Boundary Harnack principle and elliptic Harnack inequality

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January 6, 2017

Abstract

We prove a scale-invariant boundary Harnack principle for inner uniform domains over a large family of Dirichlet spaces. A novel feature of our work is that our assumptions are robust to time changes of the corresponding diffusions. In particular, we do not assume volume doubling property for the symmetric measure.

Keywords: Boundary Harnack principle, Elliptic Harnack inequality, Martin boundary.
Subject Classification (2010): 31B25, 31B05

1 Introduction

Let $(\mathcal{X}, d)$ be a metric space, and assume that associated with this space is a structure which gives a family of harmonic functions on domains $D \subset \mathcal{X}$. (For example, $\mathbb{R}^d$ with the usual definition of harmonic functions.) The *elliptic Harnack inequality* (EHI) holds if there exists a constant C_H such that whenever h is non-negative and harmonic in a ball $B(x,r)$ then, writing $\frac{1}{2}B = B(x, r/2)$,

$$\operatorname{esssup}_{\frac{1}{2}B} h \le C_H \operatorname{essinf}_{\frac{1}{2}B} h. \tag{1.1}$$

Thus the EHI controls harmonic functions in a domain D away from the boundary ∂D. On the other hand, the *boundary Harnack principle* (BHP) gives control of the ratio of two positive harmonic functions at boundary points of a domain. The BHP given in [Anc] states that if $D \subset \mathbb{R}^d$ is a Lipschitz domain, $\xi \in \partial D$, $r > 0$ is small enough, then for any pair u, v of harmonic functions in D which vanish on $\partial D \cap B(\xi, 2r)$,

$$\frac{u(x)}{v(x)} \le C \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(\xi, r). \tag{1.2}$$

[*]Research partially supported by NSERC (Canada)
[†]Research partially supported by NSERC (Canada) and the Pacific Institute for the Mathematical Sciences

The BHP is a key component in understanding the behaviour of harmonic functions near the boundary. It will in general lead to a characterisation of the Martin boundary, and there is a close connection between BHP and a Carleson estimate – see [ALM, Aik08]. (See [Aik08] also for a discussion of different kinds of BHP.)

The results in [Anc] have been extended in several ways. The first direction has been to weaken the smoothness hypotheses on the domain D; for example [Aik01] proves a BHP for inner uniform domains in Euclidean space. A second direction is to consider functions which are harmonic with respect to more general operators. The standard Laplacian is the (infinitesimal) generator of the semigroup for Brownian motion, and it is natural to ask about the BHP for more general Markov processes with values in a metric space $(\mathcal{X}, d)$. [GyS] prove a BHP for inner uniform domains in a measure metric space $(\mathcal{X}, d, m)$ with a Dirichlet form which satisfies the standard parabolic Harnack inequality PHI(2). These results are extended in [L] to spaces which satisfy a parabolic Harnack inequality with anomalous space-time scaling. In most case the BHP has been proved for Markov processes which are symmetric, but see [LS] for the BHP for some more general processes. All the papers cited above study the harmonic functions associated with continuous Markov processes: see [Bog, BKK] for BHP associated with jump processes.

The starting point for this paper is the observation that while the BHP is a purely elliptic result, previous proofs all use parabolic data, or more precisely information on the heat kernel of the process. This use is implicit in [Aik01], which just looks at the standard Laplacian on $\mathbb{R}^d$, but is explicit in [GyS, L], where Green's functions are controlled, from above and below by expressions of the form $\Psi(r)/\mu(B(x, r))$: here Ψ is the space-time scaling function.

The main result of this paper (Theorem 5.1) is that, provided the underlying metric space and process have enough local regularity, then the BHP holds for inner uniform domains whenever the elliptic Harnack inequality holds. Since the EHI is weaker than the PHI, our result extends the BHP to a wider class of spaces; also our approach has the advantage that we can dispense with unnecessary parabolic information. Our main result provides new examples of differential operators that satisfy BHP even in $\mathbb{R}^n$ – see [GS, eq. (2.1) and Example 6.14].

The contents of the paper are as follows. In Section 2 we give the definition and basic properties of inner uniform domains in length spaces. Section 3 reviews the properties of Dirichlet forms and the associated Hunt processes. In Section 4 we give the definition of harmonic function in our context, and state the additional regularity properties which we will need. We show that these lead to the existence of Green's functions, and we prove the essential technical result that Green's functions are locally in the domain of the Dirichlet form – see Lemma 4.17. Some key comparisons of Green's functions, which follow from the EHI, and were proved in [BM], are given in Proposition 4.18. After these rather lengthy preliminaries, in Section 5 we state and prove our main result Theorem 5.1. Our argument follows that of Aikawa [Aik01] (see also [GyS, L]), except that at key points in the argument we use Green's function comparisons from Proposition 4.18 rather than bounds which come from heat kernel estimates.

We use c, c', C, C' for strictly positive constants, which may change value from line to line. Constants with numerical subscripts will keep the same value in each argument, while those with letter subscripts will be regarded as constant throughout the paper. The notation $C_0 = C_0(a)$ means that the constant C_0 depends only on the constant a.

2 Inner uniform domains

In this section, we introduce the geometric assumptions on the underlying metric space, and the corresponding domains.

Definition 2.1 (Length space)**.** Let $(\mathcal{X}, d)$ be a metric space. The length $L(\gamma) \in [0, \infty]$ of a continuous curve $\gamma : [0, 1] \to X$ is given by

$$L(\gamma) = \sup \sum_i d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0, 1]$. Clearly $L(\gamma) \geq d(\gamma(0), \gamma(1))$. A metric space is a *length space* if $d(x, y)$ is equal to the infimum of the lengths of continuous curves joining x and y.

For the rest of this paper, we assume that $(\mathcal{X}, d)$ is a complete, separable, locally compact, length space. By the Hopf–Rinow–Cohn–Vossen theorem (cf. [BBI, Theorem 2.5.28]) every closed metric ball in $(\mathcal{X}, d)$ is compact. It also follows that there exists a geodesic path $\gamma(x, y)$ (not necessarily unique) between any two points $x, y \in \mathcal{X}$. We write $B(x, r) = \{y \in \mathcal{X} \ : \ d(x, y) < r\}$ for open balls in $(\mathcal{X}, d)$.

Next, we introduce the intrinsic distance d_U induced by an open set $U \subset \mathcal{X}$.

Definition 2.2 (Intrinsic distance)**.** Let $U \subset \mathcal{X}$ be a connected open subset. We define the *intrinsic distance* d_U by

$$d_U(x, y) = \inf \{L(\gamma) \ : \ \gamma : [0, 1] \to U \text{ continuous}, \gamma(0) = x, \gamma(1) = y\}.$$

It is well-known that (U, d_U) is a length metric space (cf. [BBI, Exercise 2.4.15]). We now consider its completion.

Definition 2.3 (Balls in intrinsic metric)**.** Let $U \subset X$ be a connected and open subset of the length space $(\mathcal{X}, d)$. Let $\widetilde{U}$ denote the completion of (U, d_U), equipped with the natural extension of d_U to $\widetilde{U} \times \widetilde{U}$. For $x \in \widetilde{U}$ we define

$$B_{\widetilde{U}}(x, r) = \left\{ y \in \widetilde{U} \ : \ d_U(x, y) < r \right\}.$$

Set

$$B_U(x, r) = U \cap B_{\widetilde{U}}(x, r).$$

If $x \in U$, then $B_U(x, r)$ simply corresponds to the open ball in (U, d_U). However, the definition of $B_U(x, r)$ also makes sense for $x \in \widetilde{U} \setminus U$.

Definition 2.4 (Boundary and distance to the boundary)**.** We denote the boundary of U with respect to the inner metric by

$$\partial_{\widetilde{U}} U = \widetilde{U} \setminus U,$$

and the distance to the boundary by

$$\delta_U(x) = \inf_{y \in \partial_{\widetilde{U}} U} d_U(x, y) = \inf_{y \in U^c} d(x, y).$$

For any open set $V \subset U$, let $\overline{V}^{d_U}$ denote the completion of V with respect to the metric d_U. We denote the boundary of V with respect to $\widetilde{U}$ by

$$\partial_{\widetilde{U}} V = \overline{V}^{d_U} \setminus V,$$

and the part of boundary of V that lies in U by

$$\partial_U V = \partial_{\widetilde{U}} V \cap U.$$

In this work, we prove the boundary Harnack principle for the following class of domains.

Definition 2.5 (Uniform and inner uniform domains)**.** Let U be a connected, open subset of a length space $(\mathcal{X}, d)$. Let $\gamma : [0, 1] \to U$ be a rectifiable, continuous curve in U. Let $c_U, C_U \in (0, \infty)$. We say γ is a (c_U, C_U)-uniform curve if

$$\delta_U(\gamma(t)) \geq c_U \min \left(d(\gamma(0), \gamma(t)), d(\gamma(1), \gamma(t))\right)$$

for all $t \in [0, 1]$, and if

$$L(\gamma) \leq C_U d(\gamma(0), \gamma(1)).$$

The domain U is called (c_U, C_U)*-uniform domain* if any two points in U can be joined by a (c_U, C_U)-uniform curve.

We say that U is an *uniform domain* if U is (c_U, C_U)-uniform domain for some constants $c_U, C_U \in (0, \infty)$. We say that U is an *inner uniform domain*, if U is an uniform domain in (U, d_U), where d_U is in the intrinsic metric corresponding to U.

Another definition of uniform domains that appear in the literature is that of length uniform domains.

Definition 2.6 (Length uniform and inner length uniform domains)**.** Let U be a connected, open subset of a length space $(\mathcal{X}, d)$. Let $\gamma : [0, 1] \to U$ be a rectifiable, continuous curve in U. Let $c_U, C_U \in (0, \infty)$. We say γ is a (c_U, C_U)-length uniform curve if

$$\delta_U(\gamma(t)) \geq c_U \min \left(L(\gamma\big|_{[0,t]}), L(\gamma\big|_{[t,1]}) \right)$$

for all $t \in [0, 1]$, and if

$$L(\gamma) \leq C_U d(\gamma(0), \gamma(1)).$$

The domain U is called (c_U, C_U)-*length uniform domain* if any two points in U can be joined by a (c_U, C_U)-length uniform curve.

We say that U is an length uniform domain if U is (c_U, C_U)-length uniform domain for some constants $c_U, C_U \in (0, \infty)$.

We say that U is an *inner length uniform domain*, if U is a length uniform domain in (U, d_U), where d_U is in the intrinsic metric corresponding to U.

The following lemma extends the existence of inner uniform curves between any two points in U in Definition 2.6 to the existence of inner uniform curves between any two points in $\widetilde{U}$.

Lemma 2.7. *Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length metric space. Let U be a (c_U, C_U)-inner uniform domain and let $\widetilde{U}$ denote the completion of U with respect to the inner metric d_U. Then for any two points x, y in $(\widetilde{U}, d_U)$, there exists a (c_U, C_U)-uniform curve in the d_U metric.*

Proof. Let $x, y \in \widetilde{U}$. There exist sequences $(x_n), (y_n)$ in U such that $x_n \to y, y_n \to y$ as $n \to \infty$ in the d_U metric. Let $\gamma_n : [0, 1] \to U, n \in \mathbb{N}$ denote the (c_U, C_U)-uniform curve in (U, d_U) from x_n to y_n with constant speed parametrization. By [BBI, Theorem 2.5.28], the above curves can be viewed to be in the compact space $\overline{B_U(x, 2C_U d_U(x, y))}^{d_U}$ for all large enough n. By a version of Arzela-Ascoli theorem the desired inner uniform curve γ from x to y can be constructed as a sub-sequential limit of the sequence of inner uniform curves γ_n– see [BBI, Theorem 2.5.14]. □

The following geometric property of a metric space $(\mathcal{X}, d)$ will play an important role in the paper.

Definition 2.8 (Metric doubling property)**.** We say that a metric space $(\mathcal{X}, d)$ satisfies the *metric doubling property* if there exists $N > 0$ such that any ball $B(x, r)$ can be covered by at most N balls of radius $r/2$.

A closely related notion is the volume doubling property.

Definition 2.9 (Volume doubling property)**.** We say that a Borel measure μ on a metric space $(\mathcal{X}, d)$ satisfies the *volume doubling property*, if there exist a constant $C_D > 0$ such that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r)) \quad \text{for all } x \in \mathcal{X} \text{ and for all } r > 0.$$

It is well known that volume doubling implies metric doubling. Further, by [LuS, Theorem 1] if a complete metric space $(\mathcal{X}, d)$ satisfies the metric doubling property then there exists a non-zero Borel measure μ satisfying the volume doubling property.

