# Geometry of uniform spanning forest components in high dimensions

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**Abstract**

We study the geometry of the component of the origin in the uniform spanning forest of $\mathbb{Z}^d$ and give bounds on the size of balls in the intrinsic metric.

*Keywords:* Uniform spanning forest, loop erased random walk
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## 1 Introduction

{sec:intro}

The uniform spanning tree (UST) on a finite graph $G$ is a random spanning tree of $G$, chosen uniformly among all spanning trees of $G$. Motivated by questions of Lyons, Pemantle [Pem91] considered the weak limit of the USTs on an increasing sequence of subgraphs $V_n \uparrow \mathbb{Z}^d$, and showed that the limit exists. The limiting random object is a random spanning forest of $\mathbb{Z}^d$, and is called the *uniform spanning forest (USF)*. Implicit in Pemantle's work is the result that an alternative choice of boundary condition yields the same limit. Namely, form the "wired" graph $G_n^W = (V_n \cup \{r_n\}, E_n)$, by collapsing all vertices in $\mathbb{Z}^d \setminus V_n$ into $r_n$, and removing self-loops created at $r_n$. Then the weak limit of the USTs on $G_n^W$ coincides with the USF. One of Pemantle's results was that the USF is connected a.s. in dimensions $1 \leq d \leq 4$, but it consists of infinitely many (infinite) trees a.s. in dimensions $d \geq 5$.

A fundamental tool in the study of the UST/USF is Wilson's algorithm [W, LP], which allows one to construct the UST/USF from Loop-Erased Random Walks (LERWs). All the necessary background about the UST/USF, that we do not give in this paper, can be found in the book [LP].

In the papers [Mas, BM1, BM2] Masson and Barlow studied the geometry of the LERW and the UST in two dimensions, and compared the sizes and geometry of balls in the intrinsic metric with Euclidean balls. Combined with resistance estimates, this gave a detailed understanding of random walk on the UST. In this note we make similar estimates on the geometry of the LERW and the USF in dimensions $d \geq 5$. We are interested in

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properties such as the length of paths and volumes of balls, both with respect to Euclidean distance and the intrinsic metric of the tree components. As well as its interest from the point of understanding the USF in high dimensions, our Theorem 5.4 is used in work of Bhupatiraju, Hanson and Járai [BHJ] on sandpiles.

Let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$ be the USF in $\mathbb{Z}^d$, viewed as a random subgraph of the nearest neighbour integer lattice. Write $\mathcal{U}(x)$ for the connected component of $\mathcal{U}$ containing $x$. Let

$$d_{\mathcal{U}}(x,y) := \text{graph distance between } x \text{ and } y \text{ in } \mathcal{U},$$

where, if $y \notin \mathcal{U}(x)$, we set $d_{\mathcal{U}}(x,y) = \infty$. We denote balls in different metrics as follows:

$$\begin{aligned} B_E(x,r) &= \{y \in \mathbb{Z}^d : |x-y| \le r\}, \\ B_n &= B_E(0,n) \\ Q(x,n) &= \{y \in \mathbb{Z}^d : ||x-y||_\infty \le n\}, \\ Q_n &= Q(0,n), \\ B_{\mathcal{U}}(x,r) &= \{y \in \mathbb{Z}^d : d_{\mathcal{U}}(x,y) \le r\}, \end{aligned}$$

Our main result is the following.

{T:main}

**Theorem 1.1.** *Let $d \ge 5$. There exist constants $c$, $C$, depending only on $d$ such that for $n \ge 1$, $\lambda \ge 1$,*

$$\mathbb{P}(|B_{\mathcal{U}}(0,n)| \ge \lambda n^2) \le Ce^{-c\lambda}, \tag{1.1}$$ {e:main-vub}

$$\mathbb{P}\big(|B_{\mathcal{U}}(0,n)| \le \lambda^{-1} n^2\big) \le Ce^{-c\lambda^{1/5}}. \tag{1.2}$$ {e:main-vlb}

An outline of this paper is as follows. Section 2 introduces our notation. In Section 3 we begin by recalling from [La2, Mas, BM1] some basic properties of LERW. We then obtain estimates on the probability a LERW hits a point (see Lemma 3.9 and (3.22)). In particular (see Theorem 3.12) we obtain a natural bound on the length of $d_{\mathcal{U}}(x,y)$. In Section 4, using tree-graph inequalities, we use the estimate in Theorem 3.12 to obtain the upper bound in Theorem 1.1. In Section 5 we prove the lower bound (1.2).

# 2 Notation

{sec:notation

For any of the cases of $\mathbb{Z}^d$, or $D \subset \mathbb{Z}^d$ finite or infinite, we let

$$d_{\mathcal{U}}(x,y) := \text{graph distance between } x \text{ and } y \text{ in } \mathcal{U},$$

where, if $y \notin \mathcal{U}(x)$, we set $d_{\mathcal{U}}(x,y) = \infty$. The meaning of $\mathcal{U}$ will always be clear from context.

**Notation for sets:** For $A \subset \mathbb{Z}^d$ we denote:

$$\begin{aligned} \partial A &= \{x \in \mathbb{Z}^d - A : x \sim y \text{ for some } y \in A\}, \\ \partial_i A &= \{x \in A : x \sim y \text{ for some } y \in A^c\}. \end{aligned}$$

Let $\pi_i$ be projection onto the $i$th coordinate axis, and $\mathbb{H}_n$ be the hyperplane

$$\mathbb{H}_n = \{x : \pi_1(x) = n\}.$$

Let $\mathcal{R}_n = \{n\} \times [-n, n]^{d-1}$ denote the "right-hand face" of $[-n, n]^d$, in the first coordinate direction.

**Notation for processes.** We write $S^x = (S^x_k, k \geq 0)$ for a simple random walk with $S^x_0 = x$, and $\mathbb{P}^x$ for its law. We let $S = S^0$, and $\mathbb{P} = \mathbb{P}^0$. When we discuss random walks $S^x$ and $S^y$ with $x \neq y$, then they will always be independent.

A path $\gamma$ is a (non-necessarily self avoiding) sequence of adjacent vertices in $\mathbb{Z}^d$ – ie $\gamma = (\gamma_0, \gamma_1, \dots)$ with $\gamma_{i-1} \sim \gamma_i$. (Sometimes we will write $\gamma(i)$ for $\gamma_i$.) Paths can be either finite or infinite. We will often need to consider the beginning or final portions of paths with respect to the first or last hit on a set. To this end, we define a number of operations on paths. Let $\gamma = (\gamma_0, \gamma_1, \dots)$ be a path. Given a set $A \subset \mathbb{Z}^d$ define $k_1 = \min\{k \geq 0 : \gamma_k \in A\}$, $k_2 = \max\{k \geq 0 : \gamma_k \in A\}$, and set

$$\begin{aligned}
\mathcal{B}^F_A\gamma &= (\gamma_{k_1}, \gamma_{k_1+1}, \dots, ),\\
\mathcal{B}^L_A\gamma &= (\gamma_{k_2}, \gamma_{k_1+1}, \dots, ),\\
\mathcal{E}^F_A\gamma &= (\gamma_0, \dots, \gamma_{k_1}),\\
\mathcal{E}^L_A\gamma &= (\gamma_0, \dots, \gamma_{k_2}),\\
\Theta_k\gamma &= (\gamma_k, \dots),\\
\Phi_k\gamma &= (\gamma_0, \dots, \gamma_k),\\
H_A(\gamma) &= \sum_i 1_{(\gamma_i \in A)}.
\end{aligned}$$

Thus $\mathcal{B}^F_A\gamma$ is the path $\gamma$ 'Beginning' at the 'First' hit on $A$, and $\mathcal{E}^L_A\gamma$ is the path $\gamma$ 'Ended' at the 'Last' hit on $A$, etc. If $\gamma$ is a finite path we write $|\gamma|$ for the length of $\gamma$. $H_A(\gamma)$ is the number of hits by $\gamma$ on the set $A$. Let $\mathcal{L}\gamma$ be the chronological loop erasure of $\gamma$ – see [La1, Law99]. If $\gamma = (\gamma_0, \dots, \gamma_n)$ is a finite path let $\mathcal{R}\gamma = (\gamma_n, \gamma_{n-1}, \dots, \gamma_0)$ be the time reversal of $\gamma$.

We define hitting times

$$\begin{aligned}
\tau_A &= \inf\{j \geq 0 : S_j \notin A\},\\
T_A &= \inf\{j \geq 0 : S_j \in A\},\\
T^+_A &= \inf\{j \geq 1 : S_j \in A\}.
\end{aligned}$$

When we need to specify the process we write $T_A[S]$ etc.

Given a domain $D \subset \mathbb{Z}^d$, we denote the Green functions

$$\begin{aligned}
G_D(x, y) &= \mathbb{E}^x\Big(\sum_{0 \leq k < \tau_D} 1_{(S^x_k = y)}\Big),\\
G(x, y) &= G_{\mathbb{Z}^d}(x, y).
\end{aligned}$$

**A note on constants.** Throughout this paper, $c$ and $C$ will denote positive finite constants that only depend on the dimension $d$, and whose value may change from line to line, and even within a single string of inequalities.

## 3 Properties of the LERW

{sec:lew}

In this section we derive a number of auxiliary estimates on LERW in dimensions $d \geq 5$. Some of these will be used in Sections 4 and 5, where we give upper and lower bounds on the volume of balls in the intrinsic metric. Two results of this section that are of interest in themselves are: (i) Proposition 3.11, that gives a large deviation upper bound on the lower tail of the number of steps in a LERW up to its exit from a large box; and (ii) Theorem 3.12, that gives an upper bound on the probability that $x, y \in \mathbb{Z}^d$ are in the same component of $\mathcal{U}$ and the path between them has length at most $n$.

The papers [Mas, BM1] give a number of properties of LERW in $\mathbb{Z}^2$, some of which hold for more general graphs. A fundamental fact about LERWs is the following "Domain Markov property" — see [La2].

{L:dmp}

**Lemma 3.1.** *Let $D \subset \mathbb{Z}^d$, let $\gamma = (\gamma_0, \dots, \gamma_n)$ be a path from $x = \gamma_0$ to $D^c$. Set $\alpha = \Phi_k \gamma$, $\beta = \Theta_k \gamma$. Let $Y$ be a random walk started at $\gamma_k$ conditioned on the event $\{\tau_D(Y) < T^+_\alpha(Y)\}$. Then*

$$\mathbb{P}\big(\mathcal{L}(\mathcal{E}^F_{D^c} S) = \gamma | \Psi_k(\mathcal{L}(\mathcal{E}^F_{D^c} S)) = \alpha\big) = \mathbb{P}(\mathcal{L}(\mathcal{E}^F_{D^c} Y) = \beta). \tag{3.1}$$

A key result in [Mas] is a 'separation lemma' when $d = 2$ – see [Mas, Theorem 4.7]. Let $S, S'$ be independent SRW in $\mathbb{Z}^d$ with $S_0 = S'_0 = 0$, and $T_n, T'_n$ be the hitting times of $\partial Q_n$. Set

$$F_n = \{S[1, T_n] \cap S'[1, T'_n] = \emptyset\},$$
$$Z_n = d(S(T_n), S'[1, T'_n]) \vee d(S'(T'_n), S[0, T_n]).$$

{L:sep}

**Lemma 3.2.** *('Separation lemma'). Let $d \geq 5$. There exists $c_1 > 0$ such that*

$$\mathbb{P}(Z_n \geq \tfrac{1}{2} n | F_n) \geq c_1.$$

*Proof.* Let $e_1 = (1, 0, \dots, 0)$. Let $X$ be a SRW started at $2ke_1$, and $A_k = \{je_1, k \leq j \leq 2k\}$. Since $d \geq 5$ two independent SRWs intersect with probability less than 1, and thus there exists $k$ (depending on $d$) such that

$$\mathbb{P}^0(S \text{ hits } X \cup A_k) \leq \tfrac{1}{16} d^{-2}.$$

Now fix this $k$, and let

$$G_1 = \{S_i = -ie_1, S'_i = ie_1, 0 \leq i \leq k\}.$$

So $\mathbb{P}(G_1) = (2d)^{-2k}$. Then writing $G_2 = \{S[1, T_{n/2}] \cap S'[1, T'_{n/2}] \neq \emptyset\}$,

$$\begin{aligned}\mathbb{P}(G_2 | G_1) &\leq \mathbb{P}(S[k+1, T_{n/2}] \cap S'[1, T'_{n/2}] \neq \emptyset | G_1) + \mathbb{P}(S[1, T_{n/2}] \cap S'[k, T'_{n/2}] \neq \emptyset | G_1) \\ &\leq \tfrac{1}{8} d^{-2}.\end{aligned}$$

Let $H_\pm$ be the left and right faces (in the $e_1$ direction) of the cube $Q_{n/2}$. We have

$$\mathbb{P}(S_{T_{n/2}} \in H_- | G_1) \geq (2d)^{-1}.$$

So if $G_3 = G_2^c \cap \{S_{T_{n/2}} \in H_-, S'_{T'_{n/2}} \in H_+\}$,

$$\begin{aligned}\mathbb{P}(G_3|G_1) &\geq \mathbb{P}(S_{T_{n/2}} \in H_-, S'_{T'_{n/2}} \in H_+|G_1) - \mathbb{P}(G_2|G_1) \\ &\geq (2d)^{-2} - (8d^2)^{-1} = (8d^2)^{-1}.\end{aligned}$$

If $G_3$ occurs then let $G_4$ be the event that $S'$ then (i.e. after time $T'_{n/2}$ leaves $Q_n$ before it hits hits $\mathbb{H}_0$, and $S$ leaves $Q_n$ before it hits $\mathbb{H}_0$. By comparison with a one-dimensional SRW each of these events has probability at least 1/3, so $\mathbb{P}(G_4|G_3) \geq 1/9$. On the event $G_1 \cap G_3 \cap G_4$ the path $S[0,T_n]$ is contained in $[-n,0]\times[-n,n]^{d-1} \cup Q_{n/2}$, and $\pi_1(S'_{T'_n} = n)$, so that $d(S'_{T'_n}, S[0,T_n]) \geq n/2$. The same bound holds if we interchange $S'$ and $S$, and so we deduce that

$$\mathbb{P}(Z_n \geq \tfrac{1}{2}n|F_n) \geq \mathbb{P}(\{Z_n \geq \tfrac{1}{2}n\} \cap F_n) \geq \mathbb{P}(G_1 \cap G_3 \cap G_4) \geq (2d)^{-2k}(8d^2)^{-1}9^{-1}.$$

□

**Remark.** The result in $d \geq 5$ is much easier than $d = 2$, since with high probability $S$ and $S'$ do not interesect. The proof for $d = 2$ uses the fact that if the two processes get too close, then by the Beurling estimate they hit with high probability.

