\psi(x)\mu_x \le c_1 h^2 \sum_{x,y\in\mathcal{B}} (f(x)-f(y))^2(\psi(x)\wedge\psi(y))\mu_{xy}. \tag{5.3}$$

Of course the value of a which minimizes the left side of (5.3) is the mean value of f on $\mathcal{B}_{\rm e}$ with respect to the measure

$$\hat\mu(A) = \sum_{x\in A} \hat\mu_x, \quad \text{where } \hat\mu_x = \psi(x)\mu_x. \tag{5.4}$$

We follow the proof in [21] that the (weak) PI implies the weighted PI. However, there are two differences from [21]: first, we only have volume doubling and the PI for balls of radius greater than α, and second the argument has to be done in a discrete setting. Together these mean that we cannot as in [21] take a simple Whitney covering with ball radii converging to zero as the balls approach the boundary of $\mathcal{B}$. These difficulties were already encountered in [3] in the context of percolation clusters in $\mathbb{Z}^d$. The argument here simplifies that in [3], which is based on Jerison's proof in [16], in two respects. First, because we look at $\mathcal{B}_{\rm e}$ rather than $\mathcal{B}$ on the left side of (5.3), we do not need to consider the covering close the boundary of $\mathcal{B}$. Second, we can avoid Jerison's delicate estimate on the structure of the ball covering near the boundary (see [3, Lemma 4.6]) by using an observation of Guozhen Lu [18]. (See p. 133 of [21] for more on the history of this improvement.) Since many of the proofs are standard, we will omit them and refer the reader to [3] or [21] for details.

We choose balls $B_i = B(x_i, r_i)$, $i = 1, \dots, N$ as follows. We take $x_1 = o$, and $r_1 = h/\lambda_1$. Given B_i for $i = 1, \dots, k-1$ set $A_k = \mathcal{B}_e - \cup_{i=1}^{k-1} B_i$. If A_k is empty then we let $N = k-1$ and stop; otherwise we choose $x_k \in A_k$ to minimise $d(o, x_k)$, and set

$$r_k = \frac{\rho(x_k)}{\lambda_1} = \frac{h - d(o,x_k)}{\lambda_1}.$$

We also define

$$B_i' = B(x_i, \tfrac{1}{2} r_i). \tag{5.5}$$

Lemma 5.2 *(a) The balls B_i cover $\mathcal{B}_e$.*
(b) We have $r_1 \ge r_2 \ge r_3 \ge \cdots \ge r_N \ge \frac{\lambda_1}{\lambda_1 - 1} R_0$.
(c) The balls B_i' are disjoint.
(d) If $y \in B(x_i, \kappa r_i)$ then

$$(\lambda_1 - \kappa) r_i \le \rho(y) \le (\lambda_1 + \kappa) r_i. \tag{5.6}$$

Proof. See [3]. □

We now define a tree structure on the set of indices $\{1, \dots, N\}$. Let $i \ge 2$. We need to consider two cases.
Case 1: if $d(o, x_i) > 2r_i$ let z_i be the point on the shortest path from x_i to o with $d(z_i, x_i) = \lceil 2r_i \rceil$. Since $\rho(z_i) > \rho(x_i) \ge \lambda_2 R_0$, there exists j such that $z_i \in B_j$. (If there is more than one such j we choose the smallest.) We have $|\rho(z_i) - \rho(x_j)| \le r_j$, and therefore

$$2r_i + \lambda_1 r_i \le d(z_i, x_i) + \rho(x_i) = \rho(z_i) < 2r_i + \lambda_1 r_i + 1,$$

$$2r_i + \lambda_1 r_i - r_j \le \lambda_1 r_j = \rho(x_j) \le 2r_i + \lambda_1 r_i + 1 + r_j.$$

Thus

$$\frac{\lambda_1+2}{\lambda_1+1} r_i \le r_j \le \frac{\lambda_1+2}{\lambda_1-1} r_i + \frac{1}{\lambda_1-1}. \tag{5.7}$$

In particular we have $r_j > r_i$, so $j < i$. We define an ancestor function $\mathfrak{a} : \{2, \dots, N\} \to \{1, \dots, N\}$ by taking $\mathfrak{a}(i) = j$.
Case 2: if $d(o, x_i) \le 2r_i$ then we set $\mathfrak{a}(i) = 1$. Since $2r_i \ge d(x_i, o) > r_1$, we obtain

$$\lambda_1 r_1 - 2r_i = h - 2r_i \le \rho(x_i) = \lambda_1 r_i \le \lambda_1 r_1 - r_1,$$

and hence

$$\frac{\lambda_1}{\lambda_1 - 1} r_i \le r_1 \le \frac{\lambda_1+2}{\lambda_1} r_i.$$

Thus in both cases there exist δ_1 and δ_2 depending only on λ_1 so that if $j = \mathfrak{a}(i)$ then

$$(1+\delta_1) r_i \le r_j \le (1+\delta_2) r_i; \tag{5.8}$$

further we can take $\delta_1 = (1+\lambda_1)^{-1}$ and $\delta_2 = 4/9$.

We write $\mathfrak{a}_k$ for the k-th iterate of $\mathfrak{a}$. Using this ancestor function, starting in B_i we obtain a sequence $B_i, B_{\mathfrak{a}(i)}, B_{\mathfrak{a}_2(i)}, \dots B_{\mathfrak{a}_m(i)}$ of successively larger balls which ends with the ball B_1.

Lemma 5.3 *(a) If $j = \mathfrak{a}(i)$ then $d(x_i, x_j) \le 1 + 2r_i + r_j$, and*

$$\mu\big(B(x_i, 2r_i) \cap B(x_j, 2r_j)\big) \ge C_V^{-3}\big(\mu(B_i') \vee \mu(B_j')\big). \tag{5.9}$$

(b) If $j = \mathfrak{a}_k(i)$ for some $k \ge 1$ then

$$d(x_i, x_j) \le \frac{4(1+\delta_1)}{\delta_1} r_j. \tag{5.10}$$

Consequently $B_i \subset B(x_j, K r_j) \cap \mathcal{B}$, where $K = 5\lambda_1$.

Proof. For (a) see the proof of (4.22) in [3], or [21, Lemma 5.3.7]. (b) follows easily from (5.8). □

For $1 \le i \le N$ let

$$B_i^* = B(x_i, 2r_i), \quad D_i = B(x_i, \tfrac{1}{2}\lambda_1 r_i), \quad D_i^* = B(x_i, K r_i) \cap \mathcal{B}. \tag{5.11}$$

Note that for each i,

$$B_i' \subseteq B_i \subseteq B_i^* \subseteq D_i \subseteq D_i^* \subseteq \mathcal{B}.$$

Lemma 5.4 *There exists a constant K_2 such that any $x \in \mathcal{B}$ is contained in at most K_2 of the sets D_i.*

Proof. The result follows directly from Lemma 2.2 on using Lemma 5.2(d) to control r_i. □

Let $f : \mathbb{V} \to \mathbb{R}$. For each finite subset $A \subset \mathbb{V}$, we set

$$\overline{f}_A = \mu(A)^{-1} \sum_{x \in A} f(x)\mu_x$$

for the mean value of f on A. For finite sets $A \subseteq A^*$ we write $P(A, A^*)$ for the smallest real number C such that the following PI holds:

$$\sum_{x \in A} (f(x) - \overline{f}_A)^2 \mu_x \leq C \tfrac{1}{2} \sum_{x,y \in A^*} (f(x) - f(y))^2 \mu_{xy}. \tag{5.12}$$

We also write

$$\begin{aligned} \mathcal{E}_{A^*}(f, f) &= \tfrac{1}{2} \sum_{x,y \in A^*} (f(x) - f(y))^2 \mu_{xy}, \\ \widetilde{\mathcal{E}}_{A^*}(f, f) &= \tfrac{1}{2} \sum_{x,y \in A^*} (f(x) - f(y))^2 \mu_{xy}(\psi(x) \wedge \psi(y)). \end{aligned}$$

The next result is used in chaining the Poincaré inequality.