Proposition 2.10. *([GyS, Proposition 3.3]) Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length metric space satisfying the metric doubling property. Then an open set U is a (resp. inner) length uniform domain if and only if U is (resp. inner) uniform domain.*

Remark 2.11. The proof in [GyS] is inaccurate because the first displayed equation in [GyS, p. 82] which states $B(x_j, \varepsilon r_j/2) \subset B(x, (1+\varepsilon/2)r_j)$ does not follow from $r_j = \min\{r_{j-1}, \rho(x, x_j)\} \le \rho(x, x_j)$. However this mistake can be easily fixed by following the proof of [MS, Lemma 2.7] more closely, using the parametrization of the curve as given in [MS].

Let $\overline{U} \subset X$ denote the closure of U in $(\mathcal{X}, d)$. Let $p : (\widetilde{U}, d_U) \to (\overline{U}, d)$ denote the natural projection map, that is p is the unique continuous map such that p restricted to U is the identity map on U. For any $x \in \widetilde{U}$ and for any ball $D = B(p(x), r)$, let D' denote the connected component of $p^{-1}(D \cap \overline{U})$ containing x. We identify the subset $D' \cap p^{-1}(U)$ of $(\widetilde{U}, d_U)$ with the subset $p(D') \cap U$ of $(\mathcal{X}, d)$ and simply denote it by $D' \cap U$. The following lemma allows us to compare balls with respect to the d and d_U metrics.

Lemma 2.12. *Let $(\mathcal{X}, d)$ be a complete, length space satisfying the metric doubling property. Let $U \subset X$ be a connected, open, (c_U, C_U)-inner uniform domain. Then there exists $\widetilde{C_U} > 1$ such that for all balls $B(p(x), r/\widetilde{C_U})$ with $x \in \widetilde{U}$ and $r > 0$, we have*

$$B_{\widetilde{U}}(x, r/\widetilde{C_U}) \subset D' \subset B_{\widetilde{U}}(x, r),$$

where D' the connected component of $p^{-1}(D \cap \overline{U})$ containing x.

Proof. See [LS, Lemma 3.7] where this is proved under the hypothesis of volume doubling, and note that the argument only uses metric doubling. (Alternatively, a doubling measure exists by [LuS, Theorem 1], and one can then use [LS]). ☐

The following lemma shows that every point in an inner uniform domain is close to a point that is sufficiently far away from the boundary.

Lemma 2.13. *([GyS, Lemma 3.20]) Let U be a (c_U, C_U)-inner uniform domain in a length metric space $(\mathcal{X}, d)$. For every inner ball $B = B_{\widetilde{U}}(x, r)$ with the property that $B \neq B_{\widetilde{U}}(x, 2r)$ there exists a point $x_r \in B$ with*

$$d_U(x, x_r) = r/4 \quad \textit{and} \quad \delta_U(x_r) \ge \frac{c_U r}{4}.$$

Proof. We recall the proof for convenience. Since $B_{\widetilde{U}}(x, 2r) \neq B_{\widetilde{U}}(x, r)$, there exists a point $y \in B_{\widetilde{U}}(x, r)$ with $d(x, y) = r/2$. By Lemma 2.7, there exists $\gamma : [0, 1] \to \widetilde{U}$ a (c_U, C_U)-inner uniform curve joining x and y with $\gamma(0) = x, \gamma(1) = y$. By intermediate value theorem, there exists $t \in (0, 1)$ such that $d_U(x, \gamma(t)) = r/4$. Since γ is (c_U, C_U)-inner uniform, we obtain

$$\begin{aligned}\delta_U(\gamma(t)) &\ge c_U \min\left(d_U(x, \gamma(t)), d_U(y, \gamma(t))\right) \\ &\ge c_U \min\left(d_U(x, \gamma(t)), d_U(x, y) - d_U(x, \gamma(t))\right) = c_U r/4.\end{aligned}$$

Therefore $x_r = \gamma(t)$ satisfies the desired properties. ☐

Lemma 2.14. *Let U be a (c_U, C_U)-inner uniform domain in a length metric space $(\mathcal{X}, d)$. If $x, y \in U$ the there exists a (c_U, C_U)-inner uniform curve γ connecting x and y with $\delta_U(z) \geq \frac{1}{2}c_U\,(\delta_U(x) \wedge \delta_U(y))$ for all $z \in \gamma$.*

Proof. Write $t = \delta_U(x) \wedge \delta_U(y)$. Let γ be an inner uniform curve from x to y and $z \in \gamma$. $d(z, x) \leq \frac{1}{2}t$ then $\delta_U(z) \geq \delta_U(x) - d(x, z) \geq \frac{1}{2}t$, and the same bound holds if $d(z, y) \leq \frac{1}{2}t$. Finally if $d(z, x) \wedge d(z, y) \geq \frac{1}{2}t$ then $\delta_U(z) \geq \frac{1}{2}c_U t$. □

3 Dirichlet space and Hunt process

Let $(\mathcal{X}, d)$ be a locally compact, separable, metric space and let μ be a Radon measure with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form on $L^2(X, \mu)$ – see [FOT]. Recall that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *strongly local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports, such that f is constant in a neighbourhood of $\operatorname{supp}(g)$. We call $(\mathcal{X}.d, \mu, \mathcal{E}, \mathcal{F})$ a *MMD space.*

Let $\mathcal{L}$ be the generator of $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathcal{X}, \mu)$; that is $\mathcal{L}$ is a self-adjoint and non-positive-definite operator in $L^2(\mathcal{X}, \mu)$ with domain $\mathcal{D}(\mathcal{L})$ that is dense in $\mathcal{F}$ such that

$$\mathcal{E}(f, g) = -\langle \mathcal{L}f, g\rangle,$$

for all $f \in \mathcal{D}(\mathcal{L})$ and for all $g \in \mathcal{F}$; here $\langle \cdot, \cdot\rangle$, is the inner product in $L^2(\mathcal{X}, \mu)$. The associated *heat semigroup*

$$P_t = e^{t\mathcal{L}}, t \geq 0,$$

is a family of contractive, strongly continuous, Markovian, self-adjoint operators in $L^2(\mathcal{X}, \mu)$. We set

$$\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \langle f, g\rangle, \quad ||f||_{\mathcal{E}_1} = \mathcal{E}_1(f, f)^{1/2}. \tag{3.1}$$

It is known that corresponding to a regular Dirichlet form, there exists an essentially unique Hunt process $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathcal{X})$. The relation between the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$ and the associated Hunt process is given by the identity

$$P_t f(x) = \mathbb{E}^x f(X_t),$$

for all $f \in L^\infty(\mathcal{X}, \mu)$, for every $t > 0$, and for μ-almost all $x \in \mathcal{X}$.

We define capacities for $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F}^m)$ as follows. For a non-empty open subset $D \subset \mathcal{X}$, let $\mathcal{C}_0(D)$ denote the space of all continuous functions with compact support in D. Let $\mathcal{F}_D$ denote the closure of $\mathcal{F}^m \cap \mathcal{C}_0(D)$ with respect to the $\mathcal{E}_1(\cdot, \cdot)^{1/2}$ -norm. By $A \Subset D$, we mean that the closure of A is a compact subset of D. For $A \Subset D$ we set

$$\operatorname{Cap}_D(A) = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}_D \text{ and } f \geq 1 \text{ in a neighbourhood of } A\}. \tag{3.2}$$

A statement depending on $x \in B$ is said to hold quasi-everywhere on B (abbreviated as q.e. on B), if there exists a set $N \subset B$ of zero capacity such that the statement if true for every $x \in B \setminus N$.

For a Borel subset $A \subset \mathcal{X}$, we denote by

$$T_B := \inf\{t > 0 \;:\; X_t \in B\}, \quad \tau_B := T_{\mathcal{X}\setminus B} = \inf\{t > 0 \;:\; X_t \notin U\}. \tag{3.3}$$

It is known that every function $f \in \mathcal{F}$ admits a quasi continuous version (cf. [FOT, Theorem 2.1.3]), and throughout this paper, we always assume that every $f \in \mathcal{F}$ is represented by its quasi-continuous version, which is unique up to a set of zero 1-capacity.

Also associated with the Dirichlet form and $f \in \mathcal{F}$ is the energy measure $d\Gamma(f,f)$. This is defined to be the unique measure such that for all bounded $g \in \mathcal{F}$ we have

$$\int g \, d\Gamma(f,f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g).$$

We have

$$\mathcal{E}(f,f) = \int_{\mathcal{X}} d\Gamma(f,f).$$

Definition 3.1. *For an open subset of U of $\mathcal{X}$, we define the following function spaces associated with $(\mathcal{E}, \mathcal{F})$.*

$$\begin{aligned}
\mathcal{F}_{\mathrm{loc}}(U) &= \left\{u \in L^2_{\mathrm{loc}}(U) \;:\; \forall \text{ relatively compact } V \subset U, \exists u^{\#} \in \mathcal{F}, u = u^{\#}\big|_V \mu\text{-a.e.}\right\}, \\
\mathcal{F}(U) &= \left\{u \in \mathcal{F}_{\mathrm{loc}}(U) \;:\; \int_U |u|^2 \, d\mu + \int_U d\Gamma(u,u) < \infty\right\}, \\
\mathcal{F}_c(U) &= \{u \in \mathcal{F}(U) \;:\; \text{The essential support of } u \text{ is compact in } U\}, \\
\mathcal{F}^0(U) &= \text{ the closure of } \mathcal{F}_c(U) \text{ in } \mathcal{F} \text{ for the norm } \mathcal{E}_1(u,u)^{1/2}.
\end{aligned}$$

For an open set U, another equivalent definition of $\mathcal{F}^0(U)$ is given by

$$\mathcal{F}^0(U) = \{u \in \mathcal{F} \;:\; \widetilde{u} = 0 \text{ q.e. on } \mathcal{X} \setminus U\}, \tag{3.4}$$

where $\widetilde{u}$ is a quasi continuous version of u. We refer the reader to [FOT, Theorem 4.4.3(i)] for a proof of this equivalence. By the equivalent definition, we can identify the $\mathcal{F}^0(U)$ as a subset of $L^2(U,\mu)$ where $L^2(U,\mu)$ is identified with the subspace $\{u \in L^2(\mathcal{X},\mu) \;:\; u = 0 \;\mu\text{-a.e. on } \mathcal{X} \setminus U\}$.

Definition 3.2. For an open set $U \subset \mathcal{X}$, we define the Dirichlet-type Dirichlet form on U by

$$\mathcal{D}(\mathcal{E}^D_U) = \mathcal{F}^0(U) \text{ and } \mathcal{E}^D_U(f,g) = \mathcal{E}(f,g) \text{ for } f,g \in \mathcal{F}^0(U).$$

If $(\mathcal{E}, \mathcal{F})$ is a regular, strongly-local Dirichlet form on $L^2(\mathcal{X},\mu)$ then $(\mathcal{E}^D_U, \mathcal{F}^0(U))$ is a regular, strongly-local Dirichlet form on $L^2(U,\mu)$. Moreover, $(\mathcal{E}^D_U, \mathcal{F}^0(U))$ is the Dirichlet form of the process X killed upon exiting U. We write $(P^D_t, t \geq 0)$ for the associated semigroup, and call (P^D_t) the semigroup of X killed on exiting D. See [CF, Theorem 3.3.8 and Theorem 3.3.9] or [FOT, Theorem 4.4.3] for more details.

For an open set U, we will often consider functions that vanish on a portion of the boundary of U, and therefore define the local spaces associated with $(\mathcal{E}^D_U, \mathcal{F}^0(U))$.

Definition 3.3. Let V be an open subset of U, where U is an open subset of $\mathcal{X}$.

$$\mathcal{F}^0_{\mathrm{loc}}(U,V) = \{f \in L^2_{\mathrm{loc}}(V,\mu) : \forall \text{ open } A \subset V \text{ relatively compact in } \overline{U} \text{ with } d_U(A, U \setminus V) > 0,\ \exists f^\sharp \in \mathcal{F}^0(U) : f^\sharp = f\ \mu\text{-a.e. on } A\}.$$

Roughly speaking, a function in $\mathcal{F}^0_{\mathrm{loc}}(U,V)$ vanishes along the portion of boundary given by $\partial_{\widetilde{U}} V \cap \partial_{\widetilde{U}} U$.

The extended Dirichlet space $\mathcal{F}^0(U)_e$ is defined as the family of all measurable, almost everywhere finite functions u such that there exists an approximating sequence $(u_n) \subset \mathcal{F}^0(U)$ that is $\mathcal{E}^D_U$-Cauchy and $u = \lim u_n$ μ-almost everywhere. If $(\mathcal{E}^D_U, \mathcal{F}^0(U))$ is transient then $\mathcal{F}^0(U)_e$ is a Hilbert space under the $\mathcal{E}^D_U$ inner product, by [FOT, Lemma 1.5.5].

4 Harmonic functions and the elliptic Harnack inequality

4.1 Harmonic functions

We define harmonic functions for a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$.