In the remainder of this section we give some estimates on the length of LERW paths in $\mathbb{Z}^d$ with $d \geq 5$. We fix $D \subset \mathbb{Z}^d$ and $N \geq 1$ such that $Q_N = Q(0,N) \subset D$. We will be interested in the number of steps the LERW from 0 to $\partial D$ takes up to its first exit from $Q_N$. Let $S$ be SRW on $\mathbb{Z}^d$ with $S_0 = 0$. Let

$$L = \mathcal{L}(\mathcal{E}^F_{D^c}(S)).$$

In words, $L$ is the loop erasure of $S$ up to its first hit on the boundary of $D$.

Our estimate will be broken down into studying $L$ in 'shells' $Q_{n+m} \setminus Q_n$. Fix $n, m$ such that $16 \leq n < n+m \leq N$, with $m \leq n/8$. Let

$$\alpha = \mathcal{E}^F_{\partial_i Q_n} L, \quad L' = \mathcal{B}^F_{\partial_i Q_n} L.$$

So $\alpha$ is the path $L$ up to its first hit on $\partial_i Q(0,n)$, and $L'$ is the path of $L$ from this time on – see Figure 1. Let us condition on $\alpha$, and write $x_0 \in \partial_i Q_n$ for the endpoint of $\alpha$. When $x_0 \in \mathbb{H}_n$, we let $x_1 = x_0 + (m/2)e_1$ and set

$$A = A(x_0) = Q(x_1, m/4), \quad A^* = Q(x_1, 3m/8).$$

When $x_0$ lies on one of the other faces of $Q_n$, we replace $e_1$ by the unit vector pointing towards that faces to define $x_1$ and $A(x_0)$.

Set

$$\beta = \mathcal{E}^F_{\partial_i Q(x_0,m)} L';$$

thus $\beta$ is the path $L'$ run until its first exit from the cube $Q(x_0, m)$. Let $\widetilde{X}^z$ be $S^z$ conditioned on $\{\tau_D < T^+_\alpha\}$. Note that while the law of the process $\widetilde{X}^z$ depends on $\alpha$, our

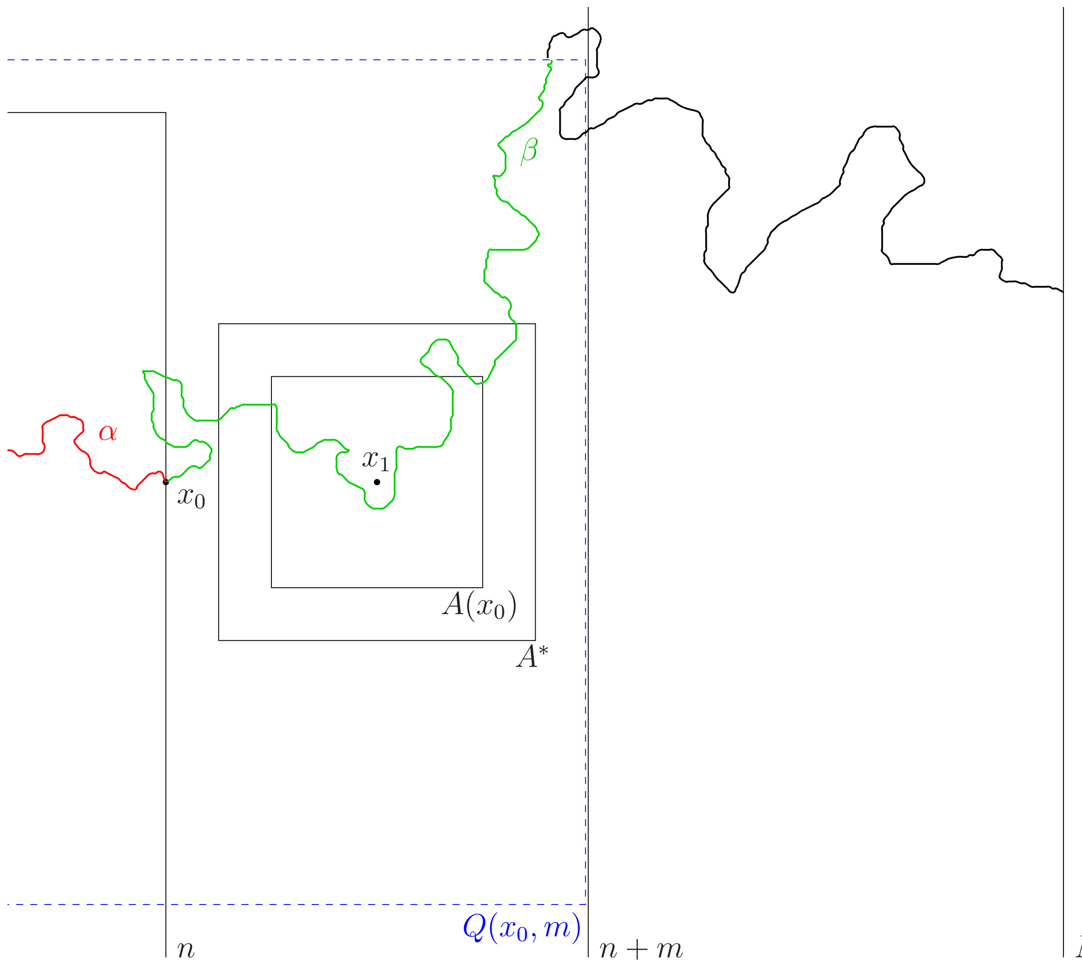


fig:box-setup

Figure 1: Setup and notation for the piece of the LERW in the shell $Q_{n+m} \setminus Q_n$.

notation does not emphasize this point. Write $\widetilde{X}$ for $\widetilde{X}^{x_0}$, and $\widetilde{G}_D(x,y)$ for the Green function for $\widetilde{X}^x$. By the domain Markov property, Lemma 3.1, we have (conditional on $\alpha$) that

$$L' \stackrel{(d)}{=} \mathcal{L}(\mathcal{E}^F_{\partial D}\widetilde{X}). \tag{3.2}$$

That is, $L'$ is distributed as the loop-erasure of $\widetilde{X}$ ended on the first hit of $\partial D$. We write $\widetilde{T}$, $\widetilde{\tau}$, etc. for hitting and exit times by $\widetilde{X}$. Set

$$h(x) = \mathbb{P}^x(\tau_D < T_\alpha).$$

Then

$$\widetilde{G}_D(x,y) = \frac{h(y)}{h(x)} G_D(x,y), \;\; x,y \in D - \alpha. \tag{3.3}$$ {e:wtG}

The standard Harnack inequality (see [La2]) gives

$$h(y) \asymp h(x_1), \quad y \in A^*, \tag{3.4}$$ {e:hiA}

and thus

$$\widetilde{G}_D(x,y) \asymp G_D(x,y), \; x,y \in A^*. \tag{3.5}$$ {e:hGn}

{L:Mub}

**Lemma 3.3.** *Let $d \geq 3$. For any $\alpha$ and $y \in A$ we have*

$$\mathbb{P}(y \in \beta | \alpha) \leq c_1 m^{2-d}, \tag{3.6}$$ {e:ub33-1}

$$\mathbb{E}(H_A(\beta)|\alpha) \leq c_1 m^2, \tag{3.7}$$ {e:ub33-2}

$$\mathbb{E}(H_A(\beta)^2|\alpha) \leq c_1 m^4. \tag{3.8}$$ {e:ub33-3}

*Proof.* This is a standard computation with Green functions. Let $B = Q(x_0, m)$. Then, since $\beta$ is a subset of the path of $\widetilde{X}$, we have

$$H_A(\beta) \leq \sum_{k=0}^{\widetilde{\tau}_B} 1_{(\widetilde{X}_k \in A)} = H_A(\mathcal{E}^F_{\partial_i B}\widetilde{X}) =: \widetilde{H}.$$

Then for $p = 1, 2$,

$$\mathbb{E}^{x_0}(\widetilde{H}^p|\alpha) = \mathbb{E}^{x_0}\Big(1_{(\widetilde{T}_{A^*} < \widetilde{\tau}_B)} \mathbb{E}^{\widetilde{X}_{\widetilde{T}_{A^*}}}(\widetilde{H}^p)\Big) \leq \max_{z \in \partial_i A^*} \mathbb{E}^z \widetilde{H}^p.$$

Let $z \in \partial_i A^*$. Then using (3.5)

$$\mathbb{E}^z(\widetilde{H}|\alpha) = \sum_{y \in A} \widetilde{G}_B(z,y) \leq c|A| \max_{y \in A} G_B(z,y) \leq c' m^d m^{2-d} = c' m^2. \tag{3.9}$$ {e:yub}

Also since on $A^*$ we have $\widetilde{G}_B \asymp G_B \leq G$,

$$\begin{aligned}
\mathbb{E}^z(\widetilde{H}^2|\alpha) &\leq 2 \sum_{k=0}^{\infty} \sum_{k=j}^{\infty} 1_{(k \leq \widetilde{\tau}_B)} 1_{(j \leq \widetilde{\tau}_B)} 1_{(\widetilde{X}_k \in A)} 1_{(\widetilde{X}_j \in A)} \\
&\leq 2 \sum_{x \in A} \sum_{y \in A} \widetilde{G}_B(z,x) \widetilde{G}_B(x,y) \\
&\leq c|A| m^{2-d} \max_{x \in A} \sum_{y \in A} \widetilde{G}(x,y) \leq c' m^4.
\end{aligned}$$

This proves (3.7)-(3.8); (3.6) follows easily from (3.9) by just considering hits on $y$. $\square$

{rem:other-se

**Remark 3.4.** The same argument works if we consider $\mathbb{E}(H_{Q(x_1,\lambda m)}(\beta)^p|\alpha)$, $p = 1, 2$, for any $\lambda \in (0, \frac{1}{2})$. Likewise for a rectangular box $A''$ whose boundary is at distance $\geq cm$ from the boundary of $Q_{n+m} \setminus Q_n$. This minor extension will be needed in Section 5: see Figure 4 for a picture of the set $A''$ we will need.

We now turn to the harder problem of obtaining a lower bound on $\mathbb{E}H_A(\beta)$, and begin with an inequality, similar to a boundary Harnack principle, which extends [Mas, Proposition 3.5] to higher dimensions. In what follows $\mathcal{R}_m = \mathbb{H}_m \cap Q_m$ is the 'right hand face' of $Q_m$.

{L:ubh}

**Lemma 3.5.** *Assume $d \geq 1$. Let $\mathcal{K}$ be an arbitrary nonempty subset of $[-m+1, 0] \times [-m+1, m-1]^{d-1}$. For all $m \geq 1$ and all $\mathcal{K}$ we have*

$$\mathbb{P}^0\big(S(\tau_{Q(0,m-1)}) \in \mathcal{R}_m \,\big|\, \tau_{Q(0,m-1)} < T^+_{\mathcal{K}}\big) \geq (2d)^{-1}. \tag{3.10}$$

*Proof.* Let $h(z) = \mathbb{P}^z\big(S_{\tau_{Q(0,m-1)}} \in \mathcal{R}_m\big)$, $z \in Q(0, m-1)$. By symmetry we have $h(0) = 1/2d$. We first show that

$$h(z) \leq h(0) \text{ for all } z \in ([-m+1, 0] \times [-m+1, m-1]^{d-1}) \cap \mathbb{Z}^d. \tag{3.11}$$ {e:max-proper

Let $z' = (0, z_2, \dots, z_d)$. Let $X^z$ and $X^{z'}$ be simple random walks with starting points $z$ and $z'$ respectively; we have $h(z) = \mathbb{P}(X^z_{\tau_A} \in \mathcal{R}_n)$, with a similar expression for $h(z')$. We couple these random walks by taking $X^z = z + S$, $X^{z'} = z' + S$, where $S$ is a SRW with $S_0 = 0$. Then $\{X^z_{\tau_A} \in \mathcal{R}_n\} \subset \{X^{z'}_{\tau_A} \in \mathcal{R}_n\}$, and so $h(z) \leq h(z')$.

