A branching process with contact tracing

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July 29, 2020

Abstract

We consider a supercritical branching process and define a contact tracing mechanism on its genealogical tree. We calculate the growth rate of the post tracing process, and give conditions under which the tracing is strong enough to drive the process to extinction.

Keywords: Galton-Watson, branching process, percolation, contact tracing

Subject Classification: Primary 60J80 Secondary 92D30

1 Introduction

In this paper we present a simple model for contact tracing during an epidemic. The epidemic is taken to be a standard (discrete time) Bienaymé-Galton-Watson branching process $(Z_n, n \geq 0)$ with mean λ; the case of interest is when the process is supercritical, so $\lambda > 1$. (See [1, 6] for background on these processes.) We assume that b generations after infection, an infected individual is detected with probability p. If an individual is detected as infected, an attempt is made to trace all this individual's contacts, both forwards and backwards. We assume that for each infection link the probability of a successful trace is α, and that it can be determined with probability 1 whether or not a traced individual has been infected. If a traced individual is detected as infected, then in turn all contacts of that individual are traced, and this process is repeated throughout the genealogy of the epidemic. It is assumed that detected individuals are quarantined and so can be removed from the pool of infectious individuals. (See the next section for a more precise definition.) We write Z^{CT} for the branching process after contact tracing.

If $\alpha = 1$ then all traces are successful, and as soon as one individual is detected every infected individual will be traced and isolated, so bringing the epidemic to an end. For smaller values of α there is the possibility that the epidemic will remain supercritical: we are interested in how large α needs to be to control the epidemic for a fixed b and p.

Our main result is as follows. Let

$$G(u) = \sum_{k=0}^{\infty} p_k u^k \tag{1.1}$$

*Research partially supported by NSERC (Canada)

be the p.g.f. of the offspring distribution. Define sequences (g_n), (h_n) by $g_0 = h_0 = 1-p$, and

$$g_n(s) = (1-p)G\big(1-\alpha+\alpha g_{n-1}\big), \tag{1.2}$$
$$h_n(s) = (1-p)\alpha G'\big(1-\alpha+\alpha g_{n-1}\big)h_{n-1}. \tag{1.3}$$

Let

$$v_n = \begin{cases} (1-\alpha)\lambda(\lambda\alpha)^{n-1} & \text{if } 1 \le n \le b, \\ (1-\alpha)\lambda(\lambda\alpha)^b h_{n-b-1} & \text{if } n \ge b+1. \end{cases} \tag{1.4}$$

Theorem 1.1. *Let $p > 0$. Then the process after contact tracing becomes extinct if and only if*

$$\sum_{n=1}^{\infty} v_n \le 1 \tag{1.5}$$

We do not know of any case where we can give exact expressions for (v_n), but unless p is small the series for (h_n) converges very rapidly.

If α is not large enough to control the epidemic, we can still ask how quickly Z^{CT} grows. Define the *Malthusian parameter* of (v_n) to be the unique θ such that

$$\sum_{n=1}^{\infty} e^{-n\theta} v_n = 1. \tag{1.6}$$

We are concerned with the case when $\sum v_n > 1$, so $\theta > 0$.

Theorem 1.2. *On the event that Z^{CT} survives, we have with probability 1,*

$$\lim_{n\to\infty} \frac{\ln Z_n^{CT}}{n} = \theta. \tag{1.7}$$

An easy argument (see Lemma 2.4) shows that there is a function $e_b(p)$ such that the epidemic becomes extinct with probability one if $\alpha > e_b(p)$, and survives with positive probability if $\alpha < e_b(p)$. (The function $e_b(p)$ depends on the offspring distribution of the original branching process.) In Section 4 we study the function $e_b(p)$ close to the critical points where it crosses the axes.

There is a very extensive applied epidemiological literature on contact tracing, but we did not find very many papers containing exact calculations. One quite closely related paper is [2], which looks at forward contact tracing for a continuous time epidemic. In the case of an exponential infection time they obtain an exact formula for the reproduction number of the epidemic after contact tracing. As in our paper, they look at the discrete time process of untraced individuals – called there unnamed individuals. [3] looks at some extensions of this model, including allowing a latent period, and tracing delays. The papers [8, 11, 10] also all consider various kinds of contact tracing for a continuous time branching processes.

2 The contact tracing process

We need to keep track of not just the size of the original branching process, but also its genealogical structure. So we write $\Lambda_0 = \{0\}$, $\Lambda_n = \mathbb{N}^n$ and set

$$\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n, \quad \Lambda[0,N] = \bigcup_{n=0}^{N} \Lambda_n.$$

A point $x = (x_1, \dots, x_n) \in \mathbb{N}^n$ represents a potential individual in the nth generation; we write $|x| = n$, say that x is in generation n, and define its ancestor to be $a(x) = (x_1, \dots x_{n-1})$. For $x \in \mathbb{N}$ we set $a(x) = 0$. We take Λ to be a graph with edge set

$$E_\Lambda = \Big\{ \{x, a(x)\} : x \in \Lambda - \{0\} \Big\}.$$

Let $(\xi_x, x \in \Lambda)$ be independent r.v. with distribution (p_k). We will define $\eta : \Lambda \to \{0,1\}$, and set $\Gamma = \{x \in \Lambda : \eta(x) = 1\}$; this will be the set of individuals in the original process. We define $\eta(0) = 1$. Once η is defined on $\Lambda[0,n]$ we extend it to $\Lambda[0,n+1]$ as follows. Let $x = (x', x_{n+1}) \in \Lambda_{n+1}$; we set $\eta(x) = 1$ if $\eta(x') = 1$ and $x_{n+1} \le \xi_{x'}$, and take $\eta(x) = 0$ otherwise. Let $Z_n = |\{x \in \Lambda_n : \eta(x) = 1\}|$, so that (Z_n) is a branching process with offspring distribution (p_k). We consider Γ as a graph with edge set $E_\Gamma = \big\{ \{x, a(x)\} : x \in \Gamma, x \ne 0 \}$.