Definition 4.1. Let $U \subset \mathcal{X}$ be open. A function $u : U \to \mathbb{R}$ is *harmonic* on U, if $u \in \mathcal{F}_{\mathrm{loc}}(U)$ and for any function $\phi \in \mathcal{F}_{\mathrm{c}}(U)$

$$\mathcal{E}(u^\#, \phi) = 0$$

where $u^\# \in \mathcal{F}$ is such that $u^\# = u$ in the essential support of ϕ.

Remark 4.2. (a) By the locality of $(\mathcal{E}, \mathcal{F})$, $\mathcal{E}(u^\#, \phi)$ does not depend on the choice of $u^\#$ in Definition 4.1.

(b) If $V \subset U$, where U and V are open subsets of $\mathcal{X}$ and if u is harmonic in U, then the restriction $u\big|_V$ is harmonic in V. This again follows from the locality of $(\mathcal{E}, \mathcal{F})$.

(c) It is known that $u \in L^\infty_{\mathrm{loc}}(U,\mu)$ is harmonic in U if and only if it satisfies the following property: for every relatively compact open subset V of U, $t \mapsto \widetilde{u}(X_{t \wedge \tau_V})$ is a uniformly integrable $\mathbb{P}^x$-martingale for q.e. $x \in V$. (Here $\widetilde{u}$ is a quasi continuous version of u on V.) This equivalence between the weak solution formulation in Definition 4.1 and the probabilistic formulation using martingales is given in [Che, Theorem 2.11].

Definition 4.3. Let $V \subset U$ be open. We say that a harmonic function $u : V \to \mathbb{R}$ satisfies Dirichlet boundary conditions along the boundary $\partial_{\widetilde{U}} U \cap \widetilde{V}^{d_U}$, if $u \in \mathcal{F}^0_{\mathrm{loc}}(U,V)$. where $\widetilde{V}^{d_U}$ is the closure of V in $(\widetilde{U}, d_U)$.

4.2 Elliptic Harnack inequality

Definition 4.4 (Elliptic Harnack inequality)**.** We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *elliptic Harnack inequality* EHI, if there exist constants $C_H < \infty$ and $\delta \in (0, 1)$ such that, for any ball $B(x, R) \subset \mathcal{X}$ and any non-negative function $u \in \mathcal{F}_{\rm loc}(B(x, R))$ that is harmonic on $B(x, R)$, we have

$$\operatorname{esssup}_{z \in B(x,\delta R)} u(z) \le C_H \operatorname{essinf}_{z \in B(x,\delta R)} u(z). \tag{EHI}$$

We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *local EHI*, denoted $(\text{EHI})_{\rm loc}$, if there exists $R_0 \in (0, \infty)$ such that the (EHI) holds for all balls $B(x, r)$ with $r < R$.

An easy chaining argument show that if the EHI holds for some $\delta \in (0, 1)$ then it holds for any other $\delta' \in (0, 1)$. Further, if the local EHI holds for some R_0 then it holds (with of course a different constant C_H) for any other $R \in (0, \infty)$.

We recall the definition of Harnack chain – see [JK, Section 3].

Definition 4.5 (Harnack chain)**.** Let $U \subsetneq \mathcal{X}$ be a connected open set. An *M-non-tangential ball* in a domain U is a ball $B(x, r)$ in U whose distance from ∂U is comparable to its radius r: $Mr > d(B(x, r), \partial U) > M^{-1} r$.

For $x_1, x_2 \in U$, a *M-Harnack chain from x_1 to x_2* in U is a sequence of M-nontangential balls such that the first ball contains x_1, the last contains x_2, and such that consecutive balls intersect. Note that consecutive balls must have comparable radius. The number of balls in a Harnack chain is called the *length* of the Harnack chain.

Remark 4.6. Assume that $(\mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality. Then if u is a positive, continuous, harmonic function in U, then

$$C_H^{-N} u(x_1) < u(x_2) < C_H^N u(x_2), \tag{4.1}$$

where N is the length of the minimal δ^{-1}-Harnack chain between x_1, and x_2 and C_H, δ are the constants in the EHI.

For a domain D write $N_D(x, y; M)$ for the length of the shortest M-Harnack chain in D connecting x and y.

Lemma 4.7. *Let $(\mathcal{X}, d)$ be a locally compact, separable, length metric space that satisfies the metric doubling property. Let $U \subsetneq \mathcal{X}$ be a (c_U, C_U)-inner uniform domain in $(\mathcal{X}, d)$. Then there exists $C \in (0, \infty)$, depending only on c_U, C_U and M, such that*

$$C^{-1} \log\left(\frac{d_U(x, y)}{\min(\delta_U(x), \delta_U(y))} + 1\right) \le N_D(x, y; M) \le C \log\left(\frac{d_U(x, y)}{\min(\delta_U(x), \delta_U(y))} + 1\right) + C$$

for all $x, y \in U$.

Proof. See [GO, Equation (1.2) and Theorem 1.1] or [Aik15, Theorem 3.8 and 3.9]) for a similar statement for the quasi-hyperbolic metric on D; the result then follows by a comparison between the quasi-hyperbolic metric and the length of Harnack chains as in [Aik01, pp. 127]. □

4.3 Green's function

We recall the definition of the smallest Dirichlet eigenvalue of an open set Ω corresponding to a Dirichlet form. Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form and let $\Omega \subsetneq X$ be open. We define

$$\lambda_{\min}(\Omega) = \inf_{u\in\mathcal{F}^0(\Omega)\setminus\{0\}} \frac{\mathcal{E}^D_\Omega(u,u)}{\|u\|_2^2}.$$

If $\mathcal{L}^D$, denotes the generator of $(\mathcal{E}^D_\Omega, \mathcal{F}^0(\Omega))$, then $\lambda_{\min}(\Omega) = \inf \operatorname{spectrum}(-\mathcal{L}^D)$.

Lemma 4.8. *([GH1, Lemma 5.1]) Let $(\mathcal{E},\mathcal{F})$ be a regular, Dirichlet form in $L^2(\mathcal{X},\mu)$ and let $\Omega \subset \mathcal{X}$ be open such that $\lambda_{\min}(\Omega) > 0$. Let $\mathcal{L}^\Omega$ be the generator of $(\mathcal{E}^\Omega_\Omega, \mathcal{F}^0(\Omega))$, and set $G^\Omega = (-\mathcal{L}^\Omega)^{-1}$, the inverse of $-\mathcal{L}^\Omega$ on $L^2(\Omega,\mu)$. Let $P^\Omega_t, t \geq 0$ denote the heat semigroup corresponding to $(\mathcal{E}^\Omega_\Omega, \mathcal{F}^0(\Omega))$. Then the following statements hold:*

(i) $\|G^\Omega\| \leq \lambda_{\min}(\Omega)^{-1}$, that is, for any $f \in L^2(\Omega,\mu)$,

$$\left\|G^\Omega f\right\|_{L^2(\Omega)} \leq \lambda_{\min}(\Omega)^{-1} \|f\|_{L^2(\Omega)};$$

(ii) for any $f \in L^2(\Omega)$, we have that $G^\Omega f \in \mathcal{F}^0(\Omega)$, and

$$\mathcal{E}^\Omega_\Omega(G^\Omega f, \phi) = \langle f, \phi\rangle \text{ for any } \phi \in \mathcal{F}^0(\Omega);$$

(iii) for any $f \in L^2(\Omega)$,

$$G^\Omega f = \int_0^\infty P^\Omega_s f\, ds;$$

(iv) G^Ω is non-negative definite: $G^\Omega f \geq 0$ if $f \geq 0$.

We now state our assumption on the Green's function.

Assumption 4.9. Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length metric space and let μ be a non-atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E},\mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X},\mu)$. For any non-empty bounded open set $\Omega \subset \mathcal{X}$ with $\operatorname{diameter}(\Omega, d) \leq \operatorname{diameter}(\mathcal{X}, d)/5$, $\lambda_{\min}(\Omega) > 0$, and there exists a function $g_\Omega(x,y)$ defined for $(x,y) \in \Omega\times\Omega$ with the following properties:

(i) $G^\Omega f(x) = \int_\Omega g_\Omega(x,z) f(x)\,\mu(dz)$ for all $f \in L^2(\Omega)$ and μ-a.e. $x \in \Omega$;

(ii) (Symmetry) $g_\Omega(x,y) = g_\Omega(y,x) \geq 0$ for all $(x,y) \in \Omega\times\Omega \setminus \operatorname{diag}$;

(iii) (Continuity) $g_\Omega(x,y)$ is jointly continuous in $(x,y) \in \Omega\times\Omega\setminus \operatorname{diag}$;

(iv) (Maximum principles) If $x_0 \in U \Subset \Omega$, then

$$\inf_{U\setminus\{x_0\}} g_\Omega(x_0,\cdot) = \inf_{\partial U} g_\Omega(x_0,\cdot),$$
$$\sup_{\Omega\setminus U} g_\Omega(x_0,\cdot) = \sup_{\partial U} g_\Omega(x_0,\cdot).$$

(v) for any fixed $x \in \Omega$, the function $y \mapsto g_\Omega(x,y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(\Omega, \Omega \setminus \{x\})$ and is harmonic in $\Omega \setminus \{x\}$.

We now give sufficient conditions for this assumption, and recall the definition of an ultracontractive semigroup, a notion introduced by E. B. Davies and B. Simon in [DS].

Definition 4.10. Let $(\mathcal{X}, d, \mu)$ be a metric measure space. Let $(P_t)_{t\geq 0}$ be the Markov semigroup corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(\mathcal{X}, \mu)$. We say that the semigroup $(P_t)_{t\geq 0}$ is *ultracontractive* if P_t is a bounded operator from $L^2(\mathcal{X}, \mu)$ to $L^\infty(\mathcal{X}, \mu)$ for all $t > 0$. We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$ is *ultracontractive* if the corresponding heat semigroup is ultracontractive.

We often require the weaker notion of local ultracontractivity introduced in [GT2, Definition 2.11].

Definition 4.11. We say that a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, F)$ is *locally ultracontractive* if for all open balls B, the killed heat semigroup (P^B_t) given by Definition 3.2 is ultracontractive.

It is well-known that ultracontractivity of a semigroup is equivalent to the existence of an essentially bounded heat kernel at all positive times. In particular, we recall the following.

Lemma 4.12. *([Dav, Lemma 2.1.2]) Let D be an open bounded subset of $\mathcal{X}$. Then P^D_t admits an integral kernel $p^D(t,\cdot,\cdot)$ for all $t > 0$ is jointly measurable in $\Omega \times \Omega$ and satisfies*

$$0 \leq p^D(t,x,y) \leq \left\| P^D_{t/2} \right\|^2_{L^2(\mu) \to L^\infty(\mu)}$$

for $\mu \times \mu$-a.e. $(x,y) \in D \times D$. Conversely if P^D_t has an integral kernel $p^D(t,x,y)$ satisfying

$$0 \leq p^D(t,x,y) \leq a_t < \infty$$

for all $t > 0$ and for $\mu \times \mu$-a.e. $(x,y) \in \Omega \times \Omega$, then $(P^D_t)_{t\geq 0}$ is ultracontractive with

$$\left\| P^D_t \right\|_{L^2(\mu) \to L^\infty(\mu)} \leq a_t^{1/2} \text{ for all } t > 0.$$

The issue of joint measurability is clarified in [GT2, p. 1227].

To verify Assumption 4.9 it will be helpful to introduce a second assumption on the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Assumption 4.13. Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length metric space and let μ be a non-atomic Radon measure on $(\mathcal{X}, d)$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X}, \mu)$. We assume that the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ is locally ultracontractive and that

$$\lambda_{\min}(\Omega) > 0,$$

for any non-empty bounded open set $\Omega \subset \mathcal{X}$ with $\mathrm{diameter}(\Omega, d) \leq \mathrm{diameter}(\mathcal{X}, d)/5$.

Remark 4.14. By domain monotonicity arguments, to verify the above assumption it suffices to check ultracontractivity and $\lambda_{\min} > 0$ on a suitable family of balls. Let μ be a Radon measure with full support on $(\mathcal{X}, d)$, and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $(\mathcal{X}, \mu)$ satisfying Assumption 4.13. Let Ω be a bounded non-empty open subset of $\mathcal{X}$. Then the heat semigroup P_t^Ω corresponding to $(\mathcal{E}_\Omega^D, \mathcal{F}^0(\Omega))$ is a compact operator on $L^2(\mathcal{X}, \mu)$ by [Dav, Theorem 2.1.5]. Therefore $\mathcal{L}^\Omega$ has a point spectrum and in particular $\lambda_{\min}(\Omega)$ is in the point spectrum of $-\mathcal{L}^\Omega$.

The nest two lemmas show that weak ultracontractivity leads to the existence of a Greens function satisfying Assumption 4.9.