Lemma 5.5 *Let $A_i \subset A_i^*$, $i = 1, 2$ be finite sets in $\mathbb{V}$, and $f : \mathbb{V} \to \mathbb{R}$. Then*

$$|\overline{f}_{A_1} - \overline{f}_{A_2}|^2 \leq \frac{2P(A_1, A_1^*)\mathcal{E}_{A_1^*}(f, f) + 2P(A_2, A_2^*)\mathcal{E}_{A_2^*}(f, f)}{\mu(A_1 \cap A_2)}. \tag{5.13}$$

Proof. See [21, Lemma 5.3.9]. □

Let $\widetilde{\mu}$ be the measure μ restricted to $\mathcal{B}$, that is

$$\widetilde{\mu}(A) = \mu(A \cap \mathcal{B}).$$

We use $\| \cdot \|_{L^2(\widetilde{\mu})}$ to denote the L_2 norm with respect to $\widetilde{\mu}$. The following lemma shows that $\widetilde{\mu}$ has the rVD property.

Lemma 5.6 *For any $x \in \mathcal{B}$ and $r \geq 2\alpha$,*

$$\widetilde{\mu}(B(x, 2r)) \leq c_1 \widetilde{\mu}(B(x, r)). \tag{5.14}$$

Proof. We consider two cases. Case 1: $r \geq h$. Let y be point of the shortest path from x to o which satisfies $d(x, y) = \lfloor d(o, x)/2 \rfloor$. Then $B(y, h/2) \subset B(x, h) \cap \mathcal{B}$. By Lemma 2.1,

$$\widetilde{\mu}(B(x, r)) \geq \mu(B(y, h/2)) \geq C_V^{-2} \left(\frac{d(o, y) + h}{h/2} \right)^{-\theta} \mu(B(o, h)) \geq 4^{-\theta} C_V^{-2} \widetilde{\mu}(B(x, 2r)).$$

So (5.14) holds in this case. Case 2: $2\alpha \leq r \leq h$. Similarly, there exists y such that $B(y, r/2) \subset B(x, r) \cap \mathcal{B}$, and again using Lemma 2.1, we obtain (5.14). □

Given that $\widetilde{\mu}$ is doubling, the next two lemmas are proved in the same way as in [21] – see Theorem 5.3.11 and Lemma 5.3.12.

Lemma 5.7 *Let $f : \mathbb{V} \to \mathbb{R}$. Set for each $u \in \mathcal{B}$,*

$$\mathcal{M}f(u) = \sup_{r \geq 2\alpha, v \in \mathcal{B} \text{ with } u \in B(v,r)} \frac{1}{\widetilde{\mu}(B(v,r))} \sum_{x \in B(v,r)} |f(x)|\widetilde{\mu}(x).$$

Then $\|\mathcal{M}f\|_{L^2(\widetilde{\mu})} \leq c\|f\|_{L^2(\widetilde{\mu})}$.

Lemma 5.8 *There exists a constant C such that, for any $a_i \geq 0$,*

$$\|\sum_i a_i 1_{D_i^*}\|_{L^2(\widetilde{\mu})} \leq C\|\sum_i a_i 1_{B_i'}\|_{L^2(\widetilde{\mu})}. \tag{5.15}$$

Let

$$\psi_i = \max_{D_i} \psi(x), \quad v_i = \frac{\max_{D_i} \psi(x)}{\min_{D_i} \psi(x)}. \tag{5.16}$$

Using the definitions of D_i and ψ it is easy to check that

Lemma 5.9 *There exists c_1 such that $v_i \leq c_1$ for each i.*

Using the rPI we have:

Lemma 5.10 *There exists c_1 such that*

$$P(B_i^*, D_i) \leq c_1 r_i^2, \quad 1 \leq i \leq N.$$

Set

$$P_* = \max_{1 \leq i \leq N} v_i P(B_i^*, D_i);$$

by the previous two lemmas we have

$$P_* \leq Ch^2. \tag{5.17}$$

Proof of Theorem 5.1. We can assume that $h > \lambda_2 R_0$, otherwise, the left side of the weighted PI is empty. Let $f : \mathbb{V} \to \mathbb{R}$ and write $\overline{f}_i = \overline{f}_{B_i^*}$. Then since the balls $\mathcal{B}_i^*$ cover $\mathcal{B}_{\mathrm{e}}$,

$$\begin{aligned}
\sum_{x \in \mathcal{B}_{\mathrm{e}}} |f(x) - \overline{f}_1|^2 \psi(x)\mu_x &\leq \sum_i \sum_{x \in B_i^*} |f(x) - \overline{f}_1|^2 \psi(x)\mu_x \\
&\leq 2\sum_i \sum_{x \in B_i^*} |f(x) - \overline{f}_i|^2 \psi(x)\mu_x + 2\sum_i \sum_{x \in B_i^*} |\overline{f}_i - \overline{f}_1|^2 \psi(x)\mu_x \\
&= S_1 + S_2.
\end{aligned}$$

Using the weak PI for $B_i^* \subset D_i$, and Lemma 5.4,

$$S_1 \leq 2\sum_{i=1}^N \sum_{x \in B_i^*} \psi_i |f(x) - \overline{f}_i|^2 \mu_x$$

$$\leq 2\sum_{i=1}^N \psi_i P(B_i^*, D_i)\mathcal{E}_{D_i}(f,f) \leq c\sum_{i=1}^N v_i P(B_i^*, D_i)\widetilde{\mathcal{E}}_{D_i}(f,f) \leq cK_2 P_* \widetilde{\mathcal{E}}_{\mathcal{B}}(f,f).$$

Thus by (5.17) we have

$$S_1 \leq ch^2 \widetilde{\mathcal{E}}_{\mathcal{B}}(f,f).$$

The sum S_2 is harder. Set

$$g(x) = \sum_j \psi_j^{1/2} |\overline{f}_1 - \overline{f}_j| 1_{B'_j}(x).$$

As the balls B'_j are disjoint,

$$g(x)^2 = \sum_j \psi_j |\overline{f}_1 - \overline{f}_j|^2 1_{B'_j}(x),$$

and

$$S_2 \leq 2\sum_j \psi_j |\overline{f}_1 - \overline{f}_j|^2 \mu(B_j^*) \leq c\sum_j \psi_j |\overline{f}_1 - \overline{f}_j|^2 \mu(B'_j) = c\sum_{x\in\mathcal{B}} g(x)^2 \mu_x = c\|g\|^2_{L^2(\widetilde{\mu})}.$$

To bound g we first fix $j \in \{2, \dots N\}$. Let $m = m_j$ be such that $\mathfrak{a}_m(j) = 1$, and write $j_k = \mathfrak{a}_k(j)$ for $k = 0, \dots, m$. Then by Lemma 5.5

$$\begin{aligned}\psi_j^{1/2} |\overline{f}_1 - \overline{f}_j| &\leq \sum_{k=0}^{m-1} \psi_j^{1/2} |\overline{f}_{j_k} - \overline{f}_{j_{k+1}}| \\ &\leq c\sum_{k=0}^{m-1} \left(\psi_j \frac{P(B_{j_k}^*, D_{j_k})\mathcal{E}_{D_{j_k}}(f,f) + P(B_{j_{k+1}}^*, D_{j_{k+1}})\mathcal{E}_{D_{j_{k+1}}}(f,f)}{\mu(B_{j_k}^* \cap B_{j_{k+1}}^*)} \right)^{1/2}.\end{aligned}$$