Let $b \in \mathbb{Z}_+$, and probabilities $p \in [0,1]$ and $\alpha \in [0,1]$. Define i.i.d. random variable $\eta_D(x)$ with a $\texttt{Ber}(p)$ distribution, and i.i.d. $\eta_T(x)$ with a $\texttt{Ber}(\alpha)$ distribution. If $\eta_D(x) = 1$ we say x is *detectable* or *detected*. If $\eta_T(x) = 1$ then we say the edge $\{x, a(x)\}$ is *open* or *traceable*. Thus (η_T) defines a bond percolation process on Γ – see [5]. A path (i.e. sequence of edges) in (Γ, E_Γ) is traceable if each edge in the path is traceable. For $x \in \Gamma$ we write $\mathcal{C}(x)$ for the set of $y \in \Gamma$ such that x and y are connected by an traceable path; we call $\mathcal{C}(x)$ the *connected cluster* of x (at time n). We always have $x \in \mathcal{C}(x)$.

The contact tracing procedure operates as follows. At time $n \ge 0$ we will define a subset A_n of $\Gamma \cap \Lambda[0,n]$. If $b > 0$ or $b = 0$ and $\eta_D(0) = 0$ we take $A_0 = \{0\}$. If $b = 0$ and $\eta_D(0) = 1$ then we set $A_0 = \emptyset$. (In this case the founding individual is detected at time 0 and the process immediately becomes extinct.)

To construct A_n from A_{n-1}, let

$$A^*_n = A_{n-1} \cup \{x \in \Gamma \cap \Lambda_n : a(x) \in A_{n-1}\};$$

thus A^*_n is A_{n-1} together with the offspring of the individuals in $A_{n-1} \cap \Lambda_{n-1}$. We now look at the r.v. $\eta_D(y)$ for $y \in A^*_n \cap \Lambda_{n-b}$, and if $\eta_D(y) = 1$ then we remove $\mathcal{C}(y)$ from A^*_n. Thus we set

$$A^R_n = \bigcup \{\mathcal{C}(y) \cap \Lambda^n_0 : y \in A^*_n \cap \Lambda_{n-b}, \eta_D(y) = 1\}, \quad A_n = A^*_n - A^R_n.$$

The *current generation* of the process (A_n) is $A^{\mathrm{CG}}_n = A_n \cap \Lambda_n$. The size of the current generation is

$$Z^{CT}_n = |A^{\mathrm{CG}}_n|.$$

Since $A_n^* \subset \Gamma \cap \Lambda_n$, $Z_n^{CT} \le Z_n$. We call the process $A = (A_n, n \ge 0)$ the (b,p,α)–*contact tracing process* or CTP(b,p,α). The parameter space is

$$\mathcal{P} = \{(b,p,\alpha) : b \in \mathbb{Z}_+, p \in [0,1], \alpha \in [0,1]\}. \tag{2.1}$$

We write $\mathbb{E}_{b,p,\alpha}$ for expectations when we wish to emphasize the dependence on the parameters.

Note that not all points are removed – for example if $\eta_D(y) = \eta_T(y) = 0$ and y has no descendants then $y \in A_n$ for all large n. If however for some n we have $A_n^{\text{CG}} = \emptyset$ then $A_{n+k}^{\text{CG}} = \emptyset$ for all $k \ge 0$.

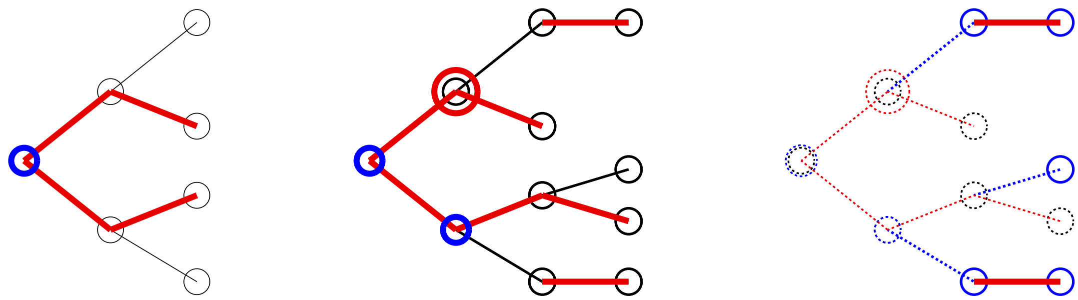

Figure 1: The process A_n at times $n = 2$ (left), $n = 3$ before tracing (center) and $n = 3$ after tracing (right). Red (thick) lines are traceable edges. At time 2 the root is not detected. At time 3 one vertex in generation 1 is detected (large red circle), and it and the 5 vertices connected to it are removed. Hence one member of the current generation is removed.

To clarify our terminology we define what we mean by survival or extinction of a random process.

Definition 2.1. Let $X = (X_n, n \ge 0)$ be a random process on $\mathbb{Z}_+$ with the property that $\{X_n = 0\} \subset \{X_{n+1} = 0\}$. We define

$$\{X \text{ becomes extinct}\} = \bigcup_{n=0}^{\infty} \{X_n = 0\}.$$

If $\mathbb{P}(X \text{ becomes extinct }) = 1$ we say X *becomes extinct*, and if $\mathbb{P}(X \text{ becomes extinct }) < 1$ we say X *survives with positive probability* or *survives wpp.* Sometimes we will shorten 'survives wpp' to 'survives'. For a set valued process such as $A^{\text{CG}} = (A_n^{\text{CG}})$ we define extinction or survival as whether or not the process $|A_n^{\text{CG}}|$ becomes extinct, or survives.

We are interested in characterizing the set of parameter values such that Z^{CT} becomes extinct, and define the following subsets of $\mathcal{P}$:

$$\mathcal{E} = \{(b,p,\alpha) : \text{ CTP}(b,p,\alpha) \text{ becomes extinct}\},$$
$$\mathcal{S} = \{(b,p,\alpha) : \text{ CTP}(b,p,\alpha) \text{ survives wpp}\}.$$

We begin with some easy properties of the sets $\mathcal{E}$ and $\mathcal{S}$.

Lemma 2.2. *(a) If $b = 0$ and $\lambda(1-p) \le 1$ then Z^{CT} becomes extinct.*
(b) If $b \ge 1$ and $\alpha = 0$ then Z^{CT} survives wpp.
(c) If $\lambda(1-p)(1-\alpha) > 1$ then Z^{CT} survives wpp.
(d) If $p = 0$ then Z^{CT} survives wpp.
(e) If $\alpha = 1$ and $p > 0$ then Z^{CT} becomes extinct.