To prove that $h(z') \leq h(0)$ we use a coupling of continuous time random walks $Y$, $Y'$ with $Y_0 = 0$, $Y'_0 = z'$; these have the same exit distribution as the discrete time walk $S$. Recall that $\pi_j$ is the projection onto the $j$th coordinate axis, so that $\pi_j(Y_t)$ gives the $j$th coordinate of $Y_t$; each coordinate is a continuous time simple random walk (run at rate $1/d$) on $\mathbb{Z}$.

The coupling is as follows. If at time $t$ we have $\pi_j(Y_t) = \pi_j(Y'_t)$ then we run the two $j$th coordinate processes together, so $\pi_j(Y_{t+s}) = \pi_j(Y'_{t+s})$ for all $s \geq 0$

Note that we have $|\pi_j(Y_t)| \leq |\pi_j(Y'_t)|$ when $t = 0$; the coupling will preserve this inequality for all $t \geq 0$. If $|\pi_j(Y_t) - \pi_j(Y'_t)| \geq 2$ then we use reflection coupling, so that $\pi_j(Y_t)$ and $\pi_j(Y'_t)$ jump at the same time, and in opposite directions. Finally, suppose that $|\pi_j(Y_t) - \pi_j(Y'_t)| = 1$, and let $a = \pi_j(Y_t)$, $a + 1 = \pi_j(Y'_t)$. We take three independent Poisson processes on $\mathbb{R}_+$, $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$; each with rate $1/2d$, and make the first jump of either $\pi_j(Y)$ or $\pi_j(Y')$ after time $t$ to be at time $t + T$, where $T$ is the first point in $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. If $T \in \mathcal{P}_1$ we set $\pi_j(Y_{t+T}) = a - 1$, $\pi_j(Y'_{t+T}) = a + 2$. If $T \in \mathcal{P}_2$ then we set $\pi_j(Y_{t+T}) = a + 1$, $\pi_j(Y'_{t+T}) = a + 1$, and if $T \in \mathcal{P}_3$ then $\pi_j(Y_{t+T}) = a$, $\pi_j(Y'_{t+T}) = a$. With this coupling we have $\{Y'_{\tau_A(Y')} \in \mathcal{R}_n\} \subset \{Y_{\tau_A(Y)} \in \mathcal{R}_n\}$, and so $h(z') \leq h(0)$.

Stopping the bounded martingale $h(S(k))$ at $\tau_{Q(0,m-1)} \wedge T_{\mathcal{K}}$, and using (3.11) we get

$$\begin{aligned} h(0) &= \sum_{y \in \mathcal{K}} h(y) \mathbb{P}^0\big(S(\tau_{Q(0,m-1)} \wedge T^+_{\mathcal{K}}) = y\big) + \mathbb{P}^0\big(\tau_{Q(0,m-1)} < T^+_{\mathcal{K}},\, S(\tau_{Q(0,m-1)}) \in \mathcal{R}_m\big) \\ &\leq h(0) \mathbb{P}\big(\tau_{Q(0,m-1)} > T^+_{\mathcal{K}}\big) + \mathbb{P}\big(\tau_{Q(0,m-1)} < T^+_{\mathcal{K}},\, S(\tau_{Q(0,m-1)}) \in \mathcal{R}_m\big). \end{aligned}$$

Rearranging gives the statement of the lemma. □

We will need a more general conditioning than is given in Lemma 3.5. First we give a preliminary estimate.

{L:ubh-ext}

**Lemma 3.6.** *Assume $d \geq 3$. Let $N \geq 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $8 \leq m \leq N/2$ and $n \leq N$. Suppose that $\mathcal{K}$ is an arbitrary nonempty subset of $Q_n$, and $x_0 \in \mathcal{K} \cap \mathbb{H}_n$. Let $z_0 = x_0 + me_1$. There exists a constant $c = c(d) > 0$ such that*

$$\mathbb{P}^{z_0}(T_{Q(x_0,m/2)} > \tau_D \,|\, T_{\mathcal{K}} > \tau_D) \geq c. \tag{3.12}$$

*Proof.* It is easy to see that the statement holds when $m \geq n/8$, since then $\mathbb{P}^{z_0}(T_{Q_{n+m/2}} > \tau_D) \geq \mathbb{P}^{z_0}(T_{Q_{n+m/2}} = \infty) \geq c$. Henceforth we assume that $m < n/8$.

Let $f(z) = \mathbb{P}^z(T_{\mathcal{K}} > \tau_D)$ and $g(z) = \mathbb{P}^z(T_{\mathcal{K}} \wedge T_{Q(x_0,m/2)} > \tau_D)$, so that we have to prove $f(z_0) \leq Cg(z_0)$. Let $z_1 = x_0 + 8me_1$. Due to the Harnack inequality, it is sufficient to show that $f(z_1) \leq Cg(z_1)$.

We first show that for all $y \in \partial Q(x_0, 8m)$ we have $g(y) \leq Cg(z_1)$. Let us write $\mathbb{H}$ for the hyperplane $\mathbb{H}_{n+4m}$, and $\mathbb{H}'$ for the hyperplane $\mathbb{H}_{n+2m}$. Observe that $\mathbb{H}$ and $\mathbb{H}'$ are both disjoint from $\mathcal{K} \cup Q(x_0, m/2)$, and they both separate $\mathcal{K} \cup Q(x_0, m/2)$ from $z_1$.

If $y \in \partial Q(x_0, 8m)$ lies on the same side of $\mathbb{H}'$ as $z_1$, then $y$ is at least distance $m$ from $\mathcal{K} \cup Q(x_0, m/2)$, and this is comparable to the distance between $y$ and $z_1$. Hence for such $y$, the Harnack inequality implies $g(y) \leq Cg(z_1)$.

Suppose now that $\mathbb{H}'$ separates $y$ from $z_1$. Let $Q^{(1)}$ and $Q^{(2)}$ be cubes that are both translates of $Q_{2N}$, such that:
(i) the right hand face of $Q^{(1)}$ and the left hand face of $Q^{(2)}$ coincide;
(ii) the common set $\mathcal{R} = Q^{(1)} \cap Q^{(2)}$, is contained in $\mathbb{H}$;
(iii) the center of $\mathcal{R}$ (viewed as a $(d-1)$-dimensional cube), is the point $x_0 + 4me_1$.
Write $\tau_i = \tau_{Q^{(i)}}$.

Since $g(S_{n\wedge\tau_1})$ is a submartingale under $\mathbb{P}^y$, we have

$$g(y) \leq \mathbb{E}^y(g(S_{\tau_1})) = \sum_{w \in \partial Q^{(1)} \setminus \mathcal{R}} g(w)\, \mathbb{P}^y(S_{\tau_1} = w) + \sum_{u \in \mathcal{R}} g(u)\, \mathbb{P}^y(S_{\tau_1} = u). \tag{3.13}$$ {e:submart}

Since $g(S_{n\wedge\tau_2})$ is a martingale under $\mathbb{P}^{z_1}$, we also have

$$g(z_1) = \mathbb{E}^{z_1}(g(S_{\tau_2})) = \sum_{w' \in \partial Q^{(2)} \setminus \mathcal{R}} g(w')\, \mathbb{P}^{z_1}(S_{\tau_1} = w) + \sum_{u \in \mathcal{R}} g(u)\, \mathbb{P}^{z_1}(S_{\tau_1} = u). \tag{3.14}$$ {e:mart}

The reflection symmetry between $Q^{(1)}$ and $Q^{(2)}$, as well as the Harnack inequality implies that

$$\mathbb{P}^y(S(\tau_{Q^{(1)}}) = u) \leq C\mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = u)$$
$$\mathbb{P}^y(S(\tau_{Q^{(1)}}) = w) \leq C\mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = w'),$$

where $w'$ is the mirror image of $w \in \partial Q^{(1)} \setminus \mathcal{R}$ in the hyperplane $\mathbb{H}$. We also have $g(w) \leq 1$, $w \in \partial Q^{(1)} \setminus \mathcal{R}$, and $g(w') \geq c$, $w' \in \partial Q^{(2)}$. These observations and (3.13) and (3.14) together imply $g(y) \leq Cg(z_1)$.

We now show the desired inequality $f(z_1) \le Cg(z_1)$. Let $1 \le R < \infty$ denote the random variable that counts the number of times $S^{z_1}$ makes a crossing from $\partial Q(x_0, 8m)$ to $Q(x_0, m/2)$ before $T_{\mathcal{K}} \wedge \tau_D$. We have

$$\mathbb{P}^{z_1}(R \ge \ell) \le \left( \max_{y \in \partial Q(x_0,8m)} \mathbb{P}^y(T_{Q(x_0,m/2)} < \infty) \right)^\ell \le p^\ell$$

with some $p = p(d) \in (0,1)$.

Using the strong Markov property at the time when the $\ell$-th crossing has occurred, we can write

$$\begin{aligned} f(z_1) &= \sum_{\ell=0}^{\infty} \mathbb{P}^{z_1}(R = \ell,\, T_{\mathcal{K}} > \tau_D) = g(z_1) + \sum_{\ell=1}^{\infty} \mathbb{P}^{z_1}(R = \ell,\, T_{\mathcal{K}} > \tau_D) \\ &\le g(z_1) + \sum_{\ell=1}^{\infty} \mathbb{P}^{z_1}(R \ge \ell) \max_{z \in Q(x_0,m/2)} \mathbb{P}^z((T_{Q(x_0,m/2)} \wedge T_{\mathcal{K}}) \circ \Theta_{\tau_{Q(x_0,8m)}} > \tau_D) \\ &\le g(z_1) + \sum_{\ell=1}^{\infty} \gamma^\ell \max_{y \in \partial Q(x_0,8m)} g(y) \\ &\le g(z_1) + Cg(z_1). \end{aligned}$$

This completes the proof of the Lemma. □

{L:ubh-tauD}

**Lemma 3.7.** *Assume $d \ge 3$. Let $N \ge 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $8 \le m \le N/2$ and $n \le N$. Suppose that $\mathcal{K}$ is an arbitrary nonempty subset of $Q_n$, and $x_0 \in \mathcal{K} \cap \mathbb{H}_n$. Let $\mathcal{R}_{n,m}$ denote the right hand face of $Q(x_0, m)$. There exists a constant $c = c(d) > 0$ such that*

$$\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m} \,\big|\, T^+_{\mathcal{K}} > \tau_D\big) \ge c. \tag{3.15}$$

*Proof.* Let $\mathcal{K}_0 = \mathcal{K} \cap Q(x_0, 2m)$ and $\mathcal{K}_1 = \mathcal{K} \setminus \mathcal{K}_0 = \mathcal{K} \setminus Q(x_0, 2m)$. Due to Lemma 3.5 we have

$$\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m} \,\big|\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)}\big) \ge (2d)^{-1}. \tag{3.16}$$ {e:simple-ubh

Let $Z$ denote the process that is $S$ conditioned on $T_{\mathcal{K}_1} > \tau_D$. Then (3.16) and an application of the Harnack inequality implies that

$$\mathbb{P}^{x_0}\big(Z(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m} \,\big|\, T^+_{\mathcal{K}}[Z] > \tau_{Q(x_0,m)}[Z]\big) \ge c. \tag{3.17}$$ {e:cond-simpl

This in turn implies that

$$\begin{aligned} &\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)},\, T_{\mathcal{K}_1} > \tau_D\big) \\ &\quad \ge c\mathbb{P}^{x_0}\big(T^+_{\mathcal{K}} > \tau_{Q(x_0,m)},\, T_{\mathcal{K}_1} > \tau_D\big) \\ &\quad \ge c\mathbb{P}^{x_0}\big(T^+_{\mathcal{K}} > \tau_D\big). \end{aligned} \tag{3.18}$$ {e:cond-ubh}

Let $z_0 = x_0 + 4me_1$. Using the Harnack inequality, the left hand side of (3.18) can be bounded from above by

$$\begin{aligned} &\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)}\big) \max_{z \in \mathcal{R}_{n,m}} \mathbb{P}^z\big(T_{\mathcal{K}_1} > \tau_D\big) \\ &\quad \le C\, \mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)}\big)\, \mathbb{P}^{z_0}\big(T_{\mathcal{K}_1} > \tau_D\big). \end{aligned} \tag{3.19}$$ {e:away}

An application of Lemma 3.6 (with $2m$ playing the role of $m/2$) shows that

$$\mathbb{P}^{z_0}\big(T_{\mathcal{K}_1} > \tau_D\big) \le C\,\mathbb{P}^{z_0}\big(T_{\mathcal{K}_1 \cup Q(x_0,2m)} > \tau_D\big) \le C\,\mathbb{P}^{z_0}\big(T_{\mathcal{K}} > \tau_D\big).$$

Substituting this into (3.19), and using the Harnack inequality again, we get that the right hand side of (3.19) is bounded above by

$$\begin{aligned} &C\,\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)}\big)\,\mathbb{P}^{z_0}\big(T_{\mathcal{K}} > \tau_D\big) \\ &\quad \le C\,\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_{Q(x_0,m)}\big) \min_{z \in \mathcal{R}_{n,m}} \mathbb{P}^{z}\big(T_{\mathcal{K}} > \tau_D\big) \\ &\quad \le C\,\mathbb{P}^{x_0}\big(S(\tau_{Q(x_0,m)}) \in \mathcal{R}_{n,m},\, T^+_{\mathcal{K}} > \tau_D\big). \end{aligned} \tag{3.20}$$ {e:bound-away

The inequalities (3.18), (3.19) and (3.20) together imply the claim of the Lemma. □

We now return to the task of giving a lower bound for $\mathbb{E}(H_A(\beta))$, the expected number of points of $\beta$ in $A$.