Since $\mathfrak{a}_k(j) < j$, we have $\psi_j \leq c\psi_{j_k}$ and so $\psi_j \mathcal{E}_{D_{j_k}}(f,f) \leq cv_{j_k} \widetilde{\mathcal{E}}_{D_{j_k}}(f,f)$. Let

$$b_i^2 = P(B_i^*, D_i) \frac{v_i \widetilde{\mathcal{E}}_{D_i}(f,f)}{\mu(B'_i)}.$$

By Lemma 5.3(a),

$$\psi_j \frac{P(B_{j_k}^*, D_{j_k})\mathcal{E}_{D_{j_k}}(f,f) + P(B_{j_{k+1}}^*, D_{j_{k+1}})\mathcal{E}_{D_{j_{k+1}}}(f,f)}{\mu(B_{j_k}^* \cap B_{j_{k+1}}^*)} \leq c(b_{j_k} + b_{j_{k+1}})^2.$$

Also, by Lemma 5.3(b) we have $B'_j \subset D_{j_k}^*$, and so

$$\begin{aligned}\psi_j^{1/2} |\overline{f}_1 - \overline{f}_j| 1_{B'_j}(x) &\leq c\sum_{k=0}^m b_{j_k} 1_{B'_j}(x) \\ &= c1_{B'_j}(x) \sum_{k=0}^m b_{j_k} 1_{D_{j_k}^*}(x) \leq c1_{B'_j}(x) \sum_{i=1}^N b_i 1_{D_i^*}(x).\end{aligned}$$

Note that the constant in the final bound does not depend on j. Using again the fact that B'_j are disjoint, so $\sum 1_{B'_j} \le 1$,

$$g(x) = \sum_j \psi_j^{1/2} |\overline{f}_1 - \overline{f}_j| 1_{B'_j}(x) \le c \sum_j 1_{B'_j}(x) \sum_{i=1}^N b_i 1_{D_i^*}(x) \le c \sum_{i=1}^N b_i 1_{D_i^*}(x).$$

Hence using Lemma 5.8,

$$\begin{aligned} S_2 &\le c\|g\|^2_{L^2(\widetilde{\mu})} \le c' \Big\| \sum_i b_i 1_{D_i^*}(x) \Big\|^2_{L^2(\widetilde{\mu})} \le c'' \Big\| \sum_i b_i 1_{B'_i}(x) \Big\|^2_{L^2(\widetilde{\mu})} \\ &= c'' \sum_i b_i^2 \mu(B'_i) \le c'' P_* \sum_i \widetilde{\mathcal{E}}_{D_i}(f,f) \le c''' K_2 h^2 \widetilde{\mathcal{E}}_{\mathcal{B}}(f,f). \end{aligned}$$

□

6 Lower bound estimates

In this section we shall use the method introduced by Fabes and Stroock [11] to obtain a 'near diagonal' lower bound. We write $p_t^A(x,y)$ for the heat kernel for X killed on exiting A. The following Lemma enables us to relax the "p_0-condition" which is used in many previous papers on lower bounds.

Lemma 6.1 *Let $1 \le \alpha \le h$, let $h^* = (2h) \vee (h + \lambda_P \alpha)$, and suppose that $\mathcal{B}^* = B(o,h^*)$ satisfies $\mathcal{G}_1(\alpha)$. Then there exists a subtree $\mathcal{T} = (V(\mathcal{T}), E(\mathcal{T}))$ such that*
(a) $V(\mathcal{T}) \supset B(o,h)$,
(b) $V(\mathcal{T}) \subset B(o, h + \lambda_P \alpha)$,
(c) $\mu_{xy} \ge c_1 h^{-c_2} \max\{\mu_x, \mu_y\}$ for any $(x,y) \in E(\mathcal{T})$.

Proof. It follows from (2.1), that $|B(o,h^*)| \le ch^\theta$. Let $\mathcal{T}_0$ be the tree which contains only the vertex $\{o\}$. Suppose we have found a sequence of trees $\mathcal{T}_0, \dots, \mathcal{T}_k$ which satisfy (b) and (c). For simplicity, we write $\mathcal{T}_k$ for the vertex set of the tree $\mathcal{T}_k$.

If $B(o,h) \subset \mathcal{T}_k$ then we set $\mathcal{T} = \mathcal{T}_k$ and stop. Otherwise, since $B(o,h)$ is connected, there exist $x, y \in B(o,h)$ with $x \sim y$, $x \in \mathcal{T}_k$, and $y \notin \mathcal{T}_k$. Set $B_0 = B(x,\alpha)$, $B_1 = B(x, \lambda_P \alpha)$ and $W = B_0 - \mathcal{T}_k$.

Letting $f = 1_{\mathcal{T}_k}$, and employing the PI on $B(x,\alpha)$, we have

$$\frac{\mu(W)(V(x,\alpha) - \mu(W))}{V(x,\alpha)} \le C_P \alpha^2 \sum_{w \in B_1 \cap \mathcal{T}_k, z \in B_1 - \mathcal{T}_k} \mu_{wz}.$$

Choose $u \in B_1 \cap \mathcal{T}_k$ and $v \in B_1 - \mathcal{T}_k$ to maximize μ_{wz} in the sum above. Then

$$\sum_{w \in B_1 \cap \mathcal{T}_k, z \in B_1 - \mathcal{T}_k} \mu_{wz} \le |B_1|^2 \mu_{uv} \le ch^{2\theta} \mu_{uv}.$$

Since $x \notin W$ and $y \in W$, using (1.4),

$$\frac{\mu(W)(V(x,\alpha)-\mu(W))}{V(x,\alpha)} \geq \frac{\mu_y \mu_x}{V(x,\alpha)} \geq c^{-2}h^{-2\theta}V(y,\alpha).$$

By the rVD we have $V(y,\alpha) \geq cV(y, 1+\lambda_P\alpha) \geq c\mu_u$, so combining the inequalities above we obtain

$$\mu_{uv} \geq ch^{-4\theta-2}\mu_u.$$

Similarly $\mu_{uv} \geq ch^{-4\theta-2}\mu_v$. Now let $V(\mathcal{T}_{k+1}) = V(\mathcal{T}_k) \cup \{v\}$ and $E(\mathcal{T}_{k+1}) = E(\mathcal{T}_k) \cup \{(u,v)\}$, so that $\mathcal{T}_{k+1}$ is a tree and still satisfies (b) and (c). As $B(o, h+\lambda_P\alpha)$ is finite, the process must terminate and we obtain a tree $\mathcal{T}$ which satisfies (a), (b) and (c). □

Using the subtree $\mathcal{T}$ we can obtain a weak uniform lower bound for $p_s^{\mathcal{B}^*}(x,y)$.

Lemma 6.2 *Let $1 \leq \alpha \leq h$, h^* and $\mathcal{B}^*$ be as in the previous Lemma, and suppose that $\mathcal{B}^*$ satisfies $\mathcal{G}_1(\alpha)$. Let $T = 10^{-5}(c_{4.1.2}h^2 \wedge 1)$. Then for any $x, y \in B(o,h)$ and $s \in [T, h^2]$,*

$$\log\left(\mu_y p_s^{\mathcal{B}^*}(x,y)\right) \geq -c_1 h^{1+\theta}. \tag{6.1}$$

Proof. By Lemma 6.1, there exists a path $x = x_0, \dots, x_m = y$ from x to y with $m \leq |\mathcal{B}^*| \leq ch^\theta$, such that at each step the probability that X jumps from x_i to x_{i+1} is at least $c_1h^{-c_2}$. Using the representation of X as a discrete time Markov chain run according to a Poisson process $(N_t, t \geq 0)$, we have

$$\begin{aligned}\mu_y p_s^{\mathcal{B}^*}(x,y) &\geq \mathbb{P}(N_s = m)(c_1h^{-c_2})^m = \frac{s^m}{m!}e^{-s}(c_1h^{-c_2})^m \\ &\geq \exp\left(m\log T - (s + m\log m + m|\log(c_1h^{-c_2})|)\right) \geq \exp(-c'h^{\theta+1}).\end{aligned}$$