Lemma 4.15. *(See [GH1, Lemma 5.2 and 5.3])). Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space satisfying EHI$_{\text{loc}}$ and Assumption 4.13. Then for any bounded, non-empty open set $\Omega \subset \mathcal{X}$ with $\lambda_{\min}(\Omega) > 0$, there exists a function $g_\Omega(x, y)$ defined for $(x, y) \in \Omega \times \Omega$ which satisfies properties (i)-(iv) of Assumption 4.9.*

Remark 4.16. [GH1, Lemma 5.2] states a similar result without assuming local ultracontractivity. Unfortunately, the proof in [GH1] of (i) has a gap, which we do not know how to fix. The problem is in the proof of [GH1, eq. (5.8)] from [GH1, eq. (5.7)]. In particular, while one has in the notation of [GH1] that $G^\Omega f_k \to G^\Omega f$ in $L^2(\Omega)$, this does not imply pointwise convergence, while the proof of (5.8) does require pointwise convergence at the specific point x.

The following example helps to illustrate the gap. Consider the Dirichlet form $\mathcal{E}(f, f) = \|f\|_2^2$ on $\mathbb{R}^n$; this satisfies (5.7) for any bounded open domain Ω with $g_x^\Omega \equiv 0$ but it fails to satisfy (5.8). This Dirichlet form does not satisfy the hypothesis of [GH1, Lemma 5.2] since it is local rather than strongly local, but it still illustrates the problem, since strong locality was not used in the proof of (5.8) from (5.7).

Proof. We use the construction in [GH1]. We denote by $g_\Omega(\cdot, \cdot)$ by the function constructed in [GH1, Lemma 5.2] off the diagonal, and extend it to $\Omega \times \Omega$ by taking g_Ω equal 0 (or any arbitrary constant) on the diagonal. By [GH1, Lemma 5.2], the function $g_\Omega(\cdot, \cdot)$ satisfies (ii) and (iii). (The proofs of (ii) and (iii) do not use [GH1, eq. (5.8)].)

Next, we show (i), using the additional hypothesis of ultracontractivity. Define the operators

$$S_t = P_t^\Omega \circ G^\Omega = G^\Omega \circ P_t^\Omega, \quad t \geq 0.$$

Formally we have $S_t = \int_t^\infty P_s ds$. Since P_t^Ω is a contraction on all $L^p(\Omega)$, we have by Lemma 4.8(i)

$$\|S_t\|_{L^2 \mapsto L^2} \leq \left\|G^\Omega\right\|_{L^2 \mapsto L^2} \left\|P_t^\Omega\right\|_{L^2 \mapsto L^2} < \infty.$$

Therefore by [FOT, Lemma 1.4.1], there exists a positive symmetric Radon measure $\sigma_t, t \geq 0$ on $\Omega \times \Omega$ such that for all functions $f_1, f_2 \in L^2(\Omega)$, we have

$$\langle f_1, S_t f_2 \rangle = \int_{\Omega \times \Omega} f_1(x) f_2(y) \sigma_t(dx, dy), \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$. Since $S_{t+r} - S_t$ is a positive symmetric operator on $L^2(\Omega)$ by [FOT, Lemma 1.4.1], the measures σ_t is a increasing sequence of symmetric positive measures on $\Omega \times \Omega$ as $t \downarrow 0$. Let A, B be measurable subsets of Ω. Note that all the measures σ_t are finite, since for all $t \geq 0$ by using Lemma 4.8(i), we have

$$\sigma_t(\Omega \times \Omega) \leq \sigma_0(\Omega \times \Omega) = \langle \mathbb{1}_\Omega, G^\Omega \mathbb{1}_\Omega \rangle \leq ||\mathbb{1}_\Omega||_2^2 \lambda_{\min}(\Omega)^{-1} = \mu(\Omega)\lambda_{\min}(\Omega)^{-1}. \tag{4.3}$$

Since P_t^Ω is a strongly continuous semigroup, we have

$$\sigma_0(A \times B) = \langle G^\Omega \mathbb{1}_A, \mathbb{1}_B \rangle = \lim_{t \downarrow 0} \langle S_t \mathbb{1}_A, \mathbb{1}_B \rangle = \lim_{t \downarrow 0} \sigma_t(A \times B).$$

The above equation implies that, for all measurable subsets $A \subset \Omega \times \Omega$, we have

$$\lim_{n \to \infty} \sigma_{1/n}(A) = \sigma_0(A). \tag{4.4}$$

For each $t > 0$ by local ultracontractivity, we have

$$\|S_t\|_{L^2 \to L^\infty} \leq \left\| G^\Omega \right\|_{L^2 \to L^2} \left\| P_t^\Omega \right\|_{L^2 \to L^\infty} < \infty. \tag{4.5}$$

By [GH2, Lemma 3.3], there exists a jointly measurable function $s_t(\cdot, \cdot)$ on $\Omega \times \Omega$ such that

$$\langle f_1, S_t f_2 \rangle_{L^2(\Omega)} = \int_\Omega \int_\Omega f_1(x) f_2(y) s_t(x, y) \mu(dx) \mu(dy), \tag{4.6}$$

for all $f_1, f_2 \in L^2(\Omega)$. By (4.2) and (4.6), we have

$$\sigma_{1/n}(dx, dy) = s_{1/n}(x, y) \mu(dx) \mu(dy)$$

for all $n \in \mathbb{N}$. Therefore by the above equation (4.3), (4.4) and Vitali-Hahn-Saks theorem (cf. [Yos, p. 70]), the measure σ_0 is absolutely continuous with respect to the product measure $\mu \times \mu$ on $\Omega \times \Omega$. Let $s(\cdot, \cdot)$ be the Radon-Nikoym derivative of σ_0 with respect to $\mu \times \mu$. By (4.2) and Fubini's theorem, for all $f \in L^2(\Omega$ and for almost all $x \in \Omega$,

$$G^\Omega f(x) = \int_\Omega s(x, y) f(y) \, \mu(dy). \tag{4.7}$$

If B and B' are disjoint open balls in Ω, then for all $f_1 \in L^2(B), f_2 \in L^2(B')$, we have

$$\langle G^\Omega f_1, f_2 \rangle = \int_B \int_{B'} f_1(x) f_2(y) s(x, y) \mu(dy) \mu(dx) = \int_B \int_{B'} f_1(x) f_2(y) g_\Omega(x, y) \mu(dy) \mu(dx).$$

We used [GH1, eq. (5.7)] along with (4.6) to obtain the above equation. By the same argument as in [GH2, Lemma 3.6(a)], the above equation implies

$$s(x, y) = g_\Omega(x, y) \tag{4.8}$$

for $\mu \times \mu$-almost every $(x, y) \in B \times B'$. By an easy covering argument $\Omega \times \Omega \setminus \text{diag}$ can be covered by countably many sets of the form $B_i \times B_i', i \in \mathbb{N}$ such that B_i and B_i' are disjoint balls contained in Ω. Therefore by (4.8), we have

$$s(x, y) = g_\Omega(x, y) \tag{4.9}$$

for almost every $(x,y) \in \Omega \times \Omega$. In the last line we used that the diagonal has measure zero by Fubini's theorem, since μ is non-atomic. By (4.9) and (4.7), we have (i).

The maximum principles in (iv) are proved in [GH1, Lemma 5.3]. $\square$

The second result is that the Green's function is locally in the domain of the Dirichlet form. This result was shown under more restrictive hypothesis (Gaussian or sub-Gaussian heat kernel estimates) in [GyS, Lemma 4.7] and by similar methods in [L, Lemma 4.3]. Our proof is based on a different approach (see [GyS, Theorem 4.16]) using Lemma 4.15. We do not require heat kernel estimates or the cutoff Sobolev inequality.

Lemma 4.17. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$, Ω be as in the previous Lemma. For any fixed $x \in \Omega$, the function $y \mapsto g_\Omega(x,y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(\Omega, \Omega\backslash\{x\})$ and is harmonic in $\Omega\backslash\{x\}$, and so satisfies Assumption 4.9(v).*

Proof. Let $x \in \Omega$ be an arbitrary point and let $V \subset \Omega$ be an open set such that $\overline{V} \subset \overline{\Omega}$. Let Ω_1, Ω_2 be precompact open sets such that $\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Let $r > 0$ be such that

$$B(x, 4r) \subset \Omega \cap V^c.$$

Let $\phi \in \mathcal{F}$ be a continuous function such that $0 \le \phi \le 1$ and

$$\phi = \begin{cases} 1 & \text{on } B(x,3r)^c \cap \Omega_1, \\ 0 & \text{on } B(x,2r) \cup \left(\overline{\Omega_2}\right)^c. \end{cases}$$

Consider the sequence of functions defined by

$$h_k(y) = \frac{1}{\mu(B(x,r/k))} \int_{B(x,r/k)} g_\Omega(z,y)\mu(dz), \tag{4.10}$$

for all $y \in \Omega$, for all $k \in \mathbb{N}$. By the maximum principle, we have

$$M := \sup_{z \in \overline{B(x,r)}, y \in B(x,2r)^c} g_\Omega(z,y) = \sup_{z \in \overline{B(x,r)}, y \in \partial B(x,2r)} g_\Omega(z,y) < \infty,$$

since the image of the compact set $\overline{B(x,r)} \times \partial B(x,2r)$ under the continuous map of g_Ω is bounded. By the continuity of $g_\Omega(\cdot,\cdot)$ on $\Omega \times \Omega \setminus \mathrm{diag}$, we have ϕh_k converges pointwise to $\phi g_\Omega(x,\cdot)$ on Ω and bounded uniformly by $M \mathbb{1}_{\Omega \setminus B(x,2r)}$. Therefore by the dominated convergence theorem, $\phi(\cdot)h_k(\cdot) \to \phi(\cdot)g_\Omega(x,\cdot)$ in $L^2(\Omega)$. It suffices to show that $\phi(\cdot)g_\Omega(x,\cdot) \in \mathcal{F}$ by [CF, Theorem 3.3.9], since $\phi \equiv 1$ on V.

For the reminder of the proof we identify $L^2(\Omega)$ with the subspace

$$\left\{ f \in L^2(\mathcal{X}) \ : \ f = 0, \quad \mu - \text{a.e. on } \Omega^c \right\}.$$

Similarly, we view $\mathcal{F}^0(\Omega)$ as a subspace of $\mathcal{F}(\Omega)$ –see [CF, eq. (3.2.2) and Theorem 3.3.9]. In particular, we view the functions ϕh_i as functions over $\mathcal{X}$, for all $i \in \mathbb{N}$. By [FOT, Theorem 1.4.2(ii),(iii)], $\phi h_i = \phi(h_i \wedge M) \in \mathcal{F}$. Note that it suffices to show that ϕh_i is

Cauchy in the seminorm induced by $\mathcal{E}(\cdot,\cdot)$. This is because ϕh_i converges pointwise and in $L^2(\Omega)$ to $\phi g_\Omega(x,\cdot)$. This would imply that $\phi g_\Omega(x,\cdot) \in \mathcal{F}$ since $(\mathcal{E},\mathcal{F})$ is a closed form.

To complete the proof that $g_\Omega(x,\cdot) \in \mathcal{F}^0_{\mathrm{loc}}(\Omega, \Omega \setminus \{x\})$, we show that ϕh_i is a Cauchy sequence in the seminorm induced by $\mathcal{E}(\cdot,\cdot)$. To this end, by Leibniz rule (cf. [FOT, Lemma 3.2.5]) we have

$$\mathcal{E}(\phi(h_i - h_j), \phi(h_i - h_j)) = \int_{\mathcal{X}} (h_i - h_j)^2 \, d\Gamma(\phi,\phi) + \mathcal{E}(h_i - h_j, \phi^2(h_i - h_j)). \tag{4.11}$$

The second term in (4.11) is zero, exactly by the argument in [GyS, p. 91] which we recall for convenience. Since $h_i - h_j$ is harmonic in $\Omega \setminus \overline{B(x,r)}$ and $\phi^2(h_i - h_j) \in \mathcal{F}^0(U)$, we can approximate $\phi^2(h_i - h_j)$ in the $\left(\|\cdot\|_2^2 + \mathcal{E}(\cdot,\cdot)\right)^{1/2}$-norm by functions in $\mathcal{F}_c(U)$ that vanish in $B(x, 3r/2)$. For the first term in (4.11), we use the fact that h_i vanish in Ω^c and strong locality to obtain

$$\int_{\mathcal{X}} (h_i - h_j)^2 \, d\Gamma(\phi,\phi) = \int_{\overline{B(x,3r)} \setminus B(x,2r)} (h_i - h_j)^2 \, d\Gamma(\phi,\phi) \tag{4.12}$$

Since the function $g_\Omega(\cdot,\cdot)$ is continuous over the compact set $\overline{B(x,r)} \times \overline{B(x,3r)} \setminus B(x,2r)$, it is uniformly continuous over $\overline{B(x,r)} \times \overline{B(x,3r)} \setminus B(x,2r)$. This in turn implies that h_i converges uniformly to $g_\Omega(x,\cdot)$ as $i \to \infty$ on $\overline{B(x,3r)} \setminus B(x,2r)$, which by (4.12) implies that ϕh_i is Cauchy in the $(\mathcal{E}(\cdot,\cdot))^{1/2}$-seminorm.