{L:wtG}

**Lemma 3.8.** *Assume $d \ge 3$. Let $z \in A$. Then*

$$\widetilde{G}_D(x_0, z) \ge c m^{2-d}.$$

*Proof.* This uses Lemma 3.7. Let $V_z$ be the number of hits on $z$ by $\widetilde{X}$ before $\widetilde{\tau}_D$. Let $\widetilde{T} = \widetilde{T}_{\partial_i Q(x_0, m/8)}$. Note that $Q(x_0, m/8)$ and $A^*$ intersect on one of the faces of $Q(x_0, m/8)$. Then since $\widetilde{T} < \widetilde{\tau}_D$,

$$\begin{aligned} \widetilde{G}_D(x_0, z) = \mathbb{E}^{x_0} V_z = \mathbb{E}^{x_0}\Big(\mathbb{E}^{\widetilde{X}_{\widetilde{T}}} V_z\Big) &\ge \mathbb{E}^{x_0}\Big(1_{(\widetilde{X}_{\widetilde{T}} \in A^*)} \min_{y \in \partial_i A^*} \mathbb{E}^{y} V_z\Big) \\ &= \mathbb{P}^{x_0}(\widetilde{X}_{\widetilde{T}} \in A^*) \min_{y \in \partial_i A^*} \widetilde{G}_D(y, z). \end{aligned}$$

Using (3.3) and (3.4) we have $\widetilde{G}_D(y, z) \asymp G_D(y, z) \asymp m^{2-d}$ if $y \in \partial_i A^*$. Let $T = T_{\partial_i Q(x_o, m/8)}$ (for $S$). Lemma 3.7 implies

$$\mathbb{P}^{x_0}(\widetilde{X}_{\widetilde{T}} \in A^*) = \mathbb{P}^{x_0}(S_T \in A^* | T^+_{\alpha} > \tau_D) \ge c,$$

and the Lemma follows. □

The key estimate for the lower bound is the following.

{L:ptlb}

**Lemma 3.9.** *Assume $d \ge 5$. Then*

$$\mathbb{E}(H_A(\beta)|\alpha) \ge c m^2. \tag{3.21}$$

*Proof.* It is enough to prove that if $z \in A$ then

$$\mathbb{P}(z \in \beta | \alpha) \ge c m^{2-d}. \tag{3.22}$$ {e:zhit}

Let $Y$ be $\widetilde{X}$ conditioned to hit $z$ before $\widetilde{T}_\alpha^+ \wedge \widetilde{\tau}_D$, and let $\widetilde{X}^z$ be independent of $Y$. Let

$$Y' = \mathcal{E}_z^L(\mathcal{E}_{\partial D}^F(Y)),$$

so $Y'$ is the path of $Y$ up to its last hit on $z$ before its first exit from $D$. Let also $X' = \Theta_1 \mathcal{E}_{\partial D}^F \widetilde{X}^z$. (We need to apply $\Theta_1$ since the last point of $Y'$ and the first point of $\widetilde{X}^z$ are both $z$.) Then as in Lemma 6.1 of [BM1] we have

$$\mathbb{P}(z \in \beta|\alpha) = \widetilde{G}_D(x_0, z)\mathbb{P}\big(\mathcal{L}Y' \cap X' = \emptyset, \mathcal{L}Y' \subset Q(x_0, m)\big). \qquad (3.23)$$ {e:3pg}

Due to Lemma 3.8, it remains to show that the probability on the right hand side is bounded away from 0. We will in fact prove the stronger statement:

$$\mathbb{P}\big(Y' \cap X' = \emptyset, Y' \subset Q(x_0, m)\big) \geq c > 0. \qquad (3.24)$$ {e:avoid}

This result is not surprising, since two independent SRW in $\mathbb{Z}^d$ (with $d \geq 5$) intersect with probability strictly less than 1.

Let us denote $A_z = Q(z, m/16)$, $B = Q(x_0, m)$ and $B' = Q(x_0, m/16)$. Note that $Y'$ starts at $x_0$ and ends at $z$. We decompose $Y'$ into four subpaths, defined below, and give separate estimates for these subpaths that together will imply the lower bound on the probability in (3.24). We define:

$$Y_1' = \mathcal{E}_{\partial B'}^F(Y') \qquad Y_2' = \mathcal{E}_{\partial A_z}^L(\mathcal{B}_{\partial B'}^F(Y')) \qquad Y_3' = \mathcal{B}_{\partial A_z}^L(Y').$$

That is, $Y_1'$ ends at the first exit from $B'$, $Y_3'$ begins at the last entrance to $A_z$ and $Y_2'$ is the portion in between. We let $y_1 = Y_1'(|Y_1'|) = Y_2'(0)$ and $y_2 = Y_2'(|Y_2'|) = Y_3'(0)$. We further decompose $Y_2'$ into the pieces:

$$Y_{2,1}' = \mathcal{E}_{y_2}^F(Y_2') \qquad Y_{2,2}' = \mathcal{B}_{y_2}^F(Y_2').$$

That is, $Y_{2,1}'$ is the piece from $y_1$ to the first hit on $y_2$, and $Y_{2,2}'$ is the remaining loop at $y_2$. Observe that conditional on $y_1$ and $y_2$, the paths $Y_1', Y_{2,1}', Y_{2,2}', Y_3'$ are independent. We now state our estimates for each piece. Our notation will assume that $x_0 \in \mathbb{H}_n$; trivial modification can be made when this is not the case.

*Claim 1.* There is constant probability that $Y_1'$ exits $B'$ on the right hand face. That is, we have $\mathbb{P}(y_1 \in \mathcal{R}_{n,m/16}) \geq c > 0$, where $\mathcal{R}_{n,m/16} = \mathbb{H}_{n+m/16} \cap Q(x_0, m/16)$.

*Proof of Claim 1.* Using Lemma 3.7 we have

$$\begin{aligned}
\mathbb{P}(y_1 \in \mathcal{R}_{n,m/16}) &= \frac{\mathbb{P}^{x_0}(\widetilde{X}(\widetilde{\tau}_{B'}) \in \mathcal{R}_{n,m/16},\, \widetilde{T}_z < \widetilde{\tau}_D)}{\mathbb{P}^{x_0}(\widetilde{T}_z < \widetilde{\tau}_D)} \\
&\geq \frac{\widetilde{G}_D(z,z)}{\widetilde{G}_D(x_0,z)} \mathbb{P}^{x_0}(\widetilde{X}(\widetilde{\tau}_{B'}) \in \mathcal{R}_{n,m/16}) \min_{w \in \mathcal{R}_{n,m/16}} \mathbb{P}^w(\widetilde{T}_z < \tau_D) \\
&\geq c \min_{w \in \mathcal{R}_{n,m/16}} \frac{\widetilde{G}_D(w,z)}{\widetilde{G}_D(x_0,z)} \geq c.
\end{aligned}$$

In the next three claims we will use the notation $B'' = x_0 + ([0, z_1 + m/32] \times [-m, m]^{d-1}) \cap \mathbb{Z}^d$.

*Claim 2.* There is constant probability that the following six events occur:
(i) $Y_3'$ starts on the left hand face of $A_z$;
(ii) $Y_3' \subset z + ([-m/16, m/32] \times [-m/16, m/16]^{d-1}) \cap \mathbb{Z}^d$;
(iii) $X'$ exits $A_z$ on the right hand face;
(iv) $X' \cap A_z \subset z + ([-m/32, m/16] \times [-m/16, m/16]^{d-1}) \cap \mathbb{Z}^d$;
(v) $Y_3' \cap (X' \cap A_z) = \emptyset$;
(vi) $\mathcal{B}^F_{\partial A_z}(X')$ is disjoint from $B''$, that is, $X'$ does not visit $B''$ after its first hit on $\partial A_z$.

*Proof of Claim 2.* Let $\widetilde{S}^z$ be the process defined as $S^z$ conditioned to hit on $x_0$ before $T_{\alpha\setminus\{x_0\}} \wedge \tau_D$. The time-reversal of $Y'$ has the law of $\widetilde{S}^z$. Therefore, the time-reversal of $Y_3'$ has the law of $\mathcal{E}^F_{\partial A_z}(\widetilde{S}^z)$. The proof of Lemma 3.2 (Separation Lemma), shows that for independent simple random walks $S^z$ and $S'^z$ there is probability $\geq c > 0$ that the analogues of the events (i)–(v) all hold. An application of the Harnack inequality then shows that in fact (i)–(v) hold with constant probability.

It is left to show that conditionally on (i)–(v), we also have (vi) with constant probability. Since $X'$ is $S$ conditioned on $T_\alpha > \tau_D$, this can be proved in the same way as Lemma 3.6. For this we merely have to replace $Q(x_0, m/2)$ in that lemma by $B''$, and make straightforward adjustments. Hence Claim 2 follows.

*Claim 3.* Conditional on $y_1$ being in the right hand face of $B'$ and $y_2$ being in the left hand face of $A_z$, there is constant probability that $Y'_{2,1} \subset B''$.

*Proof of Claim 3.* Condition on $y_1$ and $y_2$. Then $Y'_{2,1}$ has the law of $S^{y_1}$ conditioned to hit on $y_2$ before $T_\alpha \wedge \tau_D$ (stopped at the first hit on $y_2$). Since $y_1$ and $y_2$ are at least distance $cm$ from the boundary of $B''$, such a path has constant probability to stay inside $B''$. (One way to see this is to use an argument similar to that of Lemma 3.6, where we let $R$ count the number of crossings by the walk from $Q(z, m/64)$ to $\partial B''$ before time $T_z \wedge T_\alpha \wedge \tau_D$.) Hence the claim follows.

*Claim 4.* Conditional on $y_2$ being in the left hand face of $A_z$, there is constant probability that $Y'_{2,2} \subset B''$.

*Proof of Claim 4.* Condition on $y_2$. The probability that $Y'_{2,2}$ consists of a single point is $G_{D\setminus\alpha}(y_2, y_2)^{-1} \geq G(y_2, y_2)^{-1} \geq c > 0$.

When all the events in Claims 1–4 occur, the event in (3.24) occurs. Hence the Lemma follows. □

An application of Lemmas 3.3 and 3.9 and the one-sided Chebyshev inequality give the following corollary.

{C:HAlb}

**Corollary 3.10.** *When $d \geq 5$, there exists a constant $c_0 > 0$ such that*

$$\mathbb{P}(H_A(\beta) \geq c_0 m^2 | \alpha) \geq c_0.$$

{P:len-lb}

**Proposition 3.11.** *Assume $d \geq 5$. Let $N \geq 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $L = \mathcal{L}\mathcal{E}^F_{\partial D} S$ be a loop erased walk from 0 to $\partial D$, and $M_N = |\mathcal{E}^F_{\partial_i Q_N} L|$ be the number of steps in $L$ until its first hit on $\partial_i Q_N$. Then for all $\lambda > 0$ we have*

$$\mathbb{P}(M_N < \lambda N^2) \leq C \exp(-c\lambda^{-1}). \qquad (3.25) \quad \{e:Llb\}$$

*Proof.* Suppose $k \geq 1$ and $m \geq 4$ such that $N/2 \leq km < N - m$. For $j = 1, \dots, k$ let

$$\alpha_j = \mathcal{E}^F_{\partial_i Q(0,jm)} L, \quad \mathcal{F}_j = \sigma(\alpha_j).$$

Let $Y_j = \alpha_j(|\alpha_j|)$ be the last point in $\alpha_j$, and

$$\beta_j = \mathcal{E}^F_{\partial_i Q(Y_j,m)}(\mathcal{B}^F_{\partial_i Q(0,jm)} L)$$

be the path $L$ between $Y_j$ and its first hit after $Y_j$ on $\partial_i(Y_j, m)$. We have

$$M_N \geq \sum_{i=1}^{k} |\beta_j|,$$

Let $G_j = \{|\beta_j| < c_0 m^2\}$; then by Corollary 3.10

$$\mathbb{P}(G_j | \mathcal{F}_j) \leq 1 - c_0.$$

Therefore, $M_N$ stochastically dominates a sum of $k$ independent random variables that take the values $c_0 m^2$ and 0 with probabilities $c_0$ and $1 - c_0$, respectively. Hence

$$\mathbb{P}(M_N \leq (1/2) k c_0^2 m^2) \leq C \exp(-ck).$$

We now take $k \asymp \lambda^{-1}$ and $m \asymp \lambda N$ and we obtain (3.25). □

In the following theorem, we obtain a lower bound on the length of paths in the USF. We define the event:

$$F(y, x, n) = \left\{ T_x[S^y] < \infty \text{ and } |\mathcal{L}\mathcal{E}^F_x(S^y)| \leq n \right\}; \tag{3.26}$$ {e:Fndef}

note that on the event $\{T_x[S^y] < \infty\}$ the path $\mathcal{L}\mathcal{E}^F_x(S^y)$ is the loop erasure of a SRW path started at $y$ and ending at $x$

{T:Ass2}

**Theorem 3.12.** *For every $x, y \in \mathbb{Z}^d$ we have*

$$\mathbb{P}(F(y, x, n)) \leq C(1 + |x - y|)^{2-d} \exp\left[ -c \frac{|x - y|^2}{n} \right]. \tag{3.27}$$ {e:clb}

*Proof.* Using translation invariance we can assume that $y = 0$. If $|x|^2/n \leq 1$ then the term in the exponential in (3.27) is of order 1, so

$$\mathbb{P}(F(0, x, n)) \leq \mathbb{P}(T_x < \infty) \leq (1 + |x|)^{2-d} \leq e^c (1 + |x|)^{2-d} e^{-c|x|^2/n}.$$

Now assume $|x|^2 > n$, and let $N = ||x||_\infty / 4$, and $Q = Q(0, N)$. Let $X'$ be $S$ conditioned on $\{T_x < \infty\}$. Then if $h(z) = \mathbb{P}^z(T_x[S] < \infty)$, we have $h(z) \asymp N^{2-d}$ on $Q(0, N)$, and thus the processes $S$ and $X'$ have comparable laws inside $Q(0, N)$. The explicit law of a section of the loop erased random path given in [Law99] (see also (5) in [Mas]) then implies that the loop erasures of $S$ and $X'$ also have comparable laws inside $Q$.