□

Theorem 6.3 *Let $\delta \in (0,1]$. There exists a constant c_1 such that if $h \geq 1$, $(2h)^\delta \geq 2\lambda_P$, and $\mathcal{B}^* = B(o,2h)$ satisfies $\mathcal{G}_1((2h)^{1-\delta})$, then for any $x_1, x_2 \in B(o,h/2)$ and $t \in [10^{-5}h^2, h^2]$,*

$$p_t^{\mathcal{B}^*}(x_1,x_2) \geq \frac{c_1}{V(o,h)}. \tag{6.2}$$

Proof. Let $\alpha = (2h)^{1-\delta}$, so that $\alpha\lambda_P \leq h$ and hence h^* as defined in Lemma 6.1 is $2h$. If $h \leq c_0$ then by Lemma 6.2 for $x, y \in B(o,h/2)$ and $t \in [10^{-5}h^2, h^2]$,

$$\log\left(V(o,h)p_t^{\mathcal{B}^*}(x,y)\right) \geq -c_{6.2.1}h^{1+\theta} \geq -c_{6.2.1}c_0^{1+\theta},$$

so that (6.2) holds in this case. We will choose the constant c_0 during the course of the proof.

As in Section 5, we set $\lambda_1 = \lceil 10 \vee 4\lambda_P \rceil$, $R_0 = \lceil 10 \vee 4\alpha \rceil$, $\lambda_2 = \lambda_1^2/(\lambda_1 - 1)$, $\mathcal{B} = B(o,h)$, $\mathcal{B}_{\mathrm{e}} = B(o, h - \lambda_2 R_0)$ and $\rho(x) = h - d(o,x)$. Let

$$\beta = \frac{2(\theta+1)}{\delta},$$

and define ψ by (5.2).

Fix $z \in B(o,h/2)$, let $T = 10^{-5}(c_{4.1.2}\wedge 1)h^2$ and $I = [\frac{T}{2}, T]$. We choose c_0 large enough so that $B(o,3h/4) \subset \mathcal{B}_{\mathrm{e}}$ and $h \geq c_{4.1.1}$. Employing Lemma 4.1 on $\mathcal{B}^*$, for each $t \leq T$ we have

$$\mathbb{P}_z(X_t \in B(o,3h/4), \tau_{\mathcal{B}^*} > t) \geq \mathbb{P}_z(\tau_{z,h/4} > t) \geq \tfrac{1}{2}. \tag{6.3}$$

Applying Theorem 4.2 on $\mathcal{B}^*$, there exists c^* such that for each $x \in \mathcal{B}$ and $t \in I$,

$$p_t^{\mathcal{B}^*}(z,x) \leq p_t(z,x) \leq \frac{1}{c^* V(o,2h)}.$$

Let $\hat{\mu}_x = \psi(x)\mu_x$ be as in (5.4), and set $V_0 = \hat{\mu}(\mathcal{B})$. By the volume doubling property, we have

$$c_1^{-1} V(o,h) \leq 2^{-\beta} V(o,h/2) \leq V_0 \leq V(o,h). \tag{6.4}$$

Write $u(t,x) = u_t(x) = p_t^{\mathcal{B}^*}(z,x)$, and set

$$w(t,x) = w_t(x) = \log\big(c^* V_0 u(t,x)\big).$$

Thus if $x \in \mathcal{B}$ and $t \in I$ then

$$c^* V_0 p_t^{\mathcal{B}^*}(z,x) \leq 1 \quad \text{and} \quad w_t(x) \leq 0. \tag{6.5}$$

For any $t > 0$, let

$$H(t) = H(z,t) = V_0^{-1} \sum_{x\in\mathcal{B}} \psi(x) w_t(x) \mu_x.$$

As in [3], by differentiating, and then using an inequality of Stroock and Zheng [22], we obtain

$$V_0 H'(t) \geq S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
S_1 &= \tfrac{1}{4} \sum_{x\in\mathcal{B}} \sum_{y\in\mathcal{B}} (\psi(x)\wedge\psi(y))(w_t(x)-w_t(y))^2 \mu_{xy},\\
S_2 &= -\tfrac{1}{4} \sum_{x\in\mathcal{B}} \sum_{y\in\mathcal{B}} \frac{(\psi(x)-\psi(y))^2}{\psi(x)\wedge\psi(y)} \mu_{xy},\\
S_3 &= -\tfrac{1}{2} \sum_{x\in\mathcal{B}} \sum_{y\notin\mathcal{B}} \psi(x) \left(1 - \frac{u_t(y)}{u_t(x)}\right)^+ \mu_{xy}.
\end{aligned}$$

To bound S_2 note that if $x \sim y$ with $x,y \in \mathcal{B}$ and $k = 1 \vee (\rho(x)\wedge\rho(y))$, then

$$\begin{aligned}
\frac{(\psi(x)-\psi(y))^2}{\psi(x)\wedge\psi(y)} \leq& \frac{\left(\left(\frac{k+1}{h}\right)^\beta - \left(\frac{k}{h}\right)^\beta\right)^2}{\left(\frac{k}{h}\right)^\beta} \leq \frac{ch^{-2\beta}(k+1)^{2\beta-2}}{\left(\frac{k}{h}\right)^\beta}\\
\leq& c' h^{-\beta}(k+1)^{\beta-2} \leq c' h^{-2}.
\end{aligned}$$

So

$$S_2 \geq -c' h^{-2} \sum_{x\in\mathcal{B}} \mu_x = -c' h^{-2} V(o,h) \geq -c'' h^{-2} V_0.$$

Also, if $x \in \mathcal{B}$, $y \notin \mathcal{B}$ and if $\mu_{xy} > 0$, then $\psi(x) = h^{-\beta} \leq h^{-2}$, so that

$$S_3 \geq -\frac{1}{2} \sum_{x\in\mathcal{B}} \sum_{y\notin\mathcal{B}} \mu_{xy} h^{-2} \geq -h^{-2} \sum_{x\in\mathcal{B}} \mu_x \geq -c_1 h^{-2} V_0.$$

To bound the remaining term S_1, we set

$$H_{\mathrm{e}}(t) = \frac{1}{\hat{\mu}(\mathcal{B}_{\mathrm{e}})} \sum_{x\in\mathcal{B}_{\mathrm{e}}} w_t(x) \hat{\mu}_x.$$

Then using Theorem 5.1, for any $t > 0$,

$$\begin{aligned} H'(t) \geq & \frac{1}{4} V_0^{-1} \sum_{x\in\mathcal{B}} \sum_{y\in\mathcal{B}} (\psi(x) \wedge \psi(y))(w_t(x) - w_t(y))^2 \mu_{xy} - c_2 h^{-2} \\ \geq & - c_2 h^{-2} + c_3 h^{-2} V_0^{-1} \sum_{x\in\mathcal{B}_{\mathrm{e}}} (w_t(x) - H_{\mathrm{e}}(t))^2 \psi(x) \mu_x. \end{aligned} \tag{6.6}$$

Writing

$$\Psi_e = V_0^{-1} \sum_{x\in\mathcal{B}_{\mathrm{e}}} (w_t(x) - H_{\mathrm{e}}(t))^2 \psi(x) \mu_x \ \text{ and } \ \Psi = V_0^{-1} \sum_{x\in\mathcal{B}} (w_t(x) - H(t))^2 \psi(x) \mu_x,$$