Finally, we show that $g_\Omega(x,\cdot)$ is harmonic on $\Omega \setminus \{x\}$. Let $\psi \in \mathcal{F}_c(\Omega \setminus \{x\})$ an arbitrary function. Let $V \subset \Omega$ be an open set containing the $\mathrm{supp}(\psi)$ such that $d(x,V) > 0$. By the above construction, there exists a sequence ϕh_i such that $\phi \equiv 1$ on V, ϕh_i converges to ϕg_Ω in the $\left(\|\cdot\|_2^2 + \mathcal{E}(\cdot,\cdot)\right)^{1/2}$-norm and ϕh_i is harmonic on V. This immediately implies that $g_\Omega(x,\cdot)$ is harmonic on $\Omega \setminus \{x\}$. □

The elliptic Harnack inequality enables us to relate capacity and Green functions functions, and also to control their fluctuations on bounded regions of $\mathcal{X}$.

Proposition 4.18. *(See [BM, Section 3]). Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space satisfying EHI and Assumption 4.9. The following hold:*

(a) For all $A_1, A_2 \in (1,\infty)$, there exists $C_G = C_G(A_1, A_2, C_H) > 1$ such that for all bounded open sets D and for all $x_0 \in \mathcal{X}, r > 0$ that satisfy $B(x_0, A_1 r) \subset D$, we have

$$g_D(x_0, y) \le C_G g_D(x_0, z) \quad \forall y, z \in \overline{B(x_0,r)} \setminus B(x_0, r/A_2).$$

(b) For all $A_1, A_2 \in (1,\infty)$, there exists $C_0 = C_0(A, B, C_H) > 1$ such that for all bounded open sets D and for all $x_0 \in \mathcal{X}, r > 0$ that satisfy $B(x_0, A_1 r) \subset D$, we have

$$g_D(x_1, y_1) \le C_0 g_D(x_2, y_2) \quad \forall x_1, y_1, x_2, y_2 \in B(x_0, r),$$

such that $d(x_i, y_i) \ge r/A_2$, for $i = 1, 2$.

(c) *For all $A \in (1,\infty)$, there exists $C_1 = C_1(A, C_H) > 1$ such that for all bounded open sets D and for all $x_0 \in \mathcal{X}, r > 0$ that satisfy $B(x_0, Ar) \subset D$, we have*

$$\inf_{y \in \partial B(x_0,r)} g_D(x_0,y) \le \operatorname{Cap}_D\left(\overline{B(x_0,r)}\right)^{-1} \le \operatorname{Cap}_D\left(B(x_0,r)\right)^{-1} \le C_1 \inf_{y \in \partial B(x_0,r)} g_D(x_0,y).$$

(d) *For all $1 \le A_1 \le A_2 < \infty$ and $a \in (0,1]$ there exists $C_2 = C_2(a, A_1, A_2, C_H) > 1$ such that for $x \in \mathcal{X}$, and $r > 0$ with $r \le \operatorname{diameter}(\mathcal{X})/5A$,*

$$\operatorname{Cap}_{B(x_0,A_2 r)}\left(B(x_0,ar)\right) \le \operatorname{Cap}_{B(x_0,A_1 r)}\left(B(x_0,r)\right) \le C_2 \operatorname{Cap}_{B(x_0,A_2 r)}\left(B(x_0,ar)\right).$$

(e) *For all $A_2 > A_1 \ge 2$ there exists $C_3 = C_3(A_1, A_2, C_H) > 1$ such that for all $x, y \in \mathcal{X}$, with $d(x,y) = r > 0$ and such that $r \le \operatorname{diameter}(\mathcal{X})/5A$,*

$$g_{B(x,A_1 r)}(x,y) \le g_{B(x,A_2 r)}(x,y) \le C_3 g_{B(x,A_1 r)}(x,y).$$

(f) *$(\mathcal{X}, d)$ satisfies metric doubling.*

The statements above are slightly stronger than those in [BM, Section 3] but follow from there results there using straightforward chaining arguments.

4.4 Examples

In this section, we give examples of MMD spaces which satisfy Assumption 4.9: weighted Riemannian manifolds, cable systems of weighted graphs, and some regular fractals.

Let $(\mathcal{M}, g)$ be a Riemannian manifold, and ν and ∇ denote the Riemannian measure and the Riemannian gradient respectively. Write $d = d_g$ for the Riemannian distance function. A *weighted manifold* $(\mathcal{M}, g, \mu)$ is a Riemmanian manifold $(\mathcal{X}, g)$ endowed with a measure μ that has a smooth (strictly) positive density w with respect to the Riemannian measure ν. The *weighted Laplace operator* Δ_μ on $(\mathcal{M}, g, \mu)$ is given by

$$\Delta_\mu f = \Delta f + g\left(\nabla\left(\ln w\right), \nabla f\right), \quad f \in \mathcal{C}^\infty(\mathcal{M}).$$

We say that the weighted manifold (M, g, μ) has *controlled weights* if w satisfies

$$\sup_{x,y \in \mathcal{M}: d(x,y) \le 1} \frac{w(x)}{w(y)} < \infty.$$

The construction of heat kernel, Markov semigroup and Brownian motion for a weighted Riemannian manifold $(\mathcal{X}, g, \mu)$ is outlined in [Gri, Sections 3 and 8]. The corresponding Dirichlet form on $L^2(\mathcal{X}, \mu)$ given by

$$\mathcal{E}_w(f_1, f_2) = \int_{\mathcal{X}} g(\nabla f_1, \nabla f_2)\, d\mu, \quad f_1, f_2 \in \mathcal{F},$$

where $\mathcal{F}$ is the weighted Sobolev space of functions in $L^2(\mathcal{X}, \mu)$ whose distributional gradient is also in $L^2(\mathcal{X}, \mu)$. See [Gri] and [CF, pp. 75–76] for more details.

Our second example is the cable system of weighted graphs. Let $\mathbb{G} = (\mathbb{V}, E)$ be an infinite graph, such that each vertex x has finite degree. For $x \in V$ we write $x \sim y$ if $\{x, y\} \in E$. Let $w : E \to (0, \infty)$ be a function which assigns weight w_e to the edge e. We write w_{xy} for $w_{\{x,y\}}$, and define

$$w_x = \sum_{y \sim x} w_{xy}.$$

We call $(\mathbb{V}, E, w)$ a *weighted graph*. An *unweighted graph* has $w_e \equiv 1$. We say that $\mathbb{G}$ has *controlled weights* if there exists $p_0 > 0$ such that

$$\frac{w_{xy}}{w_x} \geq p_0 \text{ for all } x \in \mathbb{V}, y \sim x. \tag{4.13}$$

The *cable system of a weighted graph* gives a natural embedding of a graph in a connected metric length space. Choose a direction for each edge $e \in E$, let $(I_e, e \in E)$ be a collection of copies of the open unit interval, and set

$$\mathcal{X} = \mathbb{V} \cup \bigcup_{e \in E} I_e.$$

(We call the sets I_e cables). We define a metric d_c on $\mathcal{X}$ by using Euclidean distance on each cable, and then extend d_c to a metric on $\mathcal{X}$; note that this agrees with the graph metric for $x, y \in \mathbb{V}$. Let m be the measure on $\mathcal{X}$ which assigns zero mass to points in $\mathbb{V}$, and mass $w_e|s - t|$ to any interval $(s, t) \subset I_e$. It is straightforward to check that $(\mathcal{X}, d, m)$ is a MMD space. For more details on this construction see [V, BB3].

We say that a function f on $\mathcal{X}$ is piecewise differentiable if it is continuous at each vertex $x \in \mathbb{V}$, is differentiable on each cable, and has one sided derivatives at the endpoints. Let $\mathcal{F}_0$ be the set of piecewise differentiable functions f with compact support. Given two such functions we set

$$d\Gamma(f, g)(t) = w_e f'(t) g'(t) m(dt),$$
$$\mathcal{E}(f, g) = \int_{\mathcal{X}} d\Gamma(f, g)(t), \quad f, g \in \mathcal{F}_0,$$

and let $\mathcal{F}$ be the completion of $\mathcal{F}_0$ with respect to the $\mathcal{E}_1^{1/2}$norm. We extend $\mathcal{E}$ to $\mathcal{F}$, and it is straightforward to verify that $(\mathcal{E}, \mathcal{F})$ is a closed regular strongly local Dirichlet form. We call $(\mathcal{X}, d_c, m, \mathcal{E}, \mathcal{F})$ the *cable system* of the graph $\mathbb{G}$.

We say a function $u = u(x, t)$ is *caloric* in a region $Q \subset \mathcal{X} \times (0\infty)$ if u is a weak solution of $(\partial_t + \mathcal{L})u = 0$ in Q; here $\mathcal{L}$ is the generator corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F}, L^2(\mathcal{X}, \mu))$.

Definition 4.19. *We say a MMD space* $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ *satisfies the local volume doubling property* $(\mathrm{VD})_{\mathrm{loc}}$*, if there exists* $R, C_R > 0$ *such that*

$$V(x, 2r) \leq C_R V(x, r) \tag{VD$_{\mathrm{loc}}$}$$

for all $x \in \mathcal{X}$ *and for all* $0 < r \leq R$.

We say a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local Poincaré inequality $(\mathrm{PI}(2))_{\mathrm{loc}}$*, if there exists $R, C_R > 0$ and $A \geq 1$ such that*

$$\int_{B(x,r)} \left|f - f_{B(x,r}\right|^2 d\mu \leq C_R r^2 \int_{B(x,Ar)} d\Gamma(f,f) \qquad (\mathrm{PI}(2))_{\mathrm{loc}}$$

for all $x \in \mathcal{X}$ and for all $0 < r \leq R$, where $\Gamma(f,f)$ denotes the energy measure and $f_{B(x,r} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu$.

We say a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies the local parabolic Harnack inequality $(\mathrm{PHI}(2))_{\mathrm{loc}}$*, if there exists $R > 0, C_R > 0$ such that for all $x \in \mathcal{X}$, for all $0 < r \leq R$, any non-negative caloric function on $(0, r^2) \times B(x,r)$ satisfies*

$$\sup_{(r^2/4, r^2/2) \times B(x,r/2)} u \leq C_R \inf_{(3r^2/4, r^2) \times B(x,r/2)} u, \qquad (\mathrm{PHI}(2))_{\mathrm{loc}}$$

We will now show that the two examples given above satisfy Assumption 4.13, and therefore Assumption 4.9.

Lemma 4.20. *(a) Let $(\mathcal{M}, g, \mu)$ be a weighted Riemmanian manifold with controlled weights w which is quasi isometric to a Riemannian manifold $(\mathcal{M}', g')$ with Ricci curvature bounded below. Then the MMD space $(\mathcal{M}, d_g, \mu, \mathcal{E}_w)$ satisfies $(\mathrm{PHI}(2))_{\mathrm{loc}}$.*
(b) If $\mathbb{G}$ is a weighted graph with controlled weights then the corresponding cable system satisfies $(\mathrm{PHI}(2))_{\mathrm{loc}}$.

Proof. (a) The properties $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$ for $(\mathcal{M}', g')$ follow from the Bishop-Gromov volume comparison theorem [Cha, Theorem III.4.5] and Buser's Poincaré inequality (see [Sal02, Lemma 5.3.2]) respectively. Since quasi isometry only changes distances and volumes by at most a constant factor, we have that $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$ also hold for $(\mathcal{M}, g)$. The controlled weights condition on w then implies that these two conditions also hold for $(\mathcal{M}, g, \mu)$.
(b) Using the controlled weights condition and uniform bound on vertex degree, one can easily obtain $(\mathrm{VD})_{\mathrm{loc}}$ and $(\mathrm{PI}(2))_{\mathrm{loc}}$.