Let

$$F_1(x,n) = \left\{ |\mathcal{E}^F_{\partial_i Q}(\mathcal{L}\mathcal{E}^F_x S)| \le n, T_x(S) < \infty \right\}. \tag{3.28}$$

Thus $F(0,x,n) \subset F_1(x,n)$. Then

$$\begin{aligned}\mathbb{P}(F(0,x,n)) &\le \mathbb{P}(F_1(x,n)) \\ &= \mathbb{P}(|\mathcal{E}^F_{\partial_i Q}\mathcal{L}(\mathcal{E}^F_x S)| \le n | T_x < \infty)\, \mathbb{P}(T_x < \infty) \\ &\le C|x|^{2-d}\mathbb{P}(|\mathcal{E}^F_{\partial_i Q}\mathcal{L}(\mathcal{E}^F_x X')| \le n) \\ &\le C|x|^{2-d}\mathbb{P}(|\mathcal{E}^F_{\partial_i Q}\mathcal{L}(\mathcal{E}^F_x S)| \le n).\end{aligned}$$

Taking $n = \lambda N^2$, so that $\lambda^{-1} \ge c|x|^2 n^{-1}$, and using Proposition 3.11 completes the proof.
□

# 4 Upper bound on $|B_{\mathcal{U}}(0,n)|$

{sec:ub}

Recall that $\mathcal{U}(x)$ is the component of the USF containing $x \in \mathbb{Z}^d$. It is well-known [Pem91, Theorem 4.2] that for $d \ge 5$ and $x \ne y \in \mathbb{Z}^d$ we have

$$c|x-y|^{4-d} \le \mathbb{P}(y \in \mathcal{U}(x)) \le C|x-y|^{4-d}. \tag{4.1}$$ {e:cnctd-bnd}

A corollary of this bound is that the volume of $\mathcal{U}_0 \cap B(r)$ grows as $r^4$ in expectation. Our main result in the previous section, Theorem 3.12, is a variant of the upper bound in (4.1) that gives control over the length of the path connecting $x$ and $y$. Since that bound was formulated in terms of a single LERW, the exponent $4-d$ changes to $2-d$. In this section we extend Theorem 3.12 to control the volume of balls in the intrinsic metric.

{thm:B_U-mome

**Theorem 4.1.** *Assume $d \ge 5$, and let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$. There exists a constant $C_1$ such that for all $k \ge 0$ we have*

$$\mathbb{E}\big(|B_{\mathcal{U}}(0,n)|^k\big) \le C_1^k k! n^{2k}. \tag{4.2}$$ {e:ubk}

*Hence there are constants $c_1 > 0$ and $C_2$ such that*

$$\mathbb{P}(|B_{\mathcal{U}}(0,n)| \ge \lambda n^2) \le C_2 e^{-c_1\lambda}, \quad \lambda > 0,\ n \ge 1. \tag{4.3}$$ {e:ub-exp}

*Proof.* The bound (4.3) follows easily from (4.2) using Markov's inequality and the power series for $e^x$.

We prove (4.2) by induction on $k$. The case $k = 0$ holds trivially. We fix $k \ge 1$ and $y_1, \dots, y_k \in \mathbb{Z}^d$, and estimate the probability

$$\mathbb{P}\big(y_1, \dots, y_k \in B_{\mathcal{U}}(0,n)\big).$$

This can be done similarly to the "tree-graph inequalities" known in percolation [AN]. To facilitate notation, we write $y_0 = 0$. On the event $y_1, \dots, y_k \in \mathcal{U}_0$ consider the minimal subtree $T(y_0, \dots, y_k) \subset \mathcal{U}_0$ that contains the vertices $y_0, \dots, y_k$. This tree is finite. Since $\mathcal{U}_0$ has one end [BLPS], [LP], there is a unique infinite path in $\mathcal{U}_0$, whose only vertex in

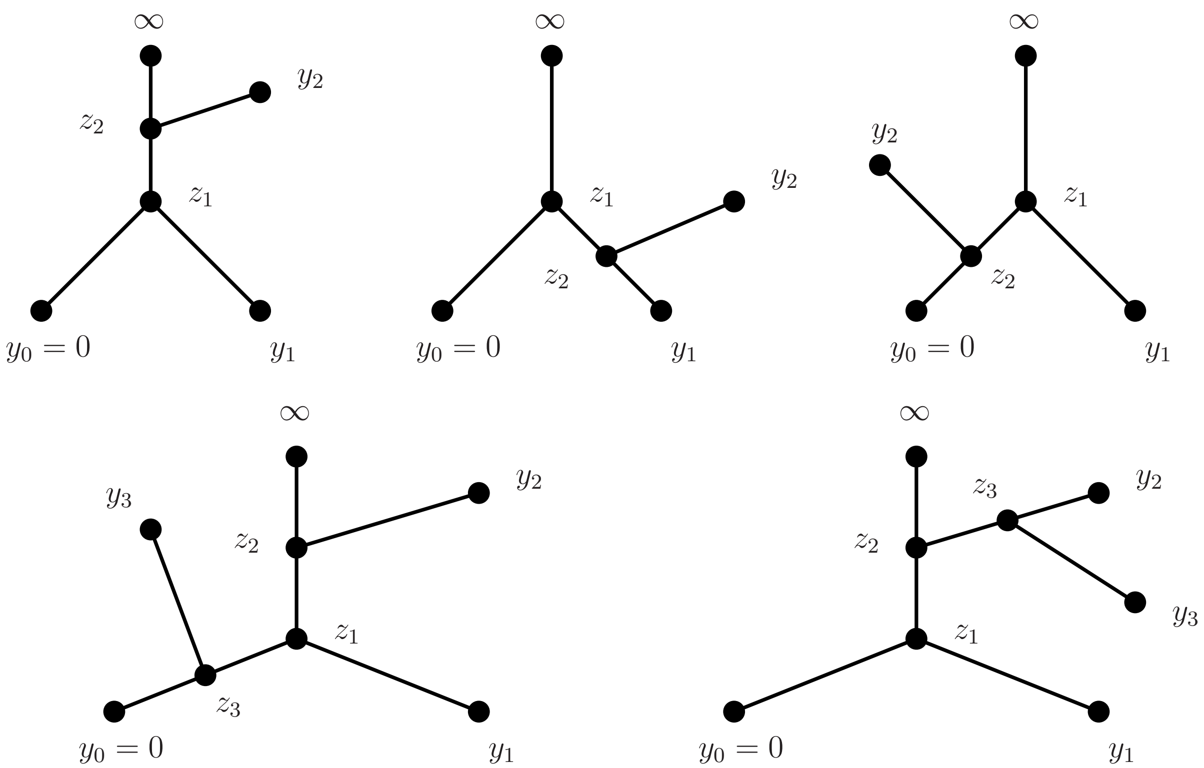


fig:taus
Figure 2: All three labelled tree graphs with $k = 2$, and two of the five possible labelled tree graphs with $k = 3$.

$T(y_0, \ldots, y_k)$ is its starting vertex. Let us write $T(y_0, \ldots, y_k, \infty)$ for the infinite subtree of $\mathcal{U}_0$ obtained by adding this infinite path to $T(y_0, \ldots, y_k)$.

Now let us consider the "topology" of $T(y_0, \ldots, y_k, \infty)$. In the case $k = 1$, it is easy to see that there exists a vertex $z_1 \in T(y_0, y_1, \infty)$ such that the paths $T(y_0, z_1)$, $T(y_1, z_1)$ and $T(z_1, \infty)$ (some of which may degenerate to a single vertex) are edge-disjoint. In the general case $k \geq 1$, we have $k$ "branch points" $z_1, \ldots, z_k$. We use a fixed rule for indexing the $z_i$'s, in requiring that for every $i \geq 1$ the path $T(y_i, z_i)$ is edge-disjoint from $T(y_0, \ldots, y_{i-1}, \infty)$. See Figure 2.

We can formalize the construction via the following recursive procedure. Let $\mathcal{T}(0)$ denote the set containing the unique tree with vertex set $\{0, \infty\}$. Assume that the collection $\mathcal{T}(k-1)$ of trees with vertex set $\{0, \ldots, k-1\} \cup \{\infty\} \cup \{\bar{1}, \ldots, \overline{k-1}\}$ has been defined for some $k \geq 1$. Let $\mathcal{T}(k)$ denote the collection of trees with vertex set $\{0, \ldots, k\} \cup \{\infty\} \cup \{\bar{1}, \ldots, \bar{k}\}$ that can be obtained in the following way. Pick some $\tau' \in \mathcal{T}(k-1)$, and pick one of the edges of $\tau'$. Split this edge into two by introducing a new vertex $\bar{k}$ on the edge, and add the new edge $\{k, \bar{k}\}$ to $\tau'$. It is easy to see that any $\tau \in \mathcal{T}(k)$ has the following properties (see Figure 2):

(i) $\deg_\tau(\infty) = 1 = \deg_\tau(y_i)$, $i = 0, \ldots, k$.

(ii) $\deg_\tau(\bar{i}) = 3$, $i = 1, \ldots, k$.

With the above definitions, the event $\{y_1, \ldots, y_k \in \mathcal{U}_0\}$ implies that there exist

$z_1, \ldots, z_k \in T(y_0, \ldots, y_k, \infty)$ and $\tau \in \mathcal{T}(k)$ such that $T(y_0, \ldots, y_k, \infty)$ is the edge-disjoint union of paths $T(\varphi(r), \varphi(s))$, where $\{r, s\} \in E(\tau)$, and $\varphi : V(\tau) \to \mathbb{Z}^d \cup \{\infty\}$ is defined by

$$\begin{cases} \varphi(i) = y_i & i = 0, \ldots, k; \\ \varphi(\infty) = \infty; \\ \varphi(\bar{i}) = z_i & i = 1, \ldots, k. \end{cases} \tag{4.4}$$

{e:def-varphi

Note that the choice of $\tau$ is not unique, due to possible coincidences between the vertices $y_0, \ldots, y_k, z_1, \ldots, z_k$. We neglect the overcounting resulting from this, for an upper bound.

If the additional restriction $d_{\mathcal{U}}(0, y_i) \leq n$, $i = 1, \ldots, k$ is in place, we must also have $d_{\mathcal{U}}(\varphi(r), \varphi(s)) \leq n$ for all $\{r, s\} \in E(\tau)$ such that $r, s \neq \infty$. We define the event

$$\begin{aligned} &E(y_1, \ldots, y_k, z_1, \ldots, z_k, \tau, n) \\ &\quad = \left\{ \begin{matrix} T(y_0, \ldots, y_k, \infty) \;=\; \cup_{\{r,s\} \in E(\tau)} T(\varphi(r), \varphi(s)) \text{ as} \\ \text{an edge-disjoint union and } d_{\mathcal{U}}(\varphi(r), \varphi(s)) \leq n \\ \text{for all } \{r, s\} \in E(\tau) \text{ such that } r, s \neq \infty \end{matrix} \right\}. \end{aligned}$$

Considering all possible choices of $\tau$ and $z_1, \ldots, z_k$, we get

$$\begin{aligned} \mathbb{E}\big(|B_{\mathcal{U}}(0, n)|^k\big) &= \sum_{y_1, \ldots, y_k \in \mathbb{Z}^d} \mathbb{P}\big(y_1, \ldots, y_k \in B_{\mathcal{U}}(0, n)\big) \\ &\leq \sum_{\tau \in \mathcal{T}(k)} \sum_{y_1, \ldots, y_k \in \mathbb{Z}^d} \sum_{z_1, \ldots, z_k \in \mathbb{Z}^d} \mathbb{P}\big(E(y_1, \ldots, y_k, z_1, \ldots, z_k, \tau, n)\big). \end{aligned}$$

We use Wilson's algorithm [W, LP] to replace the complicated event $E(y_1, \ldots)$ by a slightly larger event that is easier to handle. For this, enumerate the edges of $\tau$ as

$$\{r_0, s_0\}, \{r_1, s_1\}, \ldots, \{r_{2k}, s_{2k}\},$$

where the labelling is chosen in such a way that the following two properties are satisfied (see Figure 3(a)):

(a) $s_0 = \infty$.

(b) For every $j = 1, \ldots, 2k$, the set of edges $\{\{r_\ell, s_\ell\} : \ell = 0, \ldots, j-1\}$ spans a subtree of $\tau$, and $s_j$ is a vertex of this subtree.