(6.6) becomes

$$h^2 H'(t) \geq -c_2 + c_3 \Psi + c_3 (\Psi_e - \Psi). \tag{6.7}$$

We estimate the terms Ψ and $(\Psi_e - \Psi)$ in (6.7) separately. For Ψ we follow [11]. Fix $t \in I$. Since $v \to (\log v - l)^2/v$ is decreasing on $[e^{2+l}, \infty)$ and (6.5), we have

$$\Psi = \sum_{x\in\mathcal{B}} \frac{\left(\log\left(\, c^* V_0 u_t(x)\right) - H(t)\right)^2}{V_0 u_t(x)} \psi(x) u_t(x) \mu_x \geq c^* H(t)^2 \sum_{x\in\mathcal{B}: c^* V_0 u_t(x) > e^{2+H(t)}} \psi(x) u_t(x) \mu_x.$$

Then since $\psi(x) \geq c$ on $B(o, 3h/4)$,

$$\begin{aligned} & \sum_{x\in\mathcal{B}: c^* V_0 u_t(x) > e^{2+H(t)}} \psi(x) u_t(x) \mu_x \\ \geq & \sum_{x\in B(o,3h/4)} \psi(x) u_t(x) \mu_x - \sum_{x\in B(o,3h/4): c^* V_0 u_t(x) \leq e^{2+H(t)}} \psi(x) u_t(x) \mu_x \\ \geq & c \sum_{x\in B(o,3h/4)} u_t(x) \mu_x - \sum_{x\in B(o,3h/4): c^* V_0 u_t(x) \leq e^{2+H(t)}} \psi(x) (c^* V_0)^{-1} e^{2+H(t)} \mu_x \\ \geq & c \mathbb{P}_z(X_t \in B(o, 3h/4), \tau_{\mathcal{B}^*} > t) - e^{2+H(t)} (c^* V_0)^{-1} V(o, 3h/4) \\ \geq & c_4 - c_5 e^{H(t)}, \end{aligned}$$

where the last inequality is by (6.3). Combining these estimates, we deduce that

$$\Psi \geq c^* H(t)^2 \left(c_4 - c_5 e^{H(t)} \right). \tag{6.8}$$

We now control $|\Psi_e - \Psi|$. Write $V_e = \hat{\mu}(\mathcal{B}_e)$, $J = V_0 H(t)$ and $J_e = V_e H_e(t)$. A direct calculation gives,

$$\begin{aligned} V_0(\Psi - \Psi_e) &= \sum_{x \in \mathcal{B}} (w_t(x) - H(t))^2 \hat{\mu}_x - \sum_{x \in \mathcal{B}_\mathrm{e}} (w_t(x) - H_\mathrm{e}(t))^2 \hat{\mu}_x \\ &= \sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x + \frac{J_e^2}{V_e} - \frac{J^2}{V_0}. \end{aligned}$$

As $w_t(x) \leq 0$ for all $x \in \mathcal{B}$ we have $|J_e| \leq |J|$ and hence

$$\begin{aligned} \Big| \frac{J_e^2}{V_e} - \frac{J^2}{V_0} \Big| &= \frac{1}{V_0 V_e} \Big| (J_e^2 - J^2) V_0 + J^2 (V_0 - V_e) \Big| \\ &\leq \frac{|J + J_e|}{V_e} |J - J_e| + \frac{J^2}{V_0 V_e} (V_0 - V_e) \\ &\leq \frac{2 V_0 |H(t)|}{V_e} |J - J_e| + \frac{V_0 (V_0 - V_e)}{V_e} H(t)^2. \end{aligned}$$

For each $x \in \mathcal{B} - \mathcal{B}_\mathrm{e}$ we have $\psi(x) \leq c \alpha^\beta h^{-\beta} = c' h^{-\beta\delta} = c' h^{-2\theta - 2}$ and therefore using (6.4)

$$\frac{V_0 - V_e}{V_0} = \frac{\hat{\mu}(\mathcal{B} - \mathcal{B}_e)}{V_0} = V_0^{-1} \sum_{x \in \mathcal{B} - \mathcal{B}_e} \psi(x) \mu_x \leq c'' h^{-2\theta - 2}. \tag{6.9}$$

We choose c_0 large enough so that the final term in (6.9) is less than $\frac{1}{4}$. Hence $4V_e \geq 3V_0$ and using Cauchy-Schwarz on the term $|J - J_e|$

$$\begin{aligned} V_0 |\Psi - \Psi_e| &\leq \sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x + 4|H(t)||J - J_e| + 2H(t)^2 (V_0 - V_e) \\ &\leq \sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x + 4|H(t)|(V_0 - V_e)^{1/2} \Big(\sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x \Big)^{1/2} + 2H(t)^2 (V_0 - V_e) \\ &\leq 3 \sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x + 4H(t)^2 (V_0 - V_e). \end{aligned}$$

By Lemma 6.2 and the choice of β we have for $t \in I$, $x \in \mathcal{B}$,

$$0 \leq -w_t(x) \leq c h^{1+\theta} = c' (h/\alpha)^{\beta/2}.$$

Therefore,

$$\sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} w_t(x)^2 \hat{\mu}_x \leq c \sum_{x \in \mathcal{B} - \mathcal{B}_\mathrm{e}} (h/\alpha)^\beta (\alpha/h)^\beta \mu_x \leq c V_0.$$

Combining the bounds above we obtain

$$V_0 |\Psi_e - \Psi| \leq c V_0 + c' V_0 H(t)^2 h^{-2\theta - 2}. \tag{6.10}$$

So from (6.7), (6.8) and (6.10) we get, for sufficiently large h,

$$\begin{aligned} h^2 H'(t) &\ge -c - c' h^{-2\theta-2} H(t)^2 + c'' H(t)^2 (c_4 - c_5 e^{H(t)}) \\ &\ge -c + c'' H(t)^2 (c_4' - c_5 e^{H(t)}). \end{aligned} \tag{6.11}$$

The inequality (6.11) implies that there exist c_6, c_7 such that if $\sup_{t\in I} H(t) < -c_6$, then

$$T H'(t) \ge c_7 H(t)^2, \quad t \in I. \tag{6.12}$$

As in [3] it is straightforward to see that (6.12) together with that fact that $ctT^{-1} + H'(t)$ is increasing, implies that there exists c_{10} such that

$$H(t) = H(z,t) \ge -c_{10}, \text{ for } T \le t \le h^2.$$

To conclude the argument, we use Jensen's inequality as in [11, 3] to obtain

$$p_{2t}^{\mathcal{B}^*}(x_1, x_2) \ge (c^* V_0)^{-1} e^{-c(H(x_1,t)+H(x_2,t))} \ge \frac{c_{11}}{V_0} \ge \frac{c_{12}}{V(o,h)},$$

and adjusting the constant we get the desired result. □

Recall from Section 1 the definition of (pGB).

Corollary 6.4 *Let $\delta \in (0,1]$. Let $h \ge 1$ and $h^* = h \vee (2\lambda_P)^{1/\delta}$. Suppose that $B(o,h^*)$ satisfies $\mathcal{G}_1((h^*)^{1-\delta})$.*
(a) If $h \ge (2\lambda_P)^{1/\delta}$, then (pGB) holds on $B(o,h)$.
(b) For $x, y \in B(o,h/2)$ and $t \in [10^{-5}h^2, h^2]$,

$$p_t(x,y) \ge \frac{c_1}{V(o,h)}. \tag{6.13}$$

Proof. (a) The upper bound (1.11) holds by Theorem 4.2, the lower bound (1.12) by Theorem 6.3.
(b) If $h \ge (2\lambda_P)^{1/\delta}$, then this is immediate from (a). Otherwise we have $h^* = (2\lambda_P)^{1/\delta}$. Employing Lemma 6.2 on $B(o,h^*/2)$, we deduce that for $x, y \in B(o,h/2)$ and $t \in [10^{-5}h^2, h^2]$,

$$\log\left(\mu_y p_t^{B(o,h^*)}(x,y)\right) \ge -c_{6.2.1}(h^*/2)^{1+\theta} \ge -c_{6.2.1}(2\lambda_P)^{(1+\theta)/\delta},$$

which gives the lower bound (6.13). □

Remark 6.5 If we just require that $B(o,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ then we would need a condition such as $h^\delta \ge c\lambda_P$ to obtain the lower bound in (pGB). To see this, consider a cycle which has vertex set $\mathbb{V} = \{-4h, -4h+1, \cdots, 4h\}$ and edge weights $\mu_{0,1} = \varepsilon, \mu_{-4h,4h} = \varepsilon^{-1}$, and $\mu_{n,n+1} = 1$ for $-4h \le n < 4h$. We will fix $h \in \mathbb{N}$, let $\varepsilon \in (0,1)$ be arbitrarily small and consider $B(0,h) = \{-h, \cdots, h\}$.