Finally, it is known that $(PHI(2))_{\mathrm{loc}}$ is equivalent to the conjunction of $(PI(2))_{\mathrm{loc}}$ and $(\mathrm{VD})_{\mathrm{loc}}$ – see [CS, Theorem 8.1] and [HS, Theorem 2.7], so it follows in both cases (a) and (b) that $(PHI(2))_{\mathrm{loc}}$ holds. □

Lemma 4.21. *Let $(\mathcal{X}, d)$ be a complete, locally compact, separable, length metric space with $\mathrm{diameter}(\mathcal{X}) = \infty$, μ be a Radon measure on $(\mathcal{X}, d)$ with full support and $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form on $L^2(\mathcal{X}, \mu)$. Suppose that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{PHI}(2))_{\mathrm{loc}}$. Then $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies $\mathrm{EHI}_{\mathrm{loc}}$ and Assumption 4.13.*

Proof. First, we verify the local ultracontractivity. The heat kernel corresponding to $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ satisfies Gaussian upper bounds for small times by [HS, Theorem 2.7]. Since the Dirichlet heat kernel is dominated by the heat kernel of $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$, by $(\mathrm{VD})_{\mathrm{loc}}$ we have local ultracontractivity. The fact that μ is non-atomic follows from $(\mathrm{VD})_{\mathrm{loc}}$ due to a reverse volume doubling property [HS, eq. (2.5)]. It is obvious that $(\mathrm{PHI}(2))_{\mathrm{loc}}$ implies $\mathrm{EHI}_{\mathrm{loc}}$

By domain monotonicity, it suffices to verify $\lambda_{\min}(B(x,r)) > 0$ for all balls $B := B(x,r)$ with $0 < r < \text{diameter}(X)/4$. Consider a disjoint ball $B(z,r)$ such that $B(x,r) \cap B(z,r) = \emptyset$ and $d(x,z) \leq 3r$. Let $\left(P_t^B\right)_{t\geq 0}$ denote the Dirichlet type heat semigroup as given in Definition 4.11. By the Gaussian lower bound for small times [HS, Theorem 2.7] and $(\mathrm{VD})_{\mathrm{loc}}$, there exists $t_0 > 0, \delta \in (0,1)$ such that

$$\mathbb{P}^y(X_{t_0} \in B(z,r)) \geq \delta, \quad \forall y \in B,$$

where $(X_t)_{t\geq 0}$ is the diffusion corresponding to the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$. This implies that $P_{t_0}^B \mathbb{1}_B \leq \delta \mathbb{1}_B$, which in turn implies

$$P_t^B \mathbb{1}_B \leq \delta^{\lfloor (t/t_0) \rfloor} \mathbb{1}_B, \quad \forall t \geq 0.$$

Therefore, we obtain $\lambda_{\min}(B)^{-1} = \left\| G^B \right\|_{L^2 \to L^2} \leq \left\| G^B \right\|_{L^\infty \to L^\infty} = \left\| G^B \mathbb{1}_B \right\|_{L^\infty} < \infty$. □

Remark 4.22. The hypothesis $(\mathrm{PHI}(2))_{\mathrm{loc}}$ in Lemma 4.21 can be replaced by a more general local parabolic Harnack inequality that allows for anomalous space-time scaling as given in [GHL, eq. (1.15)]. In particular the spaces considered in [L] do satisfy Assumption 4.13.

5 Proof of Boundary Harnack Principle

The main result of this paper is the following boundary Harnack principle.

Theorem 5.1. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space satisfying EHI and Assumption 4.13. Let $U \subsetneq \mathcal{X}$ be a connected, inner uniform domain. Then, there exists $A_0, C_1 \in (1,\infty), R(U) \in (0,\infty]$ such that for all $\xi \in \partial_{\widetilde{U}} U$, for all $0 < r < R(U)$ and any two non-negative functions u, v that are harmonic on $B_U(\xi, A_0 r)$ with Dirichlet boundary conditions along $\partial_{\widetilde{U}} U \cap B_{\widetilde{U}}(\xi, 2A_0 r)$, we have*

$$\frac{u(x)}{u(x')} \leq C_1 \frac{v(x)}{v(x')} \qquad \forall x, x' \in B_U(\xi, r).$$

The constant $R(U)$ depends only on the inner uniformity constants of U and diameter(U) *and can be chosen to be $+\infty$ if U is unbounded.*

Remark 5.2. The constant A_0 only on the inner uniformity constants c_U, C_U for the domain U, and C_1 depends only on c_U, C_U and the constant C_H for the EHI.

In this Section will use A_i to denote constants which just depend on the constants c_U and C_U; other constants will depend on c_U, C_U and C_H.

For the remainder of the section, we assume the hypotheses of Theorem 5.1. Since by Proposition 4.18(e) $(\mathcal{X}, d)$ has the metric doubling property, we can use Lemma 2.12. In addition we will assume that

$$\mathrm{diameter}(U) = \infty,$$

so that $R(U) = \infty$. The proof of the general case is the same except that we need to ensure that the balls $B_U(\xi, s)$ considered in the argument are all small enough so that they do not equal U.

As remarked in the Introduction, we follow Aikawa's approach in [Aik01]. This has been adapted by several authors to more general settings [ALM, GyS, LS, L]. However all these works rely on two sided heat kernel estimates, which are used to obtain two sided estimates of Green's function for the killed process expressed in terms of volume growth and a space-time scaling function. In our argument we use instead the estimates on Green's functions for spaces satisfying EHI proved in [BM, Section 2] and summarized in Proposition 4.18.

Definition 5.3 (Capacitary width)**.** For an open set $V \subset \mathcal{X}$ and $\eta \in (0, 1)$, define the capacitary width $w_\eta(V)$ by

$$w_\eta(V) = \inf \left\{ r > 0 \ : \ \frac{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)} \setminus V\right)}{\mathrm{Cap}_{B(x,2r)}\left(\overline{B(x,r)}\right)} \geq \eta \ \forall x \in V \right\}. \tag{5.1}$$

Note that $w_\eta(U)$ is an increasing function of $\eta \in (0, 1)$ and an increasing function of the set U.

Lemma 5.4. *(See [GyS, Lemma 4.12]) There exists $\eta = \eta(c_U, C_U, C_H) \in (0, 1)$ and $A_2 > 0$ such that*

$$w_\eta\left(\{x \in U \ : \ \delta_U(x) < r\}\right) \leq A_2 r.$$

Proof. Set $V_r = \{x \in U \ : \ \delta_U(x) < r\}$. Using the inner uniformity and unboundedness of U (cf. Lemma 2.13), there is a constant $A > 1$ such that for any point $x \in V_r$, there is a point $z \in U \cap B(x, Ar)$ with the property that $\delta_U(z) \geq 2r$. Set $A_2 = A + 1$. By domain monotonicity of capacity, we have

$$\mathrm{Cap}_{B(x,2Ar)}\left(\overline{B(x, Ar)} \setminus V_r\right) \geq \mathrm{Cap}_{B(x,2Ar)}\left(\overline{B(z, r)}\right) \geq \mathrm{Cap}_{B(z,3Ar)}\left(\overline{B(z, r)}\right).$$

The capacities $\mathrm{Cap}_{B(z,3Ar)}\left(\overline{B(z, r)}\right)$ and $\mathrm{Cap}_{B(x,2Ar)}\left(\overline{B(x, Ar)}\right)$ are comparable by Proposition 4.18(a)-(d), and so the condition in (5.1) holds for some $\eta > 0$, with r replaced by Ar. □

We fix $\eta \in (0, 1)$ once and for all, small enough such that the conclusion of Lemma 5.4 applies. In what follows, we write $f \asymp g$, if there exists a constant $C_1 > 1$ that depends such that $C_1^{-1} \leq f \leq C_1 g$ only on the inner uniformity constants of the domain U and the constant C_H from EHI. The corresponding one sided inequalities $f \lesssim g$ and $f \gtrsim g$ are understood analogously.

Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ denote the Dirichlet space as before, and let $(X_t, t \geq 0, \mathbb{P}^x, x \in \mathcal{X})$ be the associated (continuous) Hunt process. We recall the definition of harmonic measure below.

Definition 5.5. Let $\Omega \subset \mathcal{X}$ be open relatively compact in $\mathcal{X}$, $V := \mathcal{X} \setminus \Omega$, and recall from (3.3) that τ_Ω denotes the first exit time of X from Ω. Since X is continuous, $X_{\tau_\Omega} \in \partial_{\mathcal{X}}\Omega$ a.s. Now define the *harmonic measure* $\omega(x, \cdot, \Omega)$ on $\partial_{\mathcal{X}}\Omega$ by setting

$$\omega(x, F, \Omega) := \mathbb{P}^x(X_{\tau_\Omega} \in F) \text{ for } F \subset \partial_{\mathcal{X}}\Omega.$$

The following lemma provides an useful estimate of the harmonic measure in terms of the capacitary width.

Lemma 5.6. *(See [GyS, Lemma 4.13]) For any non-empty open set $V \subset \mathcal{X}$ and for all $x \in \mathcal{X}$, $r > 0$, there exists $a_1 \in (0, 1)$ such that*

$$\omega\left(x, V \cap \partial B(x, r), V \cap B(x, r)\right) \leq \exp\left(2 - \frac{a_1 r}{w_\eta(V)}\right)$$

Proof. The proof is same as [GyS, Lemma 4.13] except that we use Proposition 4.18(a),(c) instead of [GyS, Lemma 4.8]. $\square$

In the following lemma, we provide an upper bound of the harmonic measure in terms of the Green function. It is an analogue of [Aik01, Lemma 2].

Lemma 5.7. *(Cf. [GyS, Lemma 4.14], [LS, Lemma 4.9] and [L, Lemma 5.3]) There exists $A_3, C_4 \in (0, \infty)$ such that for all $r > 0$, $\xi \in \partial_{\widetilde{U}}U$, there exist $\xi_r, \xi_r' \in U$ that satisfy $d_U(\xi, \xi_r) = 4r$, $\delta_U(\xi_r) \geq 2c_U r$, $d(\xi_r, \xi_r') = c_U r$ and*

$$\omega\left(x, \partial_U B_U(\xi, 2r), B_U(\xi, 2r)\right) \leq C_4 \frac{g_{B_U(\xi, A_3 r)}(x, \xi_r)}{g_{B_U(\xi, A_3 r)}(\xi_r', \xi_r)}, \quad \forall x \in B_U(\xi, r).$$

Proof. Let $r > 0$, $\xi \in \partial_{\widetilde{U}}U$ be arbitrary. Fix $A_3 \geq 2(12 + C_U)$ so that all (c_U, C_U)-inner uniform curves that connect two points in $B_U(\xi, 12r)$ stay inside $B_U(\xi, A_3 r/2)$. Fix $\xi_r, \xi_r' \in U$ satisfying the given hypothesis: these points exist by Lemma 2.13. For $z \in B_U(\xi, A_3 r)$, we set

$$g'(z) = g_{B_U(\xi, A_3 r)}(z, \xi_r).$$

Set $s = \min(c_U r, 5r/C_U)$. Since $B_U(\xi_r, s) \subset B_U(\xi, A_3 r) \setminus B_U(\xi, 2r)$, by the maximum principle (Lemma 4.15(iv))

$$g'(y) \leq \sup_{z \in \partial_U B_U(\xi_r, s)} g'(z) \text{ for all } y \in B_U(\xi, 2r).$$

By Proposition 4.18(a), we have

$$\sup_{z\in\partial_U B_U(\xi_r,s)} g'(z) \asymp g'(\xi_r'),$$

and hence there exists $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \frac{g'(y)}{g'(\xi_r')} \le e^{-1} \quad \forall y \in B_U(\xi, 2r).$$

For all non-negative integers j, define

$$U_j := \left\{ x \in Y \; : \; \exp\left(-2^{j+1}\right) \le \varepsilon_1 \frac{g'(x)}{g'(\xi_r')} < \exp\left(-2^j\right) \right\},$$

so that $B_U(\xi, 2r) = \bigcup_{j\ge 0} U_j \cap B_U(\xi, 2r)$. Set $V_j = \bigcup_{k\ge j} U_k$. We claim that there exist $c_1, \sigma \in (0,\infty)$ such that for all $j \ge 0$

$$w_\eta \left(V_j \cap B_U(\xi, 2r)\right) \le c_1 r \exp\left(-2^j/\sigma\right). \tag{5.2}$$

Let x be an arbitrary point in $V_j \cap B_U(\xi, 2r)$. Let z be the first point in the inner uniform curve from x to ξ_r which is on $\partial_U B_U(\xi_r, c_U r)$. Then by Lemma 4.7 there exists a Harnack chain of balls in $B_U(\xi, A_3 r) \setminus \{\xi_r\}$ connecting x to z of length at most $c_2 \log\left(1 + c_3 r/\delta_U(x)\right)$ for some constants $c_2, c_3 \in (0,\infty)$. Hence, there are constants $\varepsilon_2, \varepsilon_3, \sigma > 0$ such that

$$\exp(-2^j) > \varepsilon_1 \frac{g'(x)}{g'(\xi_r')} \ge \varepsilon_2 \frac{g'(x)}{g'(z)} \ge \varepsilon_3 \left(\frac{\delta_U(x)}{r}\right)^\sigma.$$

The first inequality above follows from definition of V_j, the second follows from Proposition 4.18(a) and the last one follows from Harnack chaining. Therefore, we have

$$V_j \cap B_U(\xi, 2r) \subset \left\{ x \in U \; : \; \delta_U(x) \le \varepsilon_3^{-1/\sigma} r \exp\left(-2^j/\sigma\right) \right\},$$

which by Lemma 5.4 immediately implies (5.2).

Set $R_0 = 2r$ and for $j \ge 1$,

$$R_j = \left(2 - \frac{6}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2}\right) r.$$

Then $R_j \downarrow r$ and as in [GyS]

$$\sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{a_1(R_{j-1} - R_j)}{A_4 r \exp(-2^j/\sigma)}\right) < C < \infty; \tag{5.3}$$

here C depends only on σ and the constant a_1 in Lemma 5.6.