Proof. (a) Let (A'_n) be the process A, but where at time $n+1$ only the detected individuals are removed. Then $A_n \subset A'_n$, and the process $Z'_n = |A'_n \cap \Lambda_n|$ is a simple branching process with offspring distribution mean $(\lambda(1-p))$; if $\lambda(1-p) \le 1$ then Z' becomes extinct, and therefore Z^{CT} also becomes extinct.
(b) In this case detection never removes individuals in the current generation, so $|A_n^{\mathrm{CG}}|$ is just a branching process with offspring distribution (p_k), and therefore it survives wpp.
(c) If we just consider the new points y which satisfy $\eta_T(y) = \eta_D(y) = 0$, we have a process A^{UN} which is smaller than A, and is a branching process with offspring distribution with mean $\lambda(1-p)(1-\alpha)$, and if this process survives then A survives.
(d) and (e) are clear. □

We will treat the case $p = 1$ when $b \ge 1$ in Lemma 3.5 below.

Remark 2.3. The 'heavy tailed' situation when $\sum p_k = \infty$ does not seem to be interesting in this context. If $p = 0$ or $\alpha = 1$ then Lemma 2.2(d),(e) still hold. If $p = 1$ and $\alpha < 1$ the process becomes extinct if $b = 0$ and survives otherwise, while if $p \in (0,1)$ and $\alpha < 1$ then the argument of Lemma 2.2(c) implies that the process survives wpp.

Lemma 2.4. *(Monotonicity). If $(b,p,\alpha) \in \mathcal{E}$ and $b' \le b$, $p' \ge p$ and $\alpha' \ge \alpha$ then $(b',p',\alpha') \in \mathcal{E}$.*

Proof. We write $\eta_D^{(p)}(x)$ and $\eta_T^{(\alpha)}(x)$ for the detection and tracing processes with parameters p and α respectively, and couple them so that they are monotone in p and α respectively. Write $A^{(b,p,\alpha)}$ for the associated contact tracing process. Then the construction of A gives that $A_n^{(b,p',\alpha')} \subset A_n^{(b,p,\alpha)}$. Monotonicity in b is also clear. □

Using Lemma 2.4 we see there exists a function $e_b : [0,1] \to [0,1]$ such that if $\alpha < e_b(p)$ then $(b,p,\alpha) \in \mathcal{S}$ and if $\alpha > e_b(p)$ then $(b,p,\alpha) \in \mathcal{E}$. Using Lemma 2.2 we see that

$$e_b(p) \ge \begin{cases} 1 - \frac{1}{\lambda(1-p)}, & \text{if } 0 \le p \le 1 - \lambda^{-1}, \\ 0, & \text{if } 1 - \lambda^{-1} \le p \le 1. \end{cases} \tag{2.2}$$

From Proposition 2.6 we will obtain the bound

$$e_0(p) < 1 - p/\lambda \quad \text{for } 0 < p \le 1. \tag{2.3}$$

Lemma 2.2 covers the cases when $p = 0$, so from now on we assume

$$p \in (0,1]. \tag{2.4}$$

For $x \in \Gamma \cap \Lambda_n$ let V_x be the number of offspring of x, and V_x^T and V_x^U be the number of traceable and untraceable offspring. So $V_x = V_x^T + V_x^U$ and

$$\mathbb{E}V_x^T = \lambda\alpha, \quad \mathbb{E}V_x^U = \lambda(1-\alpha).$$

Write $p_k^T = \mathbb{P}(V_x^T = k)$, and let $G_T(u)$ be the associated p.g.f. Then since the law of V_x^T conditional on V_x is $\texttt{Binom}(V_x, \alpha)$,

$$G_T(u) = \sum_{k=0}^{\infty} u^k p_k^T = G((1-\alpha) + \alpha u), \tag{2.5}$$

We can decompose A_n into a collection of disjoint connected traceable clusters $\mathcal{C}^1, \dots, \mathcal{C}^{k_n}$, and an analysis of the evolution of a traceable cluster is a key step in our proofs.

We will call $y \in \Gamma$ with $\eta_T(y) = 0$ a *cluster seed*. The edge $\{a(y), y)\}$ is untraceable, and so if $|y| = n$ and $y \in A_n$ then y will only be removed from the process $(A_{n+k}, k \geq 0)$ if some descendent of y is detected and is connected to y by a traceable path. A key observation is that the evolution of the part of $(A_{n+k}, k \geq 0)$ containing y and its descendants is independent of the rest of $(A_{n+k}, k \geq 0)$.

We look at the evolution in time of a traceable cluster, and for simplicity consider the cluster started at the root 0. Initially we consider the growth of the cluster without any detection. Let $\mathcal{V}_0 = \{0\}$, and $(\mathcal{V}_k, k \geq 1)$ be the cluster at subsequent times, given by

$$\mathcal{V}_n = \{x \in \Gamma \cap \Lambda_n : a(x) \in \mathcal{V}_{n-1}, \eta_T(x) = 1\}.$$

Let $V_0^T = 1$, $Y_0^U = Y_0^U = 0$, and for $n \geq 1$ let

$$V_n^U = \sum_{x \in \mathcal{V}_{n-1}} V_x^U, \qquad Y_n^U = \sum_{k=1}^{n} V_k^U, \tag{2.6}$$

$$V_n^T = \sum_{x \in \mathcal{V}_{n-1}} V_x^T = |\mathcal{V}_n|, \qquad Y_n^T = \sum_{k=0}^{n} V_k^T. \tag{2.7}$$

We now introduce detection for this cluster. Let S be the generation number of the first detected vertex in the cluster. So

$$\{S = n\} = \{\eta_D(y) = 0 \text{ for all } y \in \bigcup_{k=0}^{n-1} \mathcal{V}_k\} \cap \{\sum_{y \in \mathcal{V}_n} \eta_D(y) \geq 1\}.$$

Write $\widetilde{V}_n^T$ for the number of points in the cluster taking detection into account, and $\widetilde{V}_n^U$ for the corresponding number of cluster seeds. We have

$$\widetilde{V}_n^T = V_n^T 1_{(n \leq S+b-1)}, \quad \widetilde{V}_n^U = V_n^U 1_{(n \leq S+b)}. \tag{2.8}$$