Choose $\theta = 1$ and $C_0 = 3$. While volume doubling does not hold for the whole cycle, we still do have for any $x \in B(0,h)$ and $r \le h$, that $V(x,2r) \subset [-3h,3h]$ and hence

$V(x,2r) \le 2V(x,r) = 2^\theta V(x,r)$. We also have $V(x,r) \le C_0 r$ for any $x \in B(0,h)$ and $1 \le r \le h$.

Set $\lambda_P = 10$, $C_P = 1$ and $\alpha = h/2$. Then $B(x, \lambda_P r) = \mathbb{V}$ for any $x \in B(0,h)$ and $r \ge \alpha$. Consider first the graph with modified weights $\tilde\mu_{0,1} = 0$, $\tilde\mu_{-4h,4h} = 1$, and $\tilde\mu_e = 1$ for all other edges. By the strong PI given by Lemma 2.6, $B(x,r)$ with weights $\tilde\mu_e$ satisfies (weak) PI. Therefore, since $\tilde\mu_y \asymp \mu_y$ in $B(x,r)$ and $\tilde\mu_e \le \mu_e$ in the cycle, the PI holds on $B(x,r)$ under the original weights μ_e. Thus $B(x,r)$ is good for any $x \in B(0,h)$ and $r \in [\alpha, h]$, and so $B(0,h)$ satisfies $\mathcal{G}_1(\alpha)$.

Now choose δ small enough so that $h^\delta < 2$, and thus $\alpha \le h^{1-\delta}$. Then $B(0,h)$ satisfies $\mathcal{G}_1(h^{1-\delta})$ for any $\varepsilon \in (0,1)$. Thus all the conditions of the corollary above are satisfied except for $h \ge c_0$. However $p_{h^2}(0,1) \to 0$ as ε goes to zero and the lower bound of (pGB) fails.

7 Parabolic Harnack Inequality

In this section we relate the conditions (pGB) and (PHI) defined in Section 1 with volume growth and the Poincaré inequality. We begin with some consequences of (pGB).

Lemma 7.1 *If (pGB) holds on $B(o,h)$ with constants C_1, η then for any $x \in B(o,h/2)$,*

$$V(o,h) \le C_1 e \mu_x h^\eta.$$

Proof. Let $x \in B(o,h/2)$. Using (pGB) we obtain

$$p_1(x,x) \le \frac{C_1 h^\eta}{V(o,h)}.$$

Combining this with the fact that $p_t(x,x) \ge \mu_x^{-1} e^{-t}$, we have the result. □

We now show that (pGB) implies (PHI), using the balayage argument of [5, 6].

Theorem 7.2 *Let $b > a \ge 10^3$, and $C_1 \ge 1$, $\eta \ge 1$. Suppose that for each $x \in B(o,h)$ and $h \in [a,b]$, (pGB) with constants C_1, η holds on $B(x,h)$. Then there exists $C_H = C_H(C_1,\eta)$ such that for any $x \in B(o, c_1 b)$ and $R \in [c_2 a, c_3 b]$, PHI holds on $Q(x,R,R^2)$.*

Proof. Let $B(x,h)$ satisfy (pGB). Write $B_0 = B(x, \frac{1}{8}h)$, $B_1 = B(x, \frac{1}{4}h)$, $B = B(x,h)$ and $T = \frac{1}{20}h^2$ and let $u(t,y)$ be nonnegative and caloric in $Q(x,h,T)$. Using a balayage result from potential theory we can write for any $t \in (0,T]$ and $y \in B_1$,

$$u(t,y) = \sum_{z \in B_1} p_t^B(y,z) u(0,z) \mu_z + \sum_{z \in \partial_i B_1} \int_0^t p_{t-s}^B(y,z) k_s(z) \mu_z ds, \tag{7.1}$$

where $k_s(z) \ge 0$. For the proof see [6, Proposition 3.3], and particularly equation (3.9).

Fix $t_1 \in [\frac{1}{4}T, \frac{1}{2}T], t_2 \in [\frac{3}{4}T, T]$ and $y_1, y_2 \in B_0$. Using (1.11), we get for $z \in B_1$,

$$p_{t_1}^B(y_1,z) \le \frac{C_1}{V(x,h)} \Big(\frac{h^2}{t_1}\Big)^{\eta/2} \le \frac{c}{V(x,h)},$$

and for $s \le T/2$,

$$p^B_{t_2-s}(y_2,z) \ge \frac{1}{C_1 V(x,h)}.$$

Further for $s \in [0, t_1-h]$ and $z \in \partial_i B_1$, we have $d(y_1,z) \ge \frac{h}{10}$ and

$$p^B_{t_1-s}(y_1,z) \le \frac{C_1}{V(x,h)}\Big(\frac{h^2}{t_1-s}\Big)^{\eta/2} \exp\left(-\frac{h^2/100}{C_1(t_1-s)}\right) \le \frac{c}{V(x,h)}. \tag{7.2}$$

From the long range bounds (3.8), (3.9), and Lemma 7.1, we can also obtain (7.2) for each $s \in [t_1-h, t_1]$. Hence

$$\begin{aligned} u(t_2,y_2) &= \sum_{z\in B_1} p^B_{t_2}(y_2,z)u(0,z)\mu_z + \sum_{z\in\partial_i B_1}\mu_z \int_0^{t_2} p^B_{t_2-s}(y_2,z)k_s(z)ds \\ &\ge \sum_{z\in B_1} p^B_{t_2}(y_2,z)u(0,z)\mu_z + \sum_{z\in\partial_i B_1}\mu_z \int_0^{t_1} p^B_{t_2-s}(y_2,z)k_s(z)ds \\ &\ge c_1 \sum_{z\in B_1} p^B_{t_1}(y_1,z)u(0,z)\mu_z + c_1 \sum_{z\in\partial_i B_1}\mu_z \int_0^{t_1} p^B_{t_1-s}(y_1,z)k_s(z)ds = c_1 u(t_1,y_1). \end{aligned}$$

Therefore, if u is nonnegative and caloric in $Q(x,h,T)$ then

$$\sup_{[\frac14 T,\frac12 T]\times B_0} u \le c_1 \inf_{[\frac34 T,T]\times B_0} u. \tag{7.3}$$

Given this, a standard chaining argument then proves the PHI. □

Next we look at consequences of the PHI. Our arguments follow those in [10, 21].

Lemma 7.3 *Suppose PHI holds with constant C_H for $Q(x_0,R,R^2)$. If u is non-negative and caloric in $(0,\infty)\times B(x_0,R)$ then for $k \ge 1$ there exists a constant $c_k = c_k(C_H)$ such that, writing $B' = B(x_0,R/2)$, $T = R^2$,*

$$\sup_{[\frac14 T,\frac12 T]\times B'} u \le c_k \inf_{[(k-\frac14)T,kT]\times B'} u. \tag{7.4}$$

Proof. The standard PHI gives the case $k=1$. Now setting $v(s,x) = u(s-T/2,x)$ and applying the PHI to v we obtain

$$\sup_{[\frac34 T,T]\times B'} u \le C_H \inf_{[\frac54 T,\frac32 T]\times B'} u,$$

and continuing in this way we obtain (7.4). □

Next we prove that the PHI gives volume doubling.