Let $\omega_0(\cdot) = \omega\left(\cdot, \partial_U B_U(\xi, 2r), B_U(\xi, 2r)\right)$ and set

$$d_j = \begin{cases} \sup_{x \in U_j \cap B_{\widetilde{U}}(\xi, R_j)} \frac{g'(\xi_r')\omega_0(x)}{g'(x)}, & \text{if } U_j \cap B_{\widetilde{U}}(\xi, R_j) \neq \emptyset, \\ 0, & \text{if } U_j \cap B_{\widetilde{U}}(\xi, R_j) = \emptyset. \end{cases}$$

It suffices to show that $\sup_{j \geq 0} d_j \leq C_1 < \infty$, and this is proved by iteration exactly as in [LS, Lemma 4.9] or [L, Lemma 5.3]. The only difference is that we replace $r^2/V(\xi, r)$ in [LS] (or $\Psi(r)/V(\xi, r)$ in [L]) by $g'(\xi_r')$. □

By using a balayage formula (cf. [L, Proposition 4.3]) and a standard argument (cf. [GyS, pp. 75-76], [L, Theorem 5.2]) the BHP in Theorem 5.1 follows easily given the following estimate on Green's function.

Theorem 5.8. *There exists $C_1, A_0, A_8 \in (1, \infty)$ such that for any $\xi \in \partial_{\widetilde{U}} U$ and for all $r > 0$, we have writing $D = B_U(\xi, A_0 r)$,*

$$\frac{g_D(x_1, y_1)}{g_D(x_2, y_1)} \leq C_1 \frac{g_D(x_1, y_2)}{g_D(x_2, y_2)} \text{ for all } x_1, x_2 \in B_U(\xi, r),\ y_1, y_2 \in \partial_{\widetilde{U}} B_U(\xi, A_8 r).$$

Again, our proof will follow Aikawa's approach as presented in [Aik01, Lemma 3]. We replace the use of Green's function estimates with Proposition 4.18. In addition, as we are working on a metric space, we need to be careful with Harnack chaining. Since this is the key estimate in this paper, we provide the full proof, and as the proof is long we split it into several Lemmas.

Throughout the remainder of this section, we take U to be a (c_U, C_U) inner uniform domain. We define

$$A_8 = \max(2 + 2c_U^{-1}, 7). \tag{5.4}$$

Lemma 5.9. *Let $\xi \in \partial_{\widetilde{U}} U$ and $y_1, y_2 \in \partial_{\widetilde{U}} B_U(\xi, A_8 r)$. If γ is a (c_U, C_U) inner uniform curve from y_1 to y_2 in U then $\gamma \cap B_U(\xi, 2r) = \emptyset$ and $\gamma \subset \overline{B_U\left(\xi, A_8(C_U + 1)r\right)}$.*

Proof. Let $z \in \gamma$. If $d_U(y_1, z) \wedge d_U(y_2, z) \leq (A_8 - 2)r$ then by the triangle inequality $d_U(z, \xi) \geq 2r$. If $d_U(y_1, z) \wedge d_U(y_2, z) > (A_8 - 2)r$ then using the inner uniformity of γ,

$$\delta_U(z) \geq c_U\left(d_U(y_1, z) \wedge d_U(y_2, z)\right) > c_U(A_8 - 2)r \geq 2r,$$

which implies that $z \notin B_U(\xi, 2r)$.

For the second conclusion, note that for all $z \in \gamma$,

$$d_U(\xi, z) \leq A_8 r + \min\left(d_U(y_1, z), d_U(y_2, z)\right) \leq A_8 r + L(\gamma)/2 \leq A_8(C_U + 1)r.$$

□

For $\xi \in \partial_{\widetilde{U}} U$ choose $x^* = x_\xi^* \in \partial_{\widetilde{U}} B_U(\xi, r)$ and $y^* = y_\xi^* \in \partial_{\widetilde{U}} B_U(\xi, A_8 r)$ such that $\delta_U(x_\xi^*) \geq c_U r$ and $\delta_U(y_\xi^*) \geq A_8 c_U r$. Note that we have

$$\delta_U(y_\xi^*) \geq A_8 c_U r > 2r, \tag{5.5}$$

so that $B(y^*_\xi, 2r) \subset U$. Let γ_ξ be an inner uniform curve from y^*_ξ to x^*_ξ, and let z^*_ξ be the last point of this curve which is on $\partial_{\widetilde{U}} B_U(y^*_\xi, c_U r) = \partial B(y^*, c_U r)$. We will write these points as x^*, y^*, z^* when the choice of the boundary point ξ is clear.

Let A_3 be the constant from Lemma 5.7. Define

$$A_0 = A_3 + C_U(A_8 + \frac{1}{4}c_U^2 + 8).$$

To prove Theorem 5.8 it is sufficient to prove that for all $x \in B_U(\xi, r)$, for all $y \in \partial_{\widetilde{U}} B_U(\xi, A_8 r)$

$$g_D(x, y) \asymp \frac{g_D(x^*, y)}{g_D(x^*, y^*)} g_D(x, y^*). \tag{5.6}$$

Lemma 5.10. *Let $\xi \in \partial_{\widetilde{U}} U$. If $x \in B_U(\xi, r)$ and $y \in \partial_{\widetilde{U}} B_U(\xi, A_8 r)$ with $\delta_U(y) \geq \frac{1}{4}c_U^2 r$ then (5.6) holds.*

Proof. Fix $x \in B_U(\xi, r)$. Set

$$u_1(y') = g_D(x, y'), \quad v_1(y') = \frac{g_D(x^*, y')}{g_D(x^*, y^*)} g_D(x, y^*).$$

The functions u_1 and v_1 are harmonic in $D \setminus \{x, x^*\}$, vanish quasi-everywhere on the boundary of D, and satisfy $u_1(y^*) = v_1(y^*)$. Let γ be a (c_U, C_U) inner uniform curve from y to y^*; by Lemma 5.9 this curve is contained in $U \setminus B_U(\xi, 2r)$. So by Lemma 2.14 $\delta_U(z) \geq \frac{1}{2}c_U \left(\delta_U(y) \wedge \delta_U(y^*)\right) \geq \frac{1}{8}c_U^3 r$. Thus we can find a Harnack chain of balls in $U \setminus \{x, x^*\}$ of radius $\frac{1}{8}c_U^3 r$ connecting y and y^*, and using the EHI (5.6) follows. □

Lemma 5.11. *Let $\xi \in \partial_{\widetilde{U}} U$. If $x \in B_U(\xi, r)$ and $y \in \partial_{\widetilde{U}} B_U(\xi, A_8 r)$ with $\delta_U(y) < \frac{1}{4}c_U^2 r$ then*

$$g_D(x, y) \geq c \frac{g_D(x^*, y)}{g_D(x^*, y^*)} g_D(x, y^*). \tag{5.7}$$

Proof. Fix y and call u (respectively, v) the left-hand (resp. right-hand) side of (5.15), viewed as a function of x. Then by Lemma 4.15, u is harmonic in $D \setminus \{y\}$ and v is harmonic in $D \setminus \{y^*\}$. Moreover, both u and v vanish quasi-everywhere on the boundary of D, and $u(x^*) = v(x^*)$.

Let γ_ξ and z^* be as defined above. By Lemma 2.14 we have $\delta_U(z) \geq \frac{1}{2}c_U \delta_U(x^*) \geq \frac{1}{2}c_U^2 r$ for all $z \in \gamma_\xi$, and so this curve lies a distance at least $\frac{1}{4}c_U^2 r$ from y. By the choice of z^* the part of the curve from z^* to x^* lies outside $B_U(y^*, \frac{1}{2}c_U r)$. Thus there exists c_1 such that there is a a Harnack chain of balls in $U \setminus \{y, y^*\}$ connecting x^* with z^* of length at most c_1. Using this we deduce that there exists $C < \infty$ such that

$$C^{-1}v(z^*) \leq v(x^*) \leq Cv(z^*), \quad C^{-1}u(z^*) \leq u(x^*) \leq Cu(z^*). \tag{5.8}$$

Since $B(y^*, 2c_U r) \subset U \setminus \{y, y^*\}$, we can use the the EHI and Proposition 4.18 to deduce that

$$C^{-1}v(z^*) \leq v(z) \leq Cv(z^*), \quad C^{-1}u(z^*) \leq u(z) \leq Cu(z^*) \text{ for all } z \in \partial_{\widetilde{U}} B_U(y^*, c_U r).$$

Thus there exist c_1, c_2 such that

$$c_1 u(z) \geq u(x^*) = v(x^*) \geq c_1 c_2 v(z) \text{ for all } z \in \partial_{\widetilde{U}} B_U(y^*, c_U r), \tag{5.9}$$

so that the function $w = u - c_2 v$ is non-negative on $\partial_{\widetilde{U}} B_U(y^*, c_U r)$. This function is superharmonic on $D \setminus B_U(y^*, c_U r)$, and vanishes q.e. on the boundary of D; provided we can apply the maximum principle for superharmonic functions we will then deduce that

$$u(x) \geq c_2 v(x) \quad \text{ for all } x \in B_U(\xi, r). \tag{5.10}$$

There are two technical difficulties in this application of the maximum principle. The first is that the definition of superharmonic function in [GH1, Definition 3.1] requires the function w to be in the domain of the Dirichlet form. The second is that the maximum principle stated in [GH1, Lemma 4.1] applies only to bounded superharmonic functions. To handle the first difficulty we approximate u by an appropriate sequence u_n of superharmonic functions in the domain of the Dirichlet form, in the same way as was done in Lemma 4.17 – see (4.10). The second difficulty is handled by additionally replacing u_n by $M \wedge u_n$, where $M = 2\sup_{z \in \partial_U B_U(y^*, c_U r)} v(z)$. Note that by Lemmas 4.15 and 4.17, $v \in \mathcal{F}_{\text{loc}}(D \setminus B_U(y^*, c_U r))$ and v is a bounded harmonic function in $D \setminus B_U(y^*, c_U r)$. By [FOT, Corollary 2.2.2 and Lemma 2.2.10], the functions $M \wedge u_n - \epsilon_1 v$ are bounded and superharmonic in $D \setminus B_U(y^*, c_U r)$. Hence, we can apply the maximum principle given in [GH1, Lemma 4.1(2)] to the functions $M \wedge u_n - c_2 v$ in the domain $D \setminus B_U(y^*, c_U r)$ and then let u_n converge to u to obtain (5.10). □

Let

$$F(\xi) = B_U(\xi, (A_8 + 3)r) \setminus B_U(\xi', (A_8 - 3)r).$$

Let $A_{10} = A_8 + A_0$.

Lemma 5.12. *Let $\xi \in \partial_{\widetilde{U}} U$, and $D = B_U(\xi, A_0 r)$. Then*

$$g_D(x, z) \leq C_1 g_D(x, y^*), \quad \textit{for all } x \in B_U(\xi, 2r),\ z \in F(\xi). \tag{5.11}$$

Proof. We begin by proving that

$$g_D(x, y) \leq C_1 g_D(x^*, y^*), \quad \text{for all } x \in B_U(\xi, 2r),\ y \in F(\xi). \tag{5.12}$$

Let $x \in B_U(\xi, 2r)$, $y \in F(\xi)$. We have $D \subset B(y^*, A_{10} r)$, and therefore by domain montonicity of Green's function

$$g_D(x, y) \leq g_{B(x, A_{10} r)}(x, y).$$

We first consider the case $d(x, y) \geq r/(2\widetilde{C_U})$, where $\widetilde{C_U}$ is the constant from Lemma 2.12. In this case, by domain monotonicity of Green's function and Proposition 4.18, we have

$$g_D(x, y) \leq g_{B(x,(A_0+2)r)}(x, y) \asymp g_{B(x,(A_0+2)r)}(x^*, y^*) \asymp g_D(x^*, y^*).$$

Next, we consider the case $d(x,y) < r/(2\widetilde{C_U})$. Let B_y denote the connected component of $p^{-1}\left(B(p(y), r/\widetilde{C_U}) \cap \overline{U}\right)$ that contains y. By Lemma 2.12, we have $B_y \subset B_{\widetilde{U}}(y,r)$. By Lemma 4.15, $g_D(x,\cdot)$ is harmonic in $B_y \cap U$ and therefore by the maximum principle, we have

$$g_D(x,y) \le \sup_{z \in \partial_U D} g_{B_y}(x,z) \le \sup_{z \in \partial B(y, r/\widetilde{C_U})} g_D(x,z).$$

By triangle inequality, for any $z \in \partial B(y, r/\widetilde{C_U})$, we have $d(x,z) \ge r/(2\widetilde{C_U})$. Thus we can bound $g_D(x,z)$ exactly as in the previous case. This completes the proof of (5.12).