(The difference between the two expressions above is because the detection process forces $\widetilde{V}_{S+b}^T = 0$, while the cluster may still produce cluster seeds in generation $S + b$.) Thus the total number of cluster seeds produced by the cluster starting from 0 is

$$\widetilde{Y}_\infty^U = \sum_{n=1}^{\infty} \widetilde{V}_n^U = \sum_{n=1}^{\infty} V_n^U 1_{(n \leq S+b)}. \tag{2.9}$$

If $b = 0$ and $\eta_D(0) = 1$ then 0 is detected and $S = \widetilde{Y}^U_\infty = 0$. For $x \in \cup A_n$ let $\mathcal{Y}(x)$ be the set of cluster seeds produced by the traceable cluster with cluster seed x. Set

$$v_n = \mathbb{E}(\widetilde{V}^U_n) \quad \text{for } n \geq 1. \tag{2.10}$$

Theorem 2.5. *Let $p > 0$. The process Z^{CT} becomes extinct if and only if*

$$y_b(p, \alpha) = \mathbb{E}_{b,p,\alpha}(\widetilde{Y}^U_\infty) = \sum_{n=1}^{\infty} v_n \leq 1.$$

Proof. Let $\mathcal{X}_0 = 0$, $X_0 = 1$, and define for $n \geq 1$

$$\mathcal{X}_n = \bigcup_{x \in \mathcal{X}_{n-1}} \mathcal{Y}(x), \quad X_n = |\mathcal{X}_n|.$$

Then $X = (X_n)$ is a branching process with offspring distribution equal in law to $\widetilde{Y}^U_\infty$. Since $\mathbb{P}(\widetilde{Y}^U_\infty = 1) < 1$ the process X becomes extinct with probability 1 if and only if $\mathbb{E}X_1 \leq 1$, i.e. if and only if $\mathbb{E}\widetilde{Y}^U_\infty \leq 1$.

It remains to show that A^{CG} (a.s.) becomes extinct if and only if X becomes extinct. If A^{CG} (a.s.) becomes extinct then the total family size $\sum_n |A^{\mathrm{CG}}_n|$ is finite, so the total number of cluster seeds is finite and thus X becomes extinct. On the other hand, if X becomes extinct then (as $p > 0$) each traceable cluster is finite, so the total family size $\sum_n |A^{\mathrm{CG}}_n|$ is finite, and thus A^{CG} becomes extinct. □

Proposition 2.6. *If $b = 0$ and $\alpha \geq 1 - p/\lambda$ then the process Z^{CT} becomes extinct. Hence*

$$e_0(p) \leq 1 - p/\lambda \quad \textit{for } 0 < p \leq 1. \tag{2.11}$$

Proof. We consider the traceable cluster $(\mathcal{V}_n)$ started at the origin. We explore the points in the traceable cluster, starting with the origin, and completing the exploration of each generation before starting on the next. Let $X_1, X_2, \ldots$ be the exploration process; the r.v. X_i take values in the genealogical space Λ. This sequence is finite or infinite according to whether the cluster is finite or infinite. The total family size of $(\mathcal{V}_n)$ is $Y = Y^T_\infty$.

We add a cemetery point ∂, define $\eta_D(\partial) = V^U_\partial = 0$ and if $Y < \infty$ we set $X_n = \partial$ for $n > Y$. Set

$$\mathcal{F}_n = \sigma(X_j, \eta_D(X_j), V^U_{X_j}, 1 \leq j \leq n).$$

Let $T_E = \min\{n \geq 1 : X_n = \partial\}$; then $T_E = Y + 1$ and is a stopping time with respect to $(\mathcal{F}_n)$. Let $T_D = \min\{n \geq 1 : \eta_D(X_n) = 1\}$, and $T = T_D \wedge T_E$. Set $U_0 = D_0 = M_0 = 0$, and for $n \geq 1$ let

$$U_n = \sum_{k=1}^{n} V^U_{X_k}, \quad D_n = \sum_{k=1}^{n} \eta_D(X_j), \quad M_n = pU_n - \lambda(1-\alpha)D_n.$$

Then M is a martingale with respect to $(\mathcal{F}_n)$. Hence for any $n \geq 0$,

$$0 = M_0 = \mathbb{E}(M_{T \wedge n}) = p\mathbb{E}(U_{T \wedge n}) - \lambda(1-\alpha)\mathbb{E}(D_{T \wedge n}).$$

The total number of cluster seeds produced by the traceable cluster up to its extinction due to a detection is $\widetilde{Y}^U_\infty$. Note that $\widetilde{Y}^U_\infty$ need not include the all r.v. V^U_x for x in the final generation, and so $\widetilde{Y}^U_\infty \le U_T$. We have $D_T \le 1$. Using Fatou's Lemma

$$\mathbb{E}(\widetilde{Y}^U_\infty) \le \mathbb{E}(U_T) \le \lim_{n\to\infty} \mathbb{E}U_{T\wedge n} = \lambda(1-\alpha)p^{-1} \lim_{n\to\infty} \mathbb{E}D_{T\wedge n} \le \lambda(1-\alpha)p^{-1}.$$

The conclusion is now immediate from Theorem 2.5. □

3 Calculation of the number of cluster seeds

Set

$$\lambda_T = \lambda\alpha, \quad \lambda_U = \lambda(1-\alpha),$$

and let

$$w_n = \mathbb{E}(V^T_n (1-p)^{Y^T_n}), \quad n \ge 0. \tag{3.1}$$

Lemma 3.1. *We have*

$$\begin{aligned}
\mathbb{E}_{b,p,\alpha}(\widetilde{V}^U_n) &= \lambda_U \lambda_T^{n-1} \quad \textit{for } 1 \le n \le b,\\
\mathbb{E}_{b,p,\alpha}(\widetilde{V}^T_n) &= \lambda_T^{n} \quad \textit{for } 0 \le n \le b-1,\\
\mathbb{E}_{b,p,\alpha}(\widetilde{V}^U_n) &= \lambda_U \lambda_T^{b} w_{n-b-1} \quad \textit{for } n \ge b+1,\\
\mathbb{E}_{b,p,\alpha}(\widetilde{V}^T_n) &= \lambda_T^{b} w_{n-b} \quad \textit{for } n \ge b.
\end{aligned}$$

Hence

$$y_b(p,\alpha) = \mathbb{E}_{b,p,\alpha}(\widetilde{Y}^U_\infty) = \sum_{n=1}^{b} \lambda_U \lambda_T^{n-1} + \lambda_U \lambda_T^b \sum_{n=0}^{\infty} w_n. \tag{3.2}$$

Proof. The first two expressions follow easily from (2.8). Let $\mathcal{F}^T_n = \sigma(V^T_j, 0 \le j \le n, \mathcal{V}_i, 0 \le i \le n)$. We have $\{S \ge n\} = \{\eta_D(x) = 0 \text{ for all } x \in \cup_{i=0}^{n-1} \mathcal{V}_i\}$.