Lemma 7.4 *Suppose that PHI holds with constant C_H for $Q(o, \lambda R, \lambda^2 R^2)$ for $\frac{1}{2} \le \lambda \le 2$.*
(a) We have for $1 \le s \le 2$,

$$p_{sR^2}(x,y) \le \frac{C_H}{V(o,R)} \text{ for } x \in B(o,R),\ y \in \mathbb{V}. \tag{7.5}$$

(b) There exists $c_1 = c_1(C_H)$ such that

$$p_{2R^2}^{B(o,2R)}(x,y) \ge \frac{c_1}{V(o, 2^{-1/2}R)} \text{ for } x, y \in B(o, R/4). \tag{7.6}$$

(c) In particular there exists $c_2 = c_2(C_H)$ such that

$$V(o,R) \le c_2 V(o, 2^{-1/2}R). \tag{7.7}$$

Proof. (a) Let $y \in \mathbb{V}$; applying the PHI to $p_\cdot(y,\cdot)$ in $Q(o,2R,4R^2)$ we obtain for $x, x' \in B(o,R)$ and $s \in [1,2]$,

$$p_{sR^2}(y,x) \le C_H p_{4R^2}(y,x'). \tag{7.8}$$

Hence summing over $x' \in B(o,R)$,

$$p_{sR^2}(y,x)V(o,R) \le C_H \sum_{x' \in B(o,R)} p_{R^2}(y,x')\mu_{x'} \le C_H,$$

which implies (7.5).

(b) Let $B = B(o,2R)$, $A = B(o, 2^{-1/2}R)$, $B' = B(o, R/2)$, $B'' = B(o, R/4)$, and $t_1 = R^2/8$. We define a function $u(t,x)$ which is caloric on $(0,\infty) \times B'$ by setting

$$u(t,x) = \begin{cases} 1 & \text{if } t \le t_1, \\ \sum_{z \in A} \mu_z p^B_{t-t_1}(x,z) & \text{if } t > t_1. \end{cases}$$

Apply the PHI to u in $Q(o, R/2, R^2/4)$ to obtain

$$1 = u(t_1, x) \le C_H \sum_{z \in A} \mu_z p^B_{t_1}(x,z), \text{ for } x \in B''.$$

Applying Lemma 7.3 in the region $(0,\infty) \times B'$ to $p^B_\cdot(z,\cdot)$ gives

$$p^B_{t_1}(x,z) \le c p^B_{4t_1}(x,z), \text{ for } x \in B'', z \in A.$$

Now we can use use the PHI in $Q(o, 2^{1/2}R, 2R^2)$ to obtain

$$p^B_{4t_1}(x,z) \le C_H p^B_{2R^2}(x,y) \text{ for } z \in A, x, y \in B''.$$

Hence

$$1 \le c \sum_{z \in A} \mu_z p^B_{2R^2}(x,y) = cV(o, 2^{-1/2}R) p^B_{2R^2}(x,y),$$

which proves (7.6). Combining the inequalities (7.5) (with $s = 2$) and (7.6) gives (7.7). □

Theorem 7.5 *Suppose that PHI holds with constant C_H for $Q(x_0, \lambda R, \lambda^2 R^2)$ for $\frac{1}{2} \le \lambda \le 8$. Then there exists $c_1 = c_1(C_H)$ such that for any function f on $\mathbb{V}$, we have the PI*

$$\min_{a\in\mathbb{R}} \sum_{y\in B(x_0,R)} (f(y)-a)\mu_y \le c_1 R^2 \sum_{y,z\in B(x_0,9R)} (f(y)-f(z))^2 \mu_{yz}.$$

Proof. (See [10, Theorem 3.11].) Set $B_0 = B(x_0, 9R)$, $B_1 = B(x_0, R)$ and let $T = 2(4R)^2$. Let $f : \mathbb{V} \to \mathbb{R}$. Let $p'_t(x, y)$ be the density of heat kernel of the continuous time random walk X' on the graph B_0 with edge weight $\mu' = \mu\big|_{B_0\times B_0}$. (So X' is X with reflection at the boundary of B_0.) By Lemma 7.4, for $x, y \in B(x_0, R)$,

$$p'_T(x,y) \ge p_T^{B(x_0,8R)}(x,y) \ge \frac{c}{V(x_0, 2^{3/2}R)}.$$

Let P'_s be the Markov operator

$$P'_s g(x) = \sum_{y\in B_0} p'_s(x,y) g(y) \mu'_y.$$

Let $\overline{f} = \sum_{x\in B_1} f(x)\mu_x$. For $x \in B_1$ we set $a_x = P'_T f(x)$. Then

$$\begin{aligned} P'_T[(f-a_x)^2](x) &= \sum_{y\in B_0} \mu'_y p'_T(x,y)(f(y)-a_x)^2 \\ &\ge \sum_{y\in B_1} \mu_y p'_T(x,y)(f(y)-a_x)^2 \\ &\ge \frac{c}{V(x_0, 2^{3/2}R)} \sum_{y\in B_1} (f(y)-a_x)^2 \mu_y \\ &\ge \frac{c}{V(x_0, 2^{3/2}R)} \sum_{y\in B_1} (f(y)-\overline{f})^2 \mu_y. \end{aligned}$$

So, summing over $x \in B_1$ we obtain

$$\frac{cV(x_0,R)}{V(x_0, 2^{3/2}R)} \sum_{y\in B_1} (f(y)-\overline{f})^2 \mu_y \le \sum_{x\in B_1} P'_T[(f-a_x)^2](x)\mu_x.$$

A direct calculation, followed by using (3.21) of [10] gives

$$\sum_{x\in B_0} P'_T[(f-a_x)^2](x)\mu'_x = \|f\|^2_{L^2(\mu')} - \|P'_T f\|^2_{L^2(\mu')} \le 2T\mathcal{E}'(f,f),$$

where

$$\mathcal{E}'(f,f) = \tfrac{1}{2} \sum_{x,y\in B_0} (f(x)-f(y))^2 \mu'_{xy}.$$

Therefore,

$$\sum_{y\in B_1} (f(y)-\overline{f})^2 \mu_y \le cR^2 \frac{V(x_0, 2^{3/2}R)}{V(x_0,R)} \sum_{x,y\in B_0} (f(x)-f(y))^2 \mu_{xy},$$

and using the volume ratio bound (7.7) completes the proof. □

Remark 7.6 The PHI for large cylinders $Q(o,R,R^2)$ does not give polynomial control on $V(o,h)/\mu_o$. More precisely, one might hope that given $C_H > 0$ then the PHI with constant C_H for $Q(o,h,h^2)$ for all $h \geq c\alpha$ would imply that there exist $c_i = c_i(C_H)$ such that

$$V(o,\alpha) \leq \mu_o c_1 \alpha^{c_2}. \tag{7.9}$$

However, this is not true. To see this, consider the graph obtained by adding one edge $\{0,o\}$ to $\mathbb{Z}^d$ with $\mu_{o0} = \varepsilon \ll 1$. (Here $o \notin \mathbb{Z}^d$.) Some straightforward calculations give that for $t \geq 1$,

$$\begin{aligned} p_t(o,o) &\asymp (\varepsilon^{-1}e^{-t}) \vee t^{-d/2}, \\ p_t(o,x) &\asymp t^{-d/2}, \qquad \text{if } |x| \leq t^{1/2}, x \neq o. \end{aligned}$$

It follows that the PHI with a constant C_H of order 1 holds in $Q(o,R,R^2)$ if $\varepsilon^{-1}e^{-t} \ll t^{-d/2}$ for $R^2/4 \leq t \leq R^2$, and so in particular if $\varepsilon^{-1} \leq \exp(R^2/5)$. However, if α is such that $\varepsilon^{-1} = \exp(\alpha^2/5)$, then

$$\frac{V(o,\alpha)}{\mu_o} \approx \alpha^d e^{\alpha^2/5},$$

so that (7.9) cannot hold.