Therefore, using Proposition 4.18(b), then Proposition 4.18(e), and then domain monotonicity,

$$g_{B(x,A_{10}r)}(x,y) \le c g_{B(x,A_{10}r)}(z^*,y^*) \le c' g_{B(x,2c_U r)}(z^*,y^*) \le c'' g_D(z^*,y^*).$$

Finally, as in Lemma 5.11 we can use a Harnack chain between z^* and x^* to deduce that $g_D(z^*,y^*) \le c''' g_D(x^*,y^*)$. Combining these inequalities gives (5.12).

Now let $x \in B_U(\xi, 2r)$, $z \in F(\xi)$. Since $g_D(\cdot, z)$ is harmonic in $D \setminus \{z\}$, by (5.12) and the maximum principle, we have

$$g_D(x,z) \le \omega(x, \partial_U B_U(\xi,2r), B_U(\xi,2r)) \sup_{x' \in \partial_U B_U(\xi,2r)} g_D(x',z). \tag{5.13}$$

We use Lemma 5.7 to bound the first term, and (5.12) to bound the second, and obtain

$$g_D(x,z) \le c \frac{g_{B_U(\xi,A_3 r)}(x,\xi_r)}{g_{B_U(\xi,A_3 r)}(\xi_r',\xi_r)} g_D(x^*,y^*). \tag{5.14}$$

We then have by Propostion 4.18(c)-(d), Harnack chaining and domain monotonicity

$$g_{B_U(\xi,A_3 r)}(\xi_r',\xi_r) \asymp g_D(x^*,y^*), \quad g_{B_U(\xi,A_3 r)}(x,\xi_r) \le c g_D(x,y^*),$$

and combining these inequalities completes the proof. Note that for the second inequality above, one needs to consider two different cases: $\delta_U(x) \le \frac{1}{2}c_U^2 r$ and $\delta_U(x) > \frac{1}{2}c_U^2 r$. ☐

Lemma 5.13. *Let $\xi \in \partial_{\widetilde{U}} U$, and $D = B_U(\xi, A_0 r)$. If $x \in B_U(\xi, r)$ and $y \in \partial_U B_U(\xi, A_8 r)$ with $\delta_U(y) < \frac{1}{4}c_U^2 r$ then*

$$g_D(x,y) \le c \frac{g_D(x^*,y)}{g_D(x^*,y^*)} g_D(x,y^*). \tag{5.15}$$

Proof. Let $\zeta \in \partial_{\widetilde{U}} U$ be a point such that $d_U(y,\zeta) < c_U^2 r/4$, and let ζ_r and ζ_r' be the points given by Lemma 5.7 for the boundary point ζ. Since $g_D(x,\cdot)$ is harmonic in $B_U(\zeta, 2r)$, we have

$$g_D(x,y) \le \omega(y, \partial_U B_U(\zeta,2r), B_U(\zeta,2r)) \sup_{z \in \partial_U B_U(\zeta,2r)} g_D(x,z). \tag{5.16}$$

Since $B_U(\zeta, 2r)) \subset F(\xi)$, by Lemma 5.12 the second term in (5.16) is bounded by $cg_D(x, y^*)$. Using Lemma 5.7 to control the first term, we obtain

$$g_D(x,y) \leq cg_D(x,y^*) \frac{g_{B_U(\zeta,A_3 r)}(y,\zeta_r)}{g_{B_U(\zeta,A_3 r)}(\zeta_r',\zeta_r)}. \tag{5.17}$$

Again by Harnack chaining, Proposition 4.18 and domain monotonicity we have

$$g_{B_U(\zeta,A_3 r)}(\zeta_r',\zeta_r) \asymp g_D(x^*,y^*),$$

and

$$g_{B_U(\zeta,A_3 r)}(y,\zeta_r) \leq cg_D(y,x^*),$$

and combining these estimates completes the proof. □

Proof of Theorem 5.8. (5.6) follows immediately from Lemmas 5.10, 5.11 and 5.13, and as remarked above the Theorem follows easily from (5.6). □

Remark 5.14. One might ask if the converse to Theorem 5.1 holds. That is, suppose $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ is a MMD space such that for every inner uniform domain the BHP holds. Then does the EHI hold for $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$?

The following example shows this is not the case. Consider the measures μ_α on $\mathbb{R}$ given by $\mu_\alpha(dx) = (1 + |x|^2)^{\alpha/2}\,\lambda(dx)$, where λ denotes the Lebesgue measure. (See [GS]). The Dirichlet forms

$$\mathcal{E}_\alpha(f,f) = \int_{\mathbb{R}} |f'(x)|^2 \mu_\alpha(dx),$$

do not satisfy the Liouville property if $\alpha > 1$. This is because the two ends at $\pm\infty$ are transient, so the probability that the diffusion eventually ends up in $(0, \infty)$ is a non-constant positive harmonic function. Since the Liouville property fails, so does the EHI.

On the other hand, the space of inner uniform domains in $\mathbb{R}$ is same as the space of (proper) intervals in $\mathbb{R}$. The space of harmonic functions in a bounded interval vanishing at a boundary point is one dimensional, and hence the BHP holds. We can take $R(U)$ in Theorem 5.1 as diameter(U)/4.

References

[Aik01] H. Aikawa. Boundary Harnack principle and Martin boundary for a uniform domain, *J. Math. Soc. Japan* **53** (2001), p. 119–145. MR1800526

[Aik08] H. Aikawa. Equivalence between the Boundary Harnack Principle and the Carleson estimate. *Math. Scand.* **103**, no. 1 (2008), 61–76. MR2464701

[Aik15] H. Aikawa. Potential analysis on nonsmooth domains – Martin boundary and boundary Harnack principle, Complex analysis and potential theory, 235–253, *CRM Proc. Lecture Notes*, **55**, Amer. Math. Soc., Providence, RI, 2012. MR2986906

[ALM] H. Aikawa, T. Lundh, T. Mizutani. Martin boundary of a fractal domain, *Potential Anal.* **18** (2003), p. 311–357. MR1953266

[Anc] A. Ancona. Principle de Harnack á la frontiére et theoreme de Fatou pour un opérateur elliptique dans un domain lipschitzien. *Ann. Inst. Fourier (Grenoble)* **28** (1978) no. 4., 169–213. MR0513885

[BV] Z. Balogh, A. Volberg. Boundary Harnack principle for separated semihyperbolic repellers, harmonic measure applications. *Rev. Mat. Iberoamericana* **12** (1996), no. 2, 299–336. MR1402670

[BBB] R. Bañuelos, R. Bass, K. Burdzy. Hölder domains and the boundary Harnack principle. *Duke Math. J.* **64** (1991), no. 1, 195–200. MR1131398

[BB3] M.T. Barlow, R.F. Bass. Stability of parabolic Harnack inequalities. *Trans. Amer. Math. Soc.* **356** (2003) no. 4, 1501–1533. MR2034316

[BB] R.F. Bass, K. Burdzy. A boundary Harnack principle in twisted Hölder domains. *Ann. of Math.* (2) **134** (1991), no. 2, 253–276. MR1127476

[BM] M.T. Barlow, M. Murugan. Stability of elliptic Harnack inequality, Preprint 2016. arXiv:1610.01255

[Bog] K. Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Math.* **123** (1997), no. 1, 43–80. MR1438304

[BKK] K. Bogdan, T. Kulczycki, M. Kwaśnicki. Estimates and structure of α-harmonic functions. *Probab. Theory Related Fields* **140** (2008), no. 3–4, 345–381. MR2365478

[BBI] D. Burago, Y. Burago and S. Ivanov. A course in Metric Geometry, *Graduate Studies in Mathematics*, **33**. American Mathematical Society, Providence, RI, 2001. MR1835418

[CFMS] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa. Boundary behavior of non-negative solutions of elliptic operators in divergence form, *Indiana Univ. Math. J.*, **30** (1981), 621–640. MR0620271

[Cha] I. Chavel, *Riemannian geometry. A modern introduction.* Second edition. Cambridge Studies in Advanced Mathematics, **98**. Cambridge University Press, Cambridge, 2006. MR2229062

[Che] Z.-Q. Chen. On notions of harmonicity, *Proc. Amer. Math. Soc.* **137** (2009), no. 10, 3497–3510. MR2515419

[CF] Z.-Q. Chen, M. Fukushima. Symmetric Markov processes, time change, and boundary theory, *London Mathematical Society Monographs Series*, **35**. Princeton University Press, Princeton, NJ, 2012. xvi+479 pp. MR2849840

[CS] T. Coulhon, L. Saloff-Coste. Variétés riemanniennes isométriques à l'infini, *Rev. Mat. Iberoamericana* **11** (1995), no. 3, 687–726. MR1363211

[Dav] E.B. Davies. Heat kernels and spectral theory, *Cambridge Tracts in Mathematics*, **92**. Cambridge University Press, Cambridge, 1989. x+197 pp. MR0990239

[DS] E.B. Davies, B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, *J. Funct. Anal.* **59** (1984), no. 2, 335–395. MR0766493

[FOT] M. Fukushima, Y. Oshima, M. Takeda. Dirichlet forms and symmetric Markov processes, *de Gruyter Studies in Mathematics*, **19**. Walter de Gruyter & Co., Berlin, 1994. x+392 pp. MR1303354

[GO] F. W. Gehring, B. G. Osgood. Uniform domains and the quasihyperbolic metric, *J. Analyse Math.* **36** (1979), 50–74 (1980). MR0581801

[GS] A. Grigor'yan, L. Saloff-Coste. Stability results for Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 3, 825–890. MR2149405

[Gri] A. Grigor'yan. Heat kernels on weighted manifolds and applications. The ubiquitous heat kernel, 93–191, *Contemp. Math.*,**398**, Amer. Math. Soc., Providence, RI, 2006. MR2218016

[GH1] A. Grigor'yan, J. Hu. Heat kernels and Green functions on metric measure spaces, *Canad. J. Math.* **66** (2014), no. 3, 641–699. MR3194164

[GH2] A. Grigor'yan, J. Hu. Upper bounds of heat kernels on doubling spaces. *Mosc. Math. J.* **14** (2014), no. 3, 505–563, 641–642. MR3241758

[GHL] A. Grigor'yan, J. Hu, K.-S. Lau. Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric spaces. *J. Math. Soc. Japan* **67** (2015) 1485–1549 MR3417504

[GS] A. Grigor'yan, L. Saloff-Coste. Stability results for Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 3, 825–890. MR2149405

[GT1] A. Grigor'yan, A. Telcs. Harnack inequalities and sub-Gaussian estimates for random walks, *Math. Ann.* **324**, 551–556 (2002). MR1938457

[GT2] A. Grigor'yan, A. Telcs. Two-sided estimates of heat kernels on metric measure spaces, *Ann. Probab.* **40** (2012), no. 3, 1212–1284. MR2962091

[GyS] P. Gyrya, L. Saloff-Coste. Neumann and Dirichlet heat kernels in inner uniform domains, *Astérisque* **336** (2011). MR2807275

[HS] W. Hebisch, L. Saloff-Coste. On the relation between elliptic and parabolic Harnack inequalities, *Ann. Inst. Fourier (Grenoble)* **51** (2001), no. 5, 1437–1481. MR1860672

[JK] D. S. Jerison, C. E. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), no. 1, 80–147. MR0676988

[Kem] J.T. Kemper. A boundary Harnack principle for Lipschitz domains and the principle of positive singularities. *Comm. Pure Appl. Math.* **25** (1972), 247–255. MR0293114

[L] J. Lierl. Scale-invariant boundary Harnack principle on inner uniform domains in fractal-type spaces, *Potential Anal.* **43** (2015), no. 4, 717–747. MR3432457

[LS] J. Lierl, L. Saloff-Coste. Scale-invariant Boundary Harnack Principle in Inner Uniform Domains, *Osaka J. Math*, **51** (2014), pp. 619–657. MR3272609

[LuS] J. Luukkainen, E. Saksman. Every complete doubling metric space carries a doubling measure, *Proc. of Amer. Math. Soc.* **126** (1998) pp. 531–534. MR1443161

[MS] O. Martio, J. Sarvas. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1979), no. 2, 383–401. MR0565886

[Sal02] L. Saloff-Coste. Aspects of Sobolev-Type Inequalities, *London Mathematical Society Lecture Note Series*, **289**. Cambridge University Press, Cambridge, 2002. x+190 pp. MR1872526

[V] N. Th. Varopoulos. Long range estimates for Markov chains. *Bull. Sc. math., 2^e serie* **109** (1985), 225–252. MR0822826

[W] J.B. Walsh. A diffusion with a discontinuous local time. *Temps Locaux, Astérisque* **52-53** (1978) 37–45. MR0509476

[Wu] J.M.G. Wu, Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains. *Ann. Inst. Fourier (Grenoble)* **28** (1978), no. 4, 147–167 MR0513884

[Yos] K. Yosida. Functional analysis. Sixth edition. *Grundlehren der Mathematischen Wissenschaften. Springer-Verlag*, Berlin-New York, 1980. xii+501 pp. MR0617913

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