Now let $n \ge b+1$. The r.v. $1_{(S\ge n-b)}$ and V^U_n are conditionally independent given $\mathcal{F}^T_{n-b-1}$. So

$$\mathbb{E}(\widetilde{V}^U_n | \mathcal{F}^T_{n-1}) = \mathbb{E}(V^U_{n+b} 1_{(n\le S)} | \mathcal{F}^T_{n-1}) = \mathbb{E}(V^U_{n+b} | \mathcal{F}^T_{n-1})\, \mathbb{E}(1_{(n\le S)} | \mathcal{F}^T_{n-1}).$$

Then

$$\mathbb{E}(V^U_{n+b} | \mathcal{F}^T_{n-1}) = \mathbb{E}\big(\mathbb{E}(V^U_{n+b} | \mathcal{F}^T_{n+b-1}) | \mathcal{F}^T_{n-1}\big) = \mathbb{E}\big(\lambda_U V^T_{n+b-1} | \mathcal{F}^T_{n-1}\big) = \lambda_U \lambda_T^b V^T_{n-1},$$

while

$$\mathbb{P}(S \ge n | \mathcal{F}^T_{n-1}) = (1-p)^{V^T_0 + V^T_1 + \cdots + V^T_{n-1}} = (1-p)^{Y^T_{n-1}}.$$

Combining these equalities and taking expectations gives the expression for $\mathbb{E}(\widetilde{V}^U_n)$.

Similarly if $n \geq b$ we have

$$\begin{aligned}\mathbb{E}(\widetilde{V}_n^T | \mathcal{F}_{n-b}^T) &= \mathbb{E}(V_n^T 1_{(S \geq n-b+1)} | \mathcal{F}_{n-b}^T) \\ &= \mathbb{E}(V_n^T | \mathcal{F}_{n-b}^T)\, \mathbb{E}(1_{(S \geq n-b+1)} | \mathcal{F}_{n-b}^T) = \lambda_T^b V_{n-b}^T (1-p)^{Y_n^T}.\end{aligned}$$

□

To calculate w_n set for $s, t \geq 0$, $n \geq 0$

$$H_n(s,t) = \mathbb{E}(s^{Y_n^T} t^{V_n^T}). \tag{3.3}$$

Note that $H_0(s,t) = st$. Then

$$\frac{\partial}{\partial t} H_n(s,t) = \mathbb{E}(V_n^T s^{Y_n^T} t^{V_n^T - 1}),$$

and so

$$\mathbb{E}(V_n^T (1-p)^{Y_n^T}) = \frac{\partial}{\partial t} H_n(1-p, 1). \tag{3.4}$$

Since $p > 0$ and $V_n^T \leq Y_n^T$, the series for $H_n(1-p, t)$ converges in a neighbourhood of 1, and there is no problem taking the derivative in (3.4). A standard first generation branching process decomposition gives

$$H_n(s,t) = sG_T(H_{n-1}(s,t)) = sG(1 - \alpha + \alpha H_{n-1}(s,t)).$$

Thus

$$\frac{\partial}{\partial t} H_n(s,t) = s\alpha G'(1 - \alpha + \alpha H_{n-1}(s,t)) \frac{\partial}{\partial t} H_{n-1}(s,t). \tag{3.5}$$

Setting

$$g_n(s) = H_n(s,1), \quad h_n(s) = \frac{\partial}{\partial t} H_n(s,t),$$

we have the system of equations, for $s \in [0,1]$,

$$g_0(s) = h_0(s) = s, \tag{3.6}$$

$$g_n(s) = sG\big(1 - \alpha + \alpha g_{n-1}(s)\big), \tag{3.7}$$

$$h_n(s) = s\alpha G'\big(1 - \alpha + \alpha g_{n-1}(s)\big) h_{n-1}(s), \tag{3.8}$$

and

$$w_n = h_n(1-p). \tag{3.9}$$

Proof of Theorem 1.1. This follows from Theorem 2.5, (3.2) and (3.9). We also have that v_n as defined by (2.10) satisfies (1.4). □

Note that the functions g_n and h_n depend on α but not on p. We collect some properties of these functions.

Lemma 3.2. *(a) The functions $g_n(s)$, $h_n(s)$ are strictly increasing and continuous for $s \in [0,1]$.*
(b) For $s \in [0,1)$, the sequence $g_n(s)$ is strictly decreasing in n. The limit $g_\infty(s)$ satisfies $g_\infty = sG_T(g_\infty)$ and $sG_T'(g_\infty(s)) < 1$.
(c) Let $c_1(p) = (-e\log(1-p))^{-1}$. We have

$$h_n(1-p) \le c_1(p)(1-p)^n \text{ for } n \ge 1.$$

Hence for $p > 0$

$$\sum_{n=0}^{\infty} h_n(1-p) \le \frac{1}{ep\log(1/(1-p))}. \tag{3.10}$$

(d) For each $b \in \mathbb{Z}_+$ the function $y_b(p,\alpha)$ is continuous in the region $(p,\alpha) \in (0,1]\times[0,1]$.

Proof. (a) This is clear from the definition.
(b) Note that G_T is strictly monotone. Fix $s \in (0,1)$. Then $g_1 = sG_T(s) < sG_T(1) = s$. If $g_{n-1} < g_{n-2}$ then $g_n = sG_T(g_{n-1}) < sG_T(g_{n-2}) = g_{n-1}$. Thus g_n is decreasing in n, and as G_T is continuous the limit must satisfy $g_\infty = sG_T(g_\infty)$.