Using Wilson's method with random walks started at $\varphi(r_0), \ldots, \varphi(r_{2k})$, we see that

$$E(y_1, \ldots, y_k, z_1, \ldots, z_k, \tau, n) \subset \bigcap_{j=1}^{2k} F(\varphi(s_j), \varphi(r_j), n). \tag{4.5}$$

{e:E-incl}

Here $F(\cdot, \cdot, n)$ are the events defined in (3.26). Importantly, the events on the right hand side are independent. Theorem 3.12 and the inclusion (4.5) imply that

$$\begin{aligned} &\mathbb{P}\big(E(y_1, \ldots, y_k, z_1, \ldots, z_k, \tau, n)\big) \\ &\quad \leq \prod_{j=1}^{2k} C(1 + |\varphi(s_j) - \varphi(r_j)|)^{2-d} \exp\left[-c\frac{|\varphi(s_j) - \varphi(r_j)|^2}{n}\right]. \end{aligned} \tag{4.6}$$

{e:E-bnd}

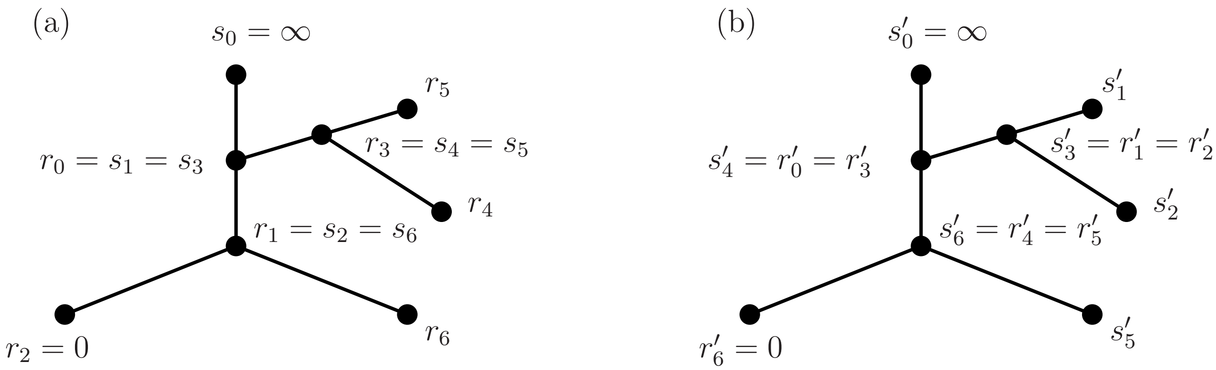


fig:enum
Figure 3: (a) A possible enumeration of edges for the application of Wilson's method. (b) A possible enumeration of edges for performing the summations using (4.8) in the order $j = 1, 2, \dots, 2k$. Summing over the spatial location $\varphi(s'_1)$ eliminates the factor involving the edge $\{s'_1, r'_1\}$. Following this, it is possible to sum over $\varphi(s'_2)$, etc.

It remains to estimate the sum of the right hand side of (4.6) over all choices of the $y_i$'s and $z_i$'s. For this it will be convenient to use a different enumeration of $E(\tau)$. Suppose that

$$\{r'_0, s'_0\}, \{r'_1, s'_1\}, \dots, \{r'_{2k}, s'_{2k}\}$$

satisfies the following properties (see Figure 3(b)).

(a′) $s'_0 = \infty$ and $r'_{2k} = 0$.

(b′) For every $j = 1, \dots, 2k$ the set $\{\{r'_\ell, s'_\ell\} : \ell = j, \dots, 2k\}$ induces a connected subtree of $\tau$, and $s'_j$ is a leaf of this subtree.

For ease of notation, let us write $u_j = \varphi(r'_j)$ and $w_j = \varphi(s'_j)$. With the new enumeration the right hand side of (4.6) takes the following form:

$$\mathbb{P}\big(E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n)\big) \leq \prod_{j=1}^{2k} C(1 + |w_j - u_j|)^{2-d} \exp\left[-c\frac{|w_j - u_j|^2}{n}\right]. \qquad (4.7)$$ {e:E-bnd2}

Note again that the $w_j$'s and $u_j$'s are $z_i$'s and $y_i$'s, determined implicitly by $\tau$. Importantly, property (b′) of the enumeration implies that if $w_j = \varphi(s'_j) = z_i$ for some $i, j$, then the variable $z_i$ does not occur in the product

$$\prod_{\ell=j+1}^{2k} C(1 + |w_j - u_j|)^{2-d} \exp\left[-c\frac{|w_j - u_j|^2}{n}\right].$$

Similar considerations apply if $w_j = \varphi(s'_j) = y_i$ for some $i, j$. The summation over $y_1, \dots, y_k$ and $z_1, \dots, z_k$ can be accomplished using the following elementary lemma.

{lem:conv-bnd

**Lemma 4.2.** *For any $u \in \mathbb{Z}^d$, we have*

$$\sum_{w \in \mathbb{Z}^d} (1 + |w - u|)^{2-d} \exp\left[-c\frac{|w - u|^2}{n}\right] \leq Cn. \qquad (4.8)$$ {e:conv-bnd}

We apply Lemma 4.2 successively to the factors with $j = 1, \dots, 2k$ on the right hand side of (4.7). See Figure 3(b) for an example of how the edges of $\tau$ are successively removed by the summations. We obtain

$$\mathbb{E}\big(|B_{\mathcal{U}}(0,n)|^k\big) \le \sum_{\tau \in \mathcal{T}(k)} (Cn)^{2k}. \tag{4.9}$$ {e:B_U-moment

Since the number of trees in $\mathcal{T}(k)$ is $1 \cdot 3 \cdot \dots \cdot (2k-1) \le 2^k k!$, this proves (4.2). □

**Remark 4.3.** The statements of Theorem 4.1 still hold, with essentially the same proof, when $\mathcal{U}$ is replaced by $\mathcal{U}_D$, the USF on a subset $D \subset \mathbb{Z}^d$. Note that $\mathcal{U}_0$ still has one end. This follows from [LMS, Proposition 3.1], and the fact that the component of 0 under the measure $\mathsf{WSF}_o$ in the domain $D$ is stochastically smaller then it is in $\mathbb{Z}^d$. Therefore, a decomposition into events $E(y_1, \dots, y_k, z_1, \dots, z_k, n)$ still holds (with $\mathcal{U} = \mathcal{U}_D$), where now all vertices are in $D$. The inclusion (4.5) still holds, with the events $F$ having the same meaning as before. This allows to bound the summations in exactly the same way as in $\mathbb{Z}^d$.

# 5 Lower bounds on volumes

{sec:vol-lb}

In this section we return to the setup of Section 3, in order to give a lower bound on the volume of $\mathcal{U}_0$. We first estimate the number of vertices of $\mathcal{U}_0$ in shells $Q_{n+m} \setminus Q_n$. Recall that $Q_N \subset D \subset \mathbb{Z}^d$, and $n, m$ satisfy $16 \le n < n + m \le N$, with $m \le n/8$. We have $L = \mathcal{L}(\mathcal{E}^F_{D^c}(S))$, a loop-erased walk from 0 to $\partial D$. We denote by $\alpha = \mathcal{E}^F_{\partial_i Q_n} L$ the portion of $L$ ended when it reaches the interior boundary of $Q_n$, and $x_0 \in \partial_i Q_n$ is the endpoint of $\alpha$. The remaining piece of $L$ is $L' = \mathcal{B}^F_{\partial_i Q_n} L$, and $\beta = \mathcal{E}^F_{\partial_i Q(x_0,m)} L'$ is the part of $L'$ until the first exit from the box of radius $m$ centred at $x_0$. See Figure 4.

Recall that when $x_0 \in \mathbb{H}_n$, we define $A = A(x_0) = Q(x_0 + (m/2)e_1, m/4)$ and $x_1 = x_0 + (m/2)e_1$, with appropriate rotations applied if $x_0$ is on a different face of $Q_n$. We will now also need a point $x_2 \in Q_{n+m} \setminus Q_n$ of order $m$ away from $A$, and further boxes contained in $Q_{n+m} \setminus Q_n$ that we define as follows. If $x_0 \in \mathbb{H}_n$ and the second coordinate of $x_0$ is negative, let

$$\begin{aligned} x_2 &= x_1 + me_2 \\ A' &= A'(x_0) = Q(x_1 + 2me_2, m/4) \\ A'' &= A''(x_0) = x_1 + [-3m/8, 3m/8] \times [-m, 3m] \times [-m, m]^{d-2} \cap \mathbb{Z}^d. \end{aligned} \tag{5.1}$$ {e:A'-A''-def

If $x_0 \in \mathbb{H}_n$ and the second coordinate of $x_0$ is positive, we replace $e_2$ by $-e_2$ and $[-m, 3m]$ by $[-3m, m]$. If $x_0$ is on a different face of $Q_n$, we replace $e_1$ and $e_2$ by two other suitable unit vectors.

The key technical estimate is to show that $\beta \cap A$ has capacity of order $m^2$ with probability bounded away from 0, which we do in the next section.

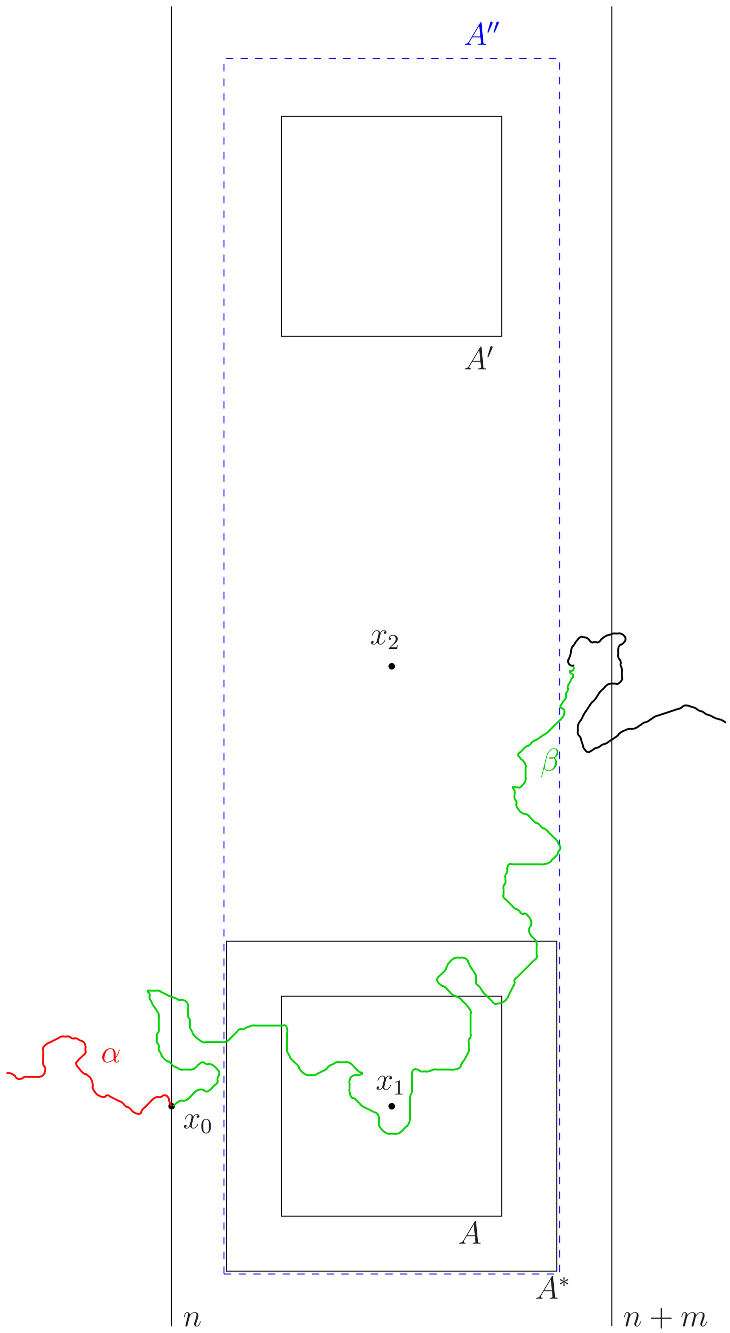


fig:boxes-cycle-pop
Figure 4: Boxes for the cycle popping argument.

## 5.1 A capacity estimate

Let $S^{x_2}$ be a random walk with $S^{x_2}(0) = x_2$, independent of $S$, $\tilde{X}$, etc.