In view of Lemma 7.1 this example shows that we cannot obtain (pGB) from (PHI) without some additional condition such as (1.4).

Proof of Theorem 1.10. (a) Choose $\delta > 0$ such that $\kappa < (1-\delta)\kappa'$. Let $y \in \mathbb{V}$, $r \geq c$ with

$$(d(x,y) \vee R(x))^{\kappa'} \leq r.$$

Let $z \in B(y,r)$ and $r^{1-\delta} \leq s \leq r$. Then $d(x,z) \leq r + d(x,y) \leq 2r^{1/\kappa'} < c_1^{-1/\kappa} r^{(1-\delta)/\kappa}$. Also, $R(x)^\kappa \leq r^{\kappa/\kappa'} < c_1^{-1} r^{1-\delta}$. Hence

$$c_1(d(x,z) \vee R(x))^\kappa \leq r^{1-\delta} \leq s,$$

so by (A_κ) the ball $B(z,s)$ is good. Thus $B(y,r)$ satisfies $\mathcal{G}_1(r^{1-\delta})$, and hence (pGB).
(b) and (c) are proved in a similar fashion, using Theorem 7.2, Lemma 7.4 and Theorem 7.5. □

Finally, we show that if (pGB) hold on enough scales, then we can recover full (upper and lower) Gaussian bounds. We need that (pGB) implies volume doubling: of course this follows from Theorem 7.2 and Lemma 7.4, but a direct proof is very simple.

Lemma 7.7 *Let $1 \leq a \leq b$ and $o \in \mathbb{V}$. Suppose that (pGB) holds on $B(x,h)$ for each $x \in B(o,b)$ and $h \in [a,b]$. Then restricted volume doubling holds, that is, for any $x \in B(o,b)$, $h \in [a, \frac{b}{2}]$,*

$$V(x,2h) \leq cV(x,h).$$

Proof. Using the lower bounds of (pGB) on $B(x,h)$, we get $p_{h^2}(x,x) \geq p^{\mathcal{B}}_{h^2}(x,x) \geq c/V(x,h)$, while the upper bounds of (pGB) on $B(x,2h)$ give $p_{h^2}(x,x) \leq c'/V(x,2h)$. Hence $V(x,2h) \leq c''V(x,h)$. □

Proposition 7.8 *Let $1 \le a \le b$ and $o \in \mathbb{V}$. Suppose that (pGB) holds on $B(z,h)$ for each $z \in B(o,b)$ and $h \in [a,b]$. Then for any $x, y \in B(o,b/8)$ and $t \in [a^2, b^2]$,*

$$p_t(x,y) \ge \frac{c_1}{V(x,\sqrt{t})} \exp\big(-c_2 \frac{d(x,y)^2}{t}\big), \text{ for } t \ge ad(x,y), \tag{7.10}$$

$$p_t(x,y) \le \frac{c_3}{V(x,\sqrt{t})} \exp\big(-c_4 \frac{d(x,y)^2}{t}\big), \text{ for } t \ge d(x,y). \tag{7.11}$$

Proof. The upper bound is proved in the same way as Theorem 1.2 is proved from Theorem 4.2.

The lower bound is proved by a standard chaining argument. Fix $x, y \in B(o, b/8)$ and $t \in [a^2, b^2]$. If $b^2 \ge t \ge \frac{d(x,y)^2}{4}$, let $h = (8t^{1/2}) \wedge b$. Then $d(x,y) \le \frac{1}{4}h$. From (1.12) and the rVD given by Lemma 7.7,

$$p_t(x,y) \ge \frac{c_1}{V(x,h)} \ge \frac{c_2}{V(x,\sqrt{t})}. \tag{7.12}$$

Otherwise, suppose $ad(x,y) \le t \le \frac{d(x,y)^2}{4}$. Then $d(x,y) \ge 4a$. Let $m = \lceil d(x,y)^2/t \rceil$ and $r = \lceil t/d(x,y) \rceil$. So there exists a sequence of vertices $z_0, \cdots, z_m$ such that $z_0 = y$, $z_m = x$ and $d(z_{i+1}, z_i) \le r$ for each i. Note that

$$d(o,x) + d(x,y) + 8r \le \frac{b}{8} + \frac{b}{4} + 8\lceil \frac{d(x,y)}{4} \rceil \le b.$$

We can further assume that $d(o, z_i) + 8r \le b$. Let $F_i = B(z_i, r)$ and $F_i^* = B(z_i, 8r)$. Since $r \ge a$, balls F_i and F_i^* satisfy (pGB). Let $s = r^2/6$; using (pGB) on F_i^* we have

$$p_s(y', x') \ge \frac{c_3}{\mu(F_i^*)} \quad \text{for } y' \in F_{i-1},\ x' \in F_i. \tag{7.13}$$

Hence by the rVD,

$$\mathbb{P}_{y'}(X_s \in F_i) \ge c_3 \frac{\mu(F_i)}{\mu(F_i^*)} \ge c_4.$$

Note that

$$t - ms \ge t - \left(\frac{d(x,y)^2}{t} + 1\right) \cdot \frac{1}{6} \left(\frac{t}{d(x,y)} + 1\right)^2 \ge \frac{t}{6}.$$

Employing (pGB) on $B(x, 4\sqrt{t})$ and the rVD, for any $y' \in F_m$ we have

$$p_{t-ms}(x, y') \ge \frac{c_5}{V(x, 4\sqrt{t})} \ge \frac{c_6}{V(x,\sqrt{t})}. \tag{7.14}$$

Therefore,

$$\begin{aligned} p_t(x,y) = p_t(y,x) &\ge \mu_x^{-1} \mathbb{P}_y(X_{is} \in F_i, 1 \le i \le m, X_t = x) \\ &\ge c_4^m \min_{y' \in F_m} \mathbb{P}_{y'}(X_{t-ms} = x) \mu_x^{-1} \end{aligned}$$

$$= c_4^m \min_{y' \in F_m} p_{t-ms}(x,y') \geq c_4^m \frac{c_6}{V(x,\sqrt{t})}$$
$$\geq \frac{c_6}{V(x,\sqrt{t})} \exp\{-c_4' m\} \geq \frac{c_7}{V(x,\sqrt{t})} \exp\{-c_4' \frac{d(x,y)^2}{t}\},$$

which completes the proof. □

Remark 7.9 Note that the argument above only requires lower bounds for $p_s(x,y)$, rather than the lower bound for the killed heat kernel $p_s^B(x,y)$ given by (1.12) given in Definition 1.7.

Proof of Theorem 1.3. The upper bound (1.6) is immediate from Theorem 1.2.

By Proposition 7.8 and Remark 7.9, to prove the lower bound (1.7) it is sufficient to prove that whenever $z \in B(o,b)$ and $a \leq h \leq b$ then

$$p_t(x,x') \geq \frac{c}{V(z,h)}, \quad x,x' \in B(z,h/4),\ 10^{-4}h^2 \leq t \leq h^2. \tag{7.15}$$

If $h \geq (2\lambda_P)^{1/\delta}$ then (7.15) holds by Corollary 6.4(a), while if $a \leq h \leq (2\lambda_P)^{1/\delta}$ then it holds by Corollary 6.4(b). □

Proof of Theorem 1.4. This follows immediately from Theorems 1.2 and 1.3 by choosing a, b appropriately. In (a) we take $a = t^{1/2}$ and $b = t^{1/2} \vee 2r$, and for (b) we use the values of a, b given in the statement of the Theorem; note that the condition on t implies that $b \geq (2\lambda_P)^{1/\delta}$. □

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