Let $f_1(x) = x$ and $f_2(x) = sG^T(x)$. Then $0 = f_1(0) < f_2(0) = sG_T(0)$, while $f_1(1) = 1 > s = f_2(1)$. Thus $f_1(x) = f_2(x)$ has a solution in $[0,1]$, and as G_T is strictly monotone it follows that this solution is unique, and therefore equals g_∞. By the mean value theorem there exists $\xi \in (g_\infty, 1)$ such that $s - g_\infty = f_2(1) - f_2(g_\infty) = (1-g_\infty)f'(\xi)$. As f_2' is monotone we have $s - g_\infty \ge (1-g_\infty)f'(g_\infty)$, and thus $f_2'(g_\infty) < 1$.
(c) Note that if $V_n^T > 0$ then $Y_n^T \ge n+1$, and that for $x \ge 0$ we have $x(1-p)^x \le c_1$. So

$$h_n(1-p) = \mathbb{E}V_n^T(1-p)^{V_n^T + Y_{n-1}^T} \le c_1(1-p)^n.$$

(d) If K is a compact subset of $(0,1]\times[0,1]$ then by (c) the functions h_n converge uniformly to 0 in K. It is straightforward to verify that each term in the sum (3.2) is continuous in p,α; as the series is uniformly convergent the limit is continuous. □

Corollary 3.3. *We have $(b,p,e_b(p)) \in \mathcal{E}$ if $p > 0$.*

Proof. As y_b is continuous in the region $p > 0$, and extinction occurs if $y_b(p,\alpha) \le 1$, extinction occurs in the critical case $\alpha = e_b(p)$. □

Remark 3.4. Note that y_b is not continuous at $(0,1)$, since $y_b(0,1) = 0$ while for $\alpha \in (\lambda^{-1}, 1)$ we have $y_b(0,\alpha) = \infty$. Further $e_b(0) = 1$, but $(b,0,1) \in \mathcal{S}$, so the restriction to $p > 0$ in the Corollary above is necessary.

The following Lemma handles the case $p = 1$ and $b \ge 1$.

Lemma 3.5. *(a) Let $b \ge 1$, and let*

$$\alpha_{\lambda,b} = \inf\Big\{\alpha \ge 0 : \lambda(1-\alpha)\sum_{n=0}^{b-1}\lambda^n\alpha^n \le 1\Big\}.$$

Then $\alpha_{\lambda,b} \in (0,1)$, and for $p \in [0,1]$ CTP$(b,1,\alpha)$ survives wpp if $\alpha \in [0, \alpha_{\lambda,b})$, and becomes extinct if $\alpha \in [\alpha_{\lambda,b}, 1]$.
(b) Let $b \geq 1$. Then $e_b(p) \geq \alpha_{\lambda,b}$ for $p \in [0,1]$.

Proof. (a) Set

$$f_b(\alpha) = \lambda(1-\alpha)\sum_{n=0}^{b-1}(\alpha\lambda)^n. \tag{3.11}$$

We have $y_b(1,\alpha) = f_b(\alpha)$. Note that $f_b(0) = \lambda$, $f_b(1) = 0$, and f_b is continuous. Hence $\alpha_{\lambda,b} \in (0,1)$, and the result follows using Theorem 1.1 and Lemma 2.4.
(b) This follows from (a) by monotonicity. □

Remark 3.6. In spite of the monotonicity given by Lemma 2.4, the function $y_b(p,\alpha)$ is not monotone in α. An easy way to see this is to note that for $b \geq 1$ we have

$$\frac{\partial}{\partial\alpha} y_b(1,\alpha)\Big|_{\alpha=0} = \lambda^2 - \lambda > 0.$$

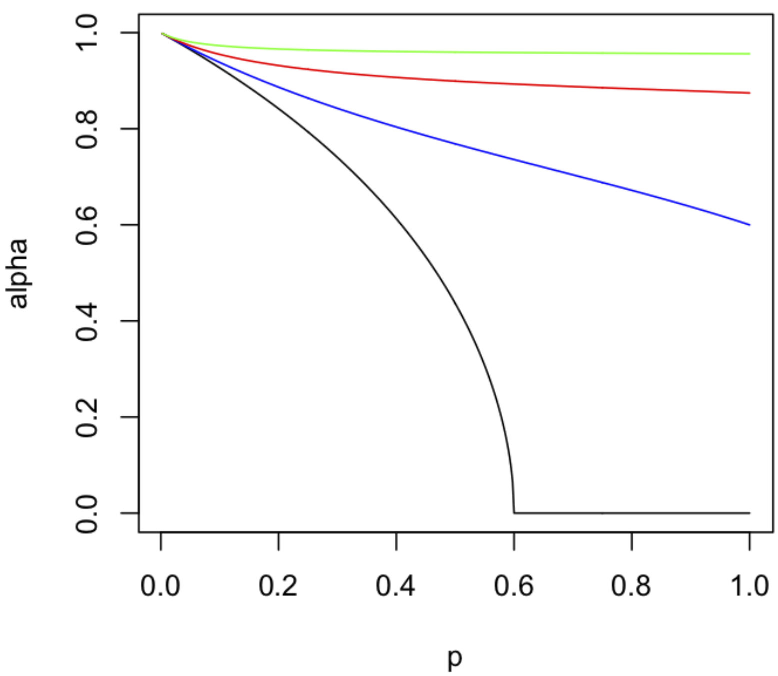

Figure 2: The functions $e_b(p)$ for $b = 0,1,2,3$, and a Poisson mean 2.5 offspring distribution.

4 Some limiting results

In the previous section we saw that we can obtain accurate numerical estimates on $e_b(p)$ in regions of the parameter space when p is bounded away from zero. In this section we look at the some limiting behaviours of this function, beginning with the easy case of $b \to \infty$.