{P:capacity}

**Proposition 5.1.** *Assume $N \geq 1$, $Q_{4N} \subset D \subset \mathbb{Z}^d$, and the setup of Section 3.*
*(a) There exists $c_1 = c_1(d) > 0$ such that*

$$m^{2-d}\,\mathbb{E}\mathrm{Cap}(A \cap \beta|\alpha) \geq c\mathbb{P}\big(S^{x_2} \text{ hits } (A \cap \beta)\,\big|\,\alpha\big) \geq c_1 m^{4-d}. \tag{5.2}$$ {e:Ecap-lb}

*(b) We have*

$$\mathbb{P}\big(c_1 m^2 \leq \mathrm{Cap}(A \cap \beta) \leq C_1 m^2\,\big|\,\alpha\big) \geq c > 0. \tag{5.3}$$ {e:capest1}

*Proof.* (a) For ease of notation, we omit the conditioning on $\alpha$. The first inequality in (5.2) is clear since $G(x_2, \cdot) \asymp m^{2-d}$ on $A$. To prove the second inequality let

$$U := \sum_{z \in A} I[z \in \beta] I[S^{x_2} \text{ hits } z],$$

so that

$$\mathbb{P}\big(S^{x_2} \text{ hits } (A \cap \beta)\big) = \mathbb{P}(U > 0).$$

Using Lemma 3.9, we have

$$\mathbb{E}(U) = \sum_{z \in A} \mathbb{P}(z \in \beta)\mathbb{P}\big(T_z[S^{x_2}] < \infty\big) \geq c m^d m^{2-d} m^{2-d} = c m^{4-d}.$$

On the other hand,

$$\mathbb{E}\big(U^2\big) = \sum_{x,y \in A} \mathbb{P}(x, y \in \beta)\,\mathbb{P}\big(T_x[S^{x_2}] < \infty,\, T_y[S^{x_2}] < \infty\big). \tag{5.4}$$ {e:U^2}

Since the process $\widetilde{X}$ generating $L'$ must pass through $\partial A^*$ in order for the event $x, y \in \beta$ to occur, we have

$$\begin{aligned}\mathbb{P}(x, y \in \beta) &\leq \max_{z \in \partial A^*} [\widetilde{G}_D(z, x)\widetilde{G}_D(x, y) + \widetilde{G}_D(z, y)\widetilde{G}_D(y, x)] \\ &\leq C m^{2-d} G(x, y).\end{aligned}$$

For the other term in the right hand side of (5.4) we have

$$\begin{aligned}\mathbb{P}\big(T_x[S^{x_2}] < \infty,\, T_y[S^{x_2}] < \infty\big) &\leq [G(x_2, x)G(x, y) + G(x_2, y)G(y, x)] \\ &\leq C m^{2-d} G(x, y).\end{aligned}$$

Since $d \geq 5$, we have $\sum_{x,y \in A} G(x, y)^2 \leq C m^d$, which gives $\mathbb{E}\big(U^2\big) \leq C m^{4-d}$.
The Paley-Zygmund inequality then gives

$$\mathbb{P}\big(S^{x_2} \text{ hits } (A \cap \beta)\big) = \mathbb{P}(U > 0) \geq \frac{\mathbb{E}(U)^2}{\mathbb{E}\big(U^2\big)} \geq c m^{4-d}.$$

(b) Since $\mathrm{Cap}(A \cap \beta) \le C|A \cap \beta| = CH_A(\beta)$, by Lemma 3.3 we have for $\lambda \ge 1$ that $\mathbb{P}(\mathrm{Cap}(A \cap \beta) > \lambda c m^2) \le \lambda^{-1}$. On the other hand, using (3.8) and the lower bound in (a), the second moment method gives that there exists $c_1 > 0$ such that

$$\mathbb{P}(\mathrm{Cap}(A \cap \beta) > c_1 m^2) > c_1;$$

taking $\lambda$ large enough then gives (5.3). $\square$

Assume now, similarly to Proposition 3.11, that $k \ge 1$ and $m \ge 4$ such that $N/2 \le km < N - m$. Recall that for $j = 1, \dots, k$ we denote $\alpha_j = \mathcal{E}^F_{\partial_i Q(0,jm)} L$, which is the initial piece of $L$ ending with the first point at radius $jm$. Let $Y_j = \alpha_j(|\alpha_j|)$ be the last point in $\alpha_j$, and $\beta_j = \mathcal{E}^F_{\partial_i Q(Y_j,m)}(\mathcal{B}^F_{\partial_i Q(0,jm)} L)$ be the path $L$ between $Y_j$ and its first hit after $Y_j$ on $\partial_i Q(Y_j, m)$. Let $Y_{j,1}$ and $Y_{j,2}$ be the points $x_1$ and $x_2$ defined with respect to $x_0 = Y_j$, respectively. Define the following event, measurable with respect to $L$:

$$G(c_1, c_2, C_1) = \left\{ \begin{array}{l} \text{there are at least } c_2 k \text{ indices } j \text{ with } 1 \le j \le k \\ \text{such that } \mathbb{P}\big(T_{A(Y_j)\cap\beta_j}[S^{Y_{j,2}}] < \infty \,\big|\, L\big) \ge c_1 m^{4-d} \\ \text{and } |A''(Y_j) \cap \beta_j| \le C_2 m^2 \end{array} \right\}. \quad (5.5)$$ {e:G-event}

Proposition 5.1 and an argument similar to that of Proposition 3.11 gives the following corollary.

{C:enough box

**Corollary 5.2.** *Under the assumptions of Proposition 5.1, there exist $c_1, c_2 > 0$ and $C_2$ such that we have*

$$\mathbb{P}\left[G(c_1, c_2, C_2)\right] \ge 1 - \exp(-ck). \quad (5.6)$$ {e:good boxes

*Proof.* Let $\mathcal{F}_j = \sigma(\alpha_j)$, and let $I_j$ be the indicator

$$I_j = \left[c_1 m^2 \le \mathrm{Cap}(A(Y_j) \cap \beta_j) \text{ and } H_{A''(Y_j)}(\beta_j) \le C_2 m^2 \right].$$

Note that $I_j$ is measurable with respect to $\mathcal{F}_{j+1} \subset \sigma(L)$. Due to Proposition 5.1 and Remark 3.4, we have

$$\mathbb{P}[I_j = 1 \,|\, \mathcal{F}_j] \ge c > 0. \quad (5.7)$$ {e:prob-lb}

When the path $L$ is such that $I_j = 1$ holds, then we have

$$\mathbb{P}\big(T_{A(Y_j)\cap\beta_j}[S^{Y_{j,2}}] < \infty \,\big|\, L\big) \ge c_1 m^{4-d},$$

and $|A''(Y_j) \cap \beta_j| \le C_2 m^2$. Therefore, we have $G(c_1, c_2, C_2) \supset \{\sum_{j=1}^k I_j \ge c_2 k\}$, and the claim follows from (5.7).

{R:cond-exit}

**Remark 5.3.** We note the following minor extension of Corollary 5.2. Assuming still that $Q_{4N} \subset D$, let $w \in \partial D$ be fixed, condition $S$ to exit $D$ at $w$, and let $L' = \mathcal{L}(\mathcal{E}^F_{D^c} S)$ be its loop-erasure. Masson [Mas] proves that the law of $\mathcal{E}^F_{Q_N^c} L'$ is comparable, up to constants factors, to the law of $\mathcal{E}^F_{Q_N^c} L$. Since the event $G(c_1, c_2, C_2)$ is measurable with respect to $\mathcal{E}^F_{Q_N^c} L$, the statement of the corollary follows also for $L'$.

## 5.2 Lower bound on $|Q_N \cap \mathcal{U}_0|$

{ssec:cycle-p

We continue with the setup of the previous section. Our argument will use the cycle popping idea of Wilson [W]; see also [LP]. The main result of this section is the following lower bound on $\mathcal{U}_0$.

{T:cyclepop}

**Theorem 5.4.** *Assume $N \geq 1$, $Q_{4N} \subset D \subset \mathbb{Z}^d$, and let $\mathcal{U} = \mathcal{U}_D$. There exist constants $C, c$, such that*

$$\mathbb{P}\big(|Q_N \cap \mathcal{U}_0| \leq \lambda N^4\big) \leq C \exp(-c\lambda^{-1/3}).$$

*Proof.* Condition on $L$, and assume that the event (5.5) occurs. Let $J = J(\omega)$ be the set of indices $1 \leq j \leq k$ (a $\sigma(L)$-measurable random set) satisfying the requirements in this event. For each $j \in J$, let

$$A'(j) = A'(Y_j) \qquad A''(j) = A''(Y_j).$$

The definitions of $A'$ and $A''$ made in (5.1) ensure that $A''(j)$, $j \in J$ are disjoint.

We now define stacks as in [W]. For each $z \in \mathbb{Z}^d - L$ we define i.i.d. stack r.v. $(\eta_{z,i}, i \geq 0)$ taking values uniformly on the vertices $z' \in \mathbb{Z}^d$ with $z' \sim z$. For $j \in J$ and $z \in (L - \beta_j) \cap A''(j)$ we define additional stack r.v. $(\eta'_{z,i}, i \geq 0)$, again taking values uniformly on the neighbours of $z$. We call Stacks I the stack r.v. given by the $\eta_{\cdot,\cdot}$, and Stacks II the stack r.v. given by the Stack I r.v. and the additional stack r.v. $\eta'_{\cdot,\cdot}$.

Assume that $j \in J$. Working with either Stacks I or Stacks II, we consider the effect of popping all cycles that are entirely contained in $A''(j)$. That is, if a cycle starts in $A''(j)$, but part of it lies outside $A''(j)$, we do not pop it. It is important to note that the order of popping cycles does not affect the final configuration on the top of the stacks.

For each $j \in J$, let

$$V_j^I = \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using Stacks I} \\ \text{reveals a path from } y \text{ to } L \end{array} \right\},$$
$$V_j^{II} = \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using Stacks II} \\ \text{reveals a path from } y \text{ to } A''(j) \cap \beta_j \end{array} \right\}.$$

{L:coupling}

**Lemma 5.5.** *We have $V_j^I \supset V_j^{II}$ for all $j \in J$.*

*Proof.* Let $y \in V_j^{II}$, and consider Stacks II. Starting from $y$, follow the arrows in Stacks II, until $A''(j) \cap \beta_j$ is hit. Removing cycles chronologically from this path pops some cycles entirely contained in $A''(j)$, and reveals a path from $y$ to $A''(j) \cap \beta_j$. Now if we follow the arrows in Stacks I instead, then the same arrows are used until the first time $L$ is hit. This guarantees that a path from $y$ to $L$ is revealed, that does not leave $A''(j)$, and hence $y \in V_j^I$. □

{L:VII-lb}

**Lemma 5.6.** *Assume $d \geq 5$. For some $c_3 > 0$ we have*

$$\mathbb{P}\big(|V_j^{II}| \geq c_3 m^4 \,\big|\, L\big) \geq c > 0 \quad \textit{on the event } \{j \in J\}.$$

*Proof.* Set $N_j = |V_j^{II}|$; we estimate the first and second moments of $N_j$.

Fix $y \in A'(j)$. Following the arrows from $y$ in Stacks II we perform a random walk until either we exit $A''(j)$, or we hit $A''(j) \cap \beta_j$. Therefore, on the event $\{j \in J\}$, we have

$$\begin{aligned}\mathbb{P}\big(y \in V_j^{II} \,\big|\, L\big) &= \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^y] < \tau_{A''(j)}[S^y] \,\big|\, L\big) \\ &\geq \mathbb{P}\big(T_{A(j)\cap\beta_j}[S^y] < \tau_{A''(j)}[S^y] \,\big|\, L\big) \\ &\geq c\mathbb{P}\big(T_{A(j)\cap\beta_j}[S^y] < \infty \,\big|\, L\big). \qquad (5.8)\end{aligned}$$

{e:hit-from-y

The final inequality is proved by an argument similar to that of Lemma 3.6, where we let $R$ count the number of crossings by the walk from a box $A^{**} \subset A''(j)$ to $\partial A''(j)$ before hitting $\beta_j \cap A(j)$, where each face of $\partial A^{**}$ is at distance $m/16$ away from the corresponding face of $\partial A''(j)$.

Summing over $y$ in (5.8), the Harnack inequality and Proposition 5.1 give that on the event $\{j \in J\}$ we have

$$\mathbb{E}\big(N_j \,\big|\, L\big) \geq cc_1 m^d m^{4-d} = cm^4.$$

We also have, on the event $\{j \in J\}$, that

$$\begin{aligned}\mathbb{P}\big(y \in V_j^{II} \,\big|\, L\big) &\leq \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^y] < \infty \,\big|\, L\big) \\ &\leq cm^{2-d}\mathrm{Cap}(A''(j) \cap \beta_j) \qquad (5.9) \\ &\leq cm^{2-d}|A''(j) \cap \beta_j| \qquad (5.10) \\ &\leq cm^{2-d}m^2 = cm^{4-d}, \qquad (5.11)\end{aligned}$$

so that summing over $y \in V_j^{II}$ we obtain

$$\mathbb{E}\big(N_j \,\big|\, L\big) \leq cm^4, \qquad \text{on } \{j \in J\}.$$

We now bound the second moment of $N_j$. For $x \in V_j^{II}$ write $\gamma_j(x)$ for the path from $x$ to $\beta_j$. Given $x, y \in A'(j)$ with $x, y \in V_j^{II}$, let $F_{xy}$ be the event that the paths $\gamma_j(x)$ and $\gamma_j(y)$ intersect. Then

$$\begin{aligned}\mathbb{E}\big(N_j^2 \,\big|\, L\big) = &\sum_{x\in A'(j)} \sum_{y\in A'(j)} \mathbb{P}\big(x \in V_j^{II}, y \in V_j^{II}, F_{xy}^c \big| L\big) \qquad (5.12) \\ &+ \sum_{x\in A'(j)} \sum_{y\in A'(j)} \mathbb{P}\big(x \in V_j^{II}, y \in V_j^{II}, F_{xy} \big| L\big). \qquad (5.13)\end{aligned}$$

{e:sumint2}

Note that as

$$\mathbb{P}\big(y \in V_j^{II}, F_{xy}^c \big| x \in V_j^{II}, L\big) \leq \mathbb{P}\big(y \in V_j^{II} \big| L\big),$$

the first sum above is bounded by $\mathbb{E}\big(N_j \,\big|\, L\big)^2$, which is in turn dominated by $cm^8$ on the event $\{j \in J\}$.