Proposition 4.1. *There exists b_0, depending only on λ, and a constant $c_3(p,\lambda)$ such that for $b \geq b_0$,*

$$c_3(p,\lambda)\lambda^{-b} \leq 1 - e_b(p) \leq 2\lambda^{-b}. \tag{4.1}$$

Proof. By (1.5) and (3.10) we have, writing $c_2(p)^{-1} = -ep\log(1-p)$,

$$f_b(\alpha) \leq y_b(p,\alpha) \leq f_b(\alpha) + \lambda(1-\alpha)(\alpha\lambda)^b \sum_{n=0}^{\infty} w_n \leq f_b(\alpha) + \lambda(1-\alpha)(\alpha\lambda)^b c_2(p). \tag{4.2}$$

Set $\alpha_1 = \alpha_1(b) = 1 - 2\lambda^{-b}$; then bounding from below the sum of the left side of (4.2) by its final term,

$$y_b(\alpha_1, p) \geq \lambda^b(1-\alpha_1)\alpha_1^b = 2(1-2\lambda^{-b})^b > 2(1-2b\lambda^{-b}) > 1$$

provided $4b < \lambda^b$. Thus $(b,p,\alpha_1) \in \mathcal{S}$ survives, and so by monotonicity $(b,p,\alpha) \in \mathcal{S}$ for all $\alpha \in [0,\alpha_1]$, and $e_b(p) \geq \alpha_1$.

For the lower bound we can assume that b is large enough so that $\alpha_1\lambda - 1 > \frac{1}{2}(\lambda-1)$. Then we obtain from (4.2) that

$$y_b(p,\alpha) \leq \lambda(1-\alpha)\lambda^b\Big(2\lambda(\lambda-1)^{-1} + c_2(p)\Big),$$

and taking $\alpha = 1 - c_3\lambda^{-b}$, with c_3 small enough, the bound follows. ☐

We now look at $e_0(p)$ close to the point $p_0 = 1 - \lambda^{-1}$; for simplicity we restrict to the case when the offspring distribution has a finite second moment.

Proposition 4.2. *(a) If*

$$\lambda(1-p)(1-\alpha)\Big(1 + \alpha(1-p)G'(1-\alpha p)\Big) > 1 \tag{4.3}$$

then (Z_n^{CT}) survives.
(b) Assume that $\sum k^2 p_k < \infty$. Then

$$\lim_{p\uparrow p_0} \frac{e_0(p)}{(p_0-p)^{1/2}} = \frac{\lambda}{p_0(1-p_0)G''(1)}. \tag{4.4}$$

Proof. (a) Note that $h_1(s) = s^2\alpha G'(1-\alpha p)$. So taking only the first two terms in the sum in (1.5) we have

$$y_b(\alpha,p) \geq (1-\alpha)\lambda(1-p)\Big(1 + (1-p)\alpha G'(1-\alpha p)\Big),$$

and the result is then immediate from Theorem 2.5.
(b) Write $t = p_0 - p$ so that $\lambda(1-p) = 1+\lambda t$. Note that $G'(1-\alpha p) = \lambda - \alpha p G''(1) + o(\alpha^2)$. We have

$$\lambda(1-\alpha)\sum_{n=0}^{1} h_n(1-p) = (1+\lambda t)(1-\alpha)\Big(1+\alpha(1-p)G'(1-\alpha p)\Big)$$
$$= 1 + \lambda t - \alpha^2 - \alpha^2 p_0(1-p_0)G''(1) + O(\alpha t) + o(\alpha^2).$$

Further

$$\lambda(1-\alpha)h_2(1-p) = \lambda(1-\alpha)\alpha^2(1-p_0)^3 G'(1-\alpha p)G'(1-\alpha+g_1(1-p)\alpha)$$
$$= \alpha^2 + O(\alpha^3, \alpha t).$$

For $n \geq 3$ we have $h_n(1-p) \leq \alpha(1-p)\lambda h_{n-1}(1-p)$, and thus we have

$$y_0(p,\alpha) = 1 + \lambda(p_0 - p) - \alpha^2 p_0(1-p_0)G''(1) + o(\alpha^2) + O(\alpha t)$$

which gives the limit (4.4). □

Proposition 4.3. *Assume that $\sum k^2 p_k < \infty$. There exists c_1, c_2 such that*

$$e_0(p) \geq 1 - c_1 p \quad \textit{for } 0 < p < c_2.$$

Proof. Let $\alpha_0 \in (0,1)$ be such that $\lambda\alpha_0 > 1$. Then it is easy to verify that there exist c_3, c_4 such that for all $n \geq 1$ and $\alpha \in [\alpha_0, 1]$,

$$\mathbb{E}V_n^T \geq c_3 \lambda_T^n, \quad \mathbb{E}(V_n^T)^2 \leq c_4 \lambda_T^{2n}, \quad \mathbb{E}(Y_n^T) \leq c_4 \lambda_T^n.$$

Write $e^{-p_1} = (1-p)$, and choose c_2 so that $2p \geq p_1 \geq p$ for $p \in [0, c_2]$. Let $\alpha \in [\alpha_0, 1]$. Choose n so that $\lambda_T^{n-1} < p^{-1} \leq \lambda_T^n$. Then

$$w_0(p,\alpha) \geq \lambda_U \mathbb{E} V_T^n (1-p)^{Y_n^T}.$$

By the second moment method

$$\mathbb{P}(V_n^T \geq \tfrac{1}{2}\lambda_T^n) \geq \frac{(\mathbb{E}V_n^T)^2}{4\mathbb{E}(V_n^T)^2} \geq \frac{c_3^2}{4c_4}.$$

Let $r = c_3^2/4c_4$. Then

$$\mathbb{P}(Y_n^T > 2c_2\lambda_T^n / r) \leq \tfrac{1}{2} r,$$

and thus

$$\mathbb{P}(V_n^T \geq \tfrac{1}{2}\lambda_T^n, Y_n^T \leq 2c_2 \lambda_T^n / r) \geq \tfrac{1}{2} r.$$

Hence writing $s = \frac{1}{2}$, $t = 2c_2/r$,

$$\mathbb{E}V_T^n(1-p)^{Y_n^T} \geq \mathbb{E}\Big(V_T^n(1-p)^{Y_n^T}; V_n^T \geq s\lambda_T^n, Y_n^T \leq t\lambda_T^n\Big)$$
$$\geq \tfrac{1}{2} r s \lambda_T^n \exp(-p_1 t \lambda_T^n) \geq c p^{-1} e^{-c'\lambda}.$$

Hence $w_0(\alpha, p) \geq c_\lambda(1-\alpha)/p$, which gives that $e_0(p) > 1 - c_1 p$. □

5 Rate of growth of the process after contact tracing

In the case when α is not large enough to make (Z_n^{CT}) extinct, it is of interest to ask how quickly this process grows. We write

$$Z_n^{CT} = |A_n^{\mathrm{CG}}| = Z_n^U + Z_n^T;$$

here Z_n^U and Z_n^T are the number of points $x \in A_n^{\mathrm{CG}}$ with $\eta_T(x) = 0$ and $\eta_T(x) = 1$ respectively. Recall from (2.10) the definition of v_n, and (1.6) that of the Malthusian parameter θ of (v_n). We will be concerned with the case when (Z_n^{CT}) survives wpp, so $\theta > 0$.