It remains to bound the sum in (5.13). If $x, y \in V_j^{II}$ and $F_{xy}$ occur, then there exists a unique $w \in A''(j)$ with the property that cycle popping reveals three edge-disjoint paths: one from $w$ to $A''(j) \cap \beta_j$, a second from $x$ to $w$ and a third from $y$ to $w$. (We allow to have $x = w$ or $y = w$ or both.) When this event happens with a fixed $w$, we can reveal the paths by first following the arrows starting from $w$ until $A''(j) \cap \beta_j$ is hit, then following the arrows starting from $x$ until $w$ is hit, then following the arrows starting from $y$ until $w$ is hit. This shows that

$$\begin{aligned}&\mathbb{P}\big(x, y \in V_j^{II} \,\big|\, L\big) \\ &\quad \le \sum_{w \in A''(j)} \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^w] < \infty \,\big|\, L\big)\, \mathbb{P}\big(T_w[S^x] < \infty\big)\, \mathbb{P}\big(T_w[S^y] < \infty\big).\end{aligned} \tag{5.14}$$

{e:VII2ndm-ub

Let $\tilde{A}(j) = Q(Y_{j,1}, (3m/2))$, and note that $\partial\tilde{A}(j)$ has distance at least $cm$ from $A''(j) \cap \beta_j$, and also distance at least $cm$ from $A'(j)$. We estimate separately the cases:
(a) $w \in A''(j) \setminus \tilde{A}(j)$; and
(b) $w \in A''(j) \cap \tilde{A}(j)$.
On the event $\{j \in J\}$, the sum of the terms in the right hand side of (5.14) corresponding to case (a) is at most:

$$\begin{aligned}Cm^{2-d}\mathrm{Cap}(A''(j) \cap \beta_j) &\sum_{w \in A''(j)\setminus\tilde{A}(j)} \sum_{x,y \in A'(j)} G(x,w)G(y,w) \\ &\le Cm^{2-d}m^2m^2m^2m^d = Cm^8.\end{aligned}$$

The sum for case (b) is at most:

$$\begin{aligned}Cm^{2-d}m^{2-d}m^dm^d &\sum_{w \in A''(j)\cap\tilde{A}(j)} \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^w] < \infty\big) \\ &\le Cm^4 \sum_{w \in \tilde{A}(j)} \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^w] < \tau_{\tilde{A}(j)}\big) \\ &\le Cm^4m^2\mathrm{Cap}(A''(j) \cap \beta_j) \le Cm^8.\end{aligned}$$

Here the last line follows from $j \in J$ and Proposition 5.1.

The moment estimates for $|V_j^{II}|$ and the one-sided Chebyshev inequality yield:

$$\mathbb{P}\big(|V_j^{II}| \ge cm^4 \,\big|\, L\big) \ge c > 0 \quad \text{on the event } \{j \in J\}.$$

This completes the proof of the Lemma. □

We can now complete the proof of Theorem 5.4. Choose $k \asymp \lambda^{-1/3}$ so that $\lambda N^4 \asymp km^4$. Then using Corollary 5.2, the conditional independence of $(V_j^{II})_{j\in J}$, and Lemma 5.5, for a suitably small $c_4 > 0$ we have

$$\begin{aligned}\mathbb{P}\big(|Q_N \cap \mathcal{U}_0| \le \lambda N^4\big) &\le C\exp(-ck) + \mathbb{E}\left(\mathbb{P}\left(\begin{matrix} V_j^{II} \;\ge\; c_3m^4 \text{ for less} \\ \text{than } c_4k \text{ indices } j \in J\end{matrix} \,\middle|\, L\right) I[G(c_1,c_2,C_2)]\right) \\ &\le C\exp(-c\lambda^{-1/3}).\end{aligned}$$

This completes the proof of the Theorem.

{T:UST-lb}

**Theorem 5.7.** *Assume $d \geq 5$ and let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$. There exist $c > 0$ and $C$ such that for all $\lambda > 0$ we have*

$$\mathbb{P}\big(|B_{\mathcal{U}}(0,n)| \leq \lambda n^2\big) \leq C \exp(-c\lambda^{-1/5}).$$

For the proof of this theorem, we assume the setting of Proposition 3.11, with $D = \mathbb{Z}^d$. Recall that $M_N = |\mathcal{E}^F_{\partial_i Q_N} L|$, that is the number of steps of $L$ until it reaches the boundary of $Q_N$.

{L:len-box-ub

**Lemma 5.8.** *We have*

$$\mathbb{E}\big(M_N^k\big) \leq C_2^k k! N^{2k}.$$

*Consequently, there exist $c > 0$ and $C$ such that for all $\lambda > 0$ we have*

$$\mathbb{P}\big(M_N \geq \lambda N^2\big) \leq C \exp(-c\lambda). \tag{5.15}$$ {e:MNexp}

**Remark 5.9.** If $M_N^S$ is the length of a simple random walk path run until its first exit from $Q_N$ then it is well known that $M_N^S/N^2$ has an exponential tail. However we do not have $M_N \leq M_N^S$, so we need an alternative argument to obtain the bound (5.15).

*Proof.* We have

$$\begin{aligned}
\mathbb{E}\big(M_N^k\big) &\leq \mathbb{E}\big(|S[0,\infty) \cap Q_N|^k\big) \\
&= k! \sum_{x_1,\dots,x_k \in Q_N} G(0,x_1)G(x_1,x_2)\dots G(x_{k-1},x_k) \\
&\leq k! \Big( \sum_{z \in Q_{2N}} G(0,z) \Big)^k = C_2^k k! N^{2k}.
\end{aligned}$$

To see the second statement:

$$\mathbb{P}\big(M_N \geq \lambda N^2\big) \leq \exp(-\lambda t N^2) \mathbb{E}\big(e^{tM_N}\big) \leq \exp(-\lambda t N^2) \frac{1}{1 - C_2 t N^2}.$$

Choosing $t = 1/(2C_2N^2)$ completes the proof of the Lemma. □

*Proof of Theorem 5.7.* It is sufficient to prove the statement for $0 < \lambda < \lambda_0$ for some fixed $\lambda_0$. Let us choose $N = \lambda^\alpha \sqrt{n}$ with some exponent $\alpha > 0$, that we will optimize over at the end of the proof. We have

$$\mathbb{P}(M_N \geq n/2) \leq C \exp\Big( -c \frac{n}{2N^2} \Big) = C \exp(-c\lambda^{-2\alpha}).$$

Condition on $L$, as in the proof of Theorem 5.4, and assume the event

$$\widetilde{G} = G(c_1, c_2, C_1) \cap \{M_N < n/2\}.$$

We set

$$\lambda n^2 = c_3 k m^4 \asymp N m^3,$$

which means we pick $m$ to be

$$m \asymp \sqrt{n}\lambda^{(1-\alpha)/3}.$$

Hence $N/m \asymp k \asymp \lambda^{(4\alpha-1)/3}$. Note that this implies that

$$\mathbb{P}\big(G(c_1, c_2, C_1)^c\big) \le C\exp(-c(N/m)) = C\exp(-c\lambda^{(4\alpha-1)/3}).$$

Since we want $N/m \gg 1$, we impose the condition $0 < \alpha < 1/4$ on $\alpha$.

For each $j \in J$, let

$$\widetilde{V}_j^I = \left\{ y \in A'(j) : \begin{array}{l}\text{cycle popping using } \mathsf{Stacks\ I} \text{ reveals} \\ \text{a path from } y \text{ to } L \text{ of length} \le n/2\end{array} \right\},$$

$$\widetilde{V}_j^{II} = \left\{ y \in A'(j) : \begin{array}{l}\text{cycle popping using } \mathsf{Stacks\ II} \text{ reveals a} \\ \text{path from } y \text{ to } A''(j) \cap \beta_j \text{ of length} \le n/2\end{array} \right\}.$$

As in Lemma 5.5 we have $\widetilde{V}_j^I \supset \widetilde{V}_j^{II}$ for all $j \in J$.

In estimating $\mathbb{E}\big(\widetilde{V}^{II}\big)$ from below, we write

$$\begin{aligned}\mathbb{P}\big(y \in \widetilde{V}_j^{II} \,\big|\, L\big) \ge\ & \mathbb{P}\big(T_{A''(j)\cap\beta_j}[S^y] < \tau_{A''(j)}[S^y] \,\big|\, L\big) \\ & - \mathbb{P}\big(|\mathcal{E}^F_{\partial A''(j)}(S^y)| > n/2,\, T_{A''(j)\cap\beta_j}[S^y] \circ \Theta_{n/2} < \infty\big).\end{aligned} \tag{5.16}$$

{e:first-mome

On the event $\{j \in J\}$, the first term on the right hand side is $\ge cm^{4-d}$ due to (5.8). We now show that the subtracted term is $\le C\exp(-cn/m^2)m^{4-d}$.

Note that we may restrict to $n/2 > 2m^2$ for convenience (although not needed for the claim), since our choice of $m$ implies that $n \asymp m^2\lambda^{-2(1-\alpha)/3}$, and we are considering small $\lambda$. Using the Markov property at time $n/2 - m^2$, the second term in the right hand side of (5.16) is at most

$$\mathbb{P}^y\big(\tau_{A''(j)} > n/2 - m^2\big) \sum_{z\in A''(j)} \mathbb{P}^z\big(T_{A''(j)\cap\beta_j} < \infty\big)\, \mathbb{P}^y\big(S(n/2) = z \,\big|\, \tau_{A''(j)} > n/2 - m^2\big).$$

The first probability can be bounded by $C\exp(-cn/m^2)$, by considering stretches of the walk of length $m^2$, in each of which there is probability $\ge c > 0$ of exiting from $A''(j)$. The conditional distribution of $S(n/2)$ is bounded above by $cm^{-d}$, due to the local CLT applied to $S(n/2 - m^2), \dots, S(n/2)$. Hence it remains to show that

$$\sum_{z\in A''(j)} \mathbb{P}^z\big(T_{A''(j)\cap\beta_j} < \infty\big) \le m^4.$$

Let us write $\widetilde{\beta}_j = A''(j) \cap \beta_j$, and $h(z) = \mathbb{P}^z(T_{\widetilde{\beta}_j} < \infty)$. By a last exit decomposition $h(z) = \sum_{u\in\widetilde{\beta}_j} G(z,u)e_{\widetilde{\beta}_j}(u)$, where $e_{\widetilde{\beta}_j}(u) = \mathbb{P}^u(T^+_{\widetilde{\beta}_j} = \infty)$. Therefore, we have

$$\begin{aligned}\sum_{z\in A''(j)} h(z) = |\widetilde{\beta}_j| + \sum_{z\in A''(j)\setminus\widetilde{\beta}_j} h(z) &\le Cm^2 + \sum_{u\in\widetilde{\beta}_j}\sum_{z\in A''(j)} G(z,u)e_{\widetilde{\beta}_j}(u) \\ &\le Cm^2 + Cm^2\sum_{u\in\widetilde{\beta}_j} e_{\widetilde{\beta}_j}(u) = Cm^2 + Cm^2\mathrm{Cap}(\widetilde{\beta}_j) \le Cm^4;\end{aligned}$$

here we used the fact that when $j \in J$ then $|\widetilde{\beta}_j| \vee \mathrm{Cap}(\widetilde{\beta}_j) \le Cm^2$.

Hence we obtain that there exists $\lambda_0 = \lambda_0(d) > 0$, such that when $0 < \lambda \le \lambda_0$, the right hand side of (5.16) is at least

$$cm^{4-d} - C\,\exp(-cn/m^2)\,m^{4-d} \ge cm^{4-d} - C\,\exp(-c\lambda^{-2(1-\alpha)/3})\,m^{4-d} \ge cm^{4-d}.$$

It follows that $\mathbb{E}\big(|\widetilde{V}_j^{II}| \,\big|\, L\big) \ge cm^4$ on the event $\{j \in J\}$.

For the second moment, we simply estimate

$$\mathbb{E}\Big( \Big|\widetilde{V}_j^{II}\Big|^2 \,\Big|\, L\Big) \le \mathbb{E}\Big( \big|V_j^{II}\big|^2 \,\Big|\, L\Big) \le Cm^8 \quad \text{on the event } \{j \in J\}.$$

The one-sided Chebyshev inequality yields that for some $c_4 = c_4(d) > 0$ we have

$$\mathbb{P}\Big( \Big|\widetilde{V}_j^{II}\Big| \ge c_4 m^4 \,\Big|\, L\Big) \ge c > 0 \quad \text{on } \{j \in J\}.$$

Therefore

$$\begin{aligned}
&\mathbb{P}\big(|B_{\mathcal{U}}(0,n)| \le \lambda n^2\big) \\
&\quad \le \mathbb{P}\big(\widetilde{G}^c\big) + \mathbb{P}\Big(\widetilde{G},\, \sum_{j\in J} \widetilde{V}_j^I \le \lambda n^2\Big) \\
&\quad \le \mathbb{P}(M_N > n/2) + \mathbb{P}\big(G(c_1,c_2,C_1)^c\big) + \mathbb{E}\Big(\mathbb{P}\Big(\sum_{j\in J} \widetilde{V}_j^{II} < c_3 k m^4 \,\Big|\, L\Big); \widetilde{G}\Big) \\
&\quad \le C\exp(-c\lambda^{-2\alpha}) + C\exp(-c\lambda^{(4\alpha-1)/3}) + \exp(-c\lambda^{(4\alpha-1)/3}).
\end{aligned}$$

We choose $\alpha$, so that $-2\alpha = (4\alpha - 1)/3$, so $\alpha = 1/10$. This completes the proof of the Theorem. □

**Remark 5.10.** We note the following minor extension of Theorem 5.4, that is needed in [BHJ]. Similarly to Remark 5.3, since the arguments of Theorem 5.4 only rely on properties of $\mathcal{E}^F_{Q_N^c} L$, the result extends to the case when the component of the origin is connected to a fixed vertex $w \in \partial D$.

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