Let (X_n) be the cluster seed process defined in the proof of Theorem 2.5. Let $\mathcal{C}_n$ be the set of cluster seeds at time n. Define a process R_n taking values in $\mathbb{Z}_+^{\mathbb{Z}_+}$ as follows. For a sequence $x \in \mathbb{Z}_+^{\mathbb{Z}_+}$ we write the ith component as $x(i)$. Set $R_0(0) = 1$, and $R_0(i) = 0$ for $i \geq 1$. Let $\mathcal{C}_n$ be the set of cluster seeds at generation n, so

$$\mathcal{C}_n = \{x \in A_n^{\mathrm{CG}} : \eta_T(x) = 0\}.$$

If x is a cluster seed and $|x| = n$ write $\widetilde{V}_k^U(x)$ for the number of cluster seeds in generation $n + k$ produced by the traceable cluster with seed x. Define

$$R_n(k) = \sum_{m=0}^{n-1} \sum_{x \in \mathcal{C}_m} \widetilde{V}_{n-m}^U(x).$$

We have

$$R_n(k) = R_{n-1}(k+1) + \sum_{x \in \mathcal{C}_{n-1}} \widetilde{V}_{k+1}^U.$$

The process R_n is a branching process, with infinitely many types. For each $k \geq 1$ an individual of type k produces with probability 1 one descendent of type $k-1$. An individual of type 0 produces a random number of offspring, with distribution $(\widetilde{V}_1^U, \dots)$. Let M_{ij} denote the mean number of offspring of a type i individual. Then

$$M_{0,k} = v_{k+1}, \quad n \geq 0, \tag{5.1}$$

$$M_{j,k} = \delta_{k,j-1}, \quad j \geq 1,\ k \geq 0. \tag{5.2}$$

It is straightforward to verify that if $u(n) = e^{-n\theta}$, for $n \geq 0$ then u is a right eigenvector for M and $Mu = e^{\theta}u$. Thus we have

$$\mathbb{E}(R_{n+1}(j)|R_n) = \sum_{k=0}^{\infty} R_n(k) M_{kj},$$

and

$$Y_n = e^{-\theta n} \sum_{k=0}^{\infty} R_n(k) u(k)$$

is a martingale with respect to $\mathcal{F}_n^R = \sigma(R_0, R_1, \dots, R_n)$. The following Lemma follows easily from Markov's inequality and the convergence of Y.

Lemma 5.1. *(a) We have*

$$\mathbb{P}(R_n(0)e^{-n\theta} > t) \le t^{-1}. \tag{5.3}$$

(b) With probability 1,

$$\limsup_{n\to\infty} \frac{\ln R_n(0)}{n} \le \theta. \tag{5.4}$$

Let (X_n) be the cluster seed process, and let F be the event that (X_n) survives; set $\mathbb{P}(F) = p_F > 0$.

Lemma 5.2. *On the event F, a.s.*

$$\liminf_{n\to\infty} \frac{\ln R_n(0)}{n} \ge \theta. \tag{5.5}$$

Proof. Let $\varepsilon > 0$. Let $N \ge 1$ and set

$$v_n^{(N)} = 1_{(n<N)}\mathbb{E}(\widetilde{V}_n^U \wedge N),$$

and let θ_N be the Malthusian parameter for $(v_n^{(N)})$. As v_n are bounded, we have for any $r > 0$ that

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} e^{-rn} v_n^{(N)} = \sum_{n=1}^{\infty} e^{-rn} v_n,$$

and it follows that we can choose N large enough so that $\theta - \varepsilon < \theta_N \le \theta$.

Let $R^{(N)}$ be the process R, but with the offspring distribution for a type 0 individual given by $(N \wedge \widetilde{V}_1^U, \dots, N \wedge \widetilde{V}_N^U, 0, \dots)$. We can do the truncation a.s. on the probability space, so that $R_n(k) \ge R_n^{(N)}(k)$ for all $n \ge 0$, $k \ge 0$. As all individuals for $R^{(N)}$ are of type $0, 1, \dots, N-1$, we can regard $R^{(N)}$ as a multi-type branching process with finitely many types. Let $M^{(N)}$ be the matrix of means of the truncated process $R^{(N)}$, and $u^{(N)}(k) = e^{-\theta_N k}$ for $0 \le k \le N-1$. Then $M^{(N)}u^{(N)} = e^{\theta_N}u^{(N)}$. A classic limit theorem (see [1, p. 192]) implies that $e^{-k\theta_N}R_k^{(N)}$ converges a.s. and in L^2 to $u_L W_N$, where u_L is the left eigenvector of $M^{(N)}$ with eigenvalue $e^{-\theta_N}$, and $\mathbb{P}(W_N > 0) > 0$. Thus a.s. on $\{W_N > 0\}$,

$$\lim_{n\to\infty} \frac{\ln R_n^{(N)}(0)}{n} \ge \theta_N > \theta - \varepsilon.$$

Let

$$G_\varepsilon = \Big\{ \lim_{n\to\infty} \frac{\ln R_n(0)}{n} \ge \theta - \varepsilon \Big\}. \tag{5.6}$$

By the above, we have $\mathbb{P}(G_\varepsilon) = q > 0$.

Let $m \ge 1$, and $T_m = \min\{k \ge 0 : X_k \ge m\}$. We have $\mathbb{P}(T_m < \infty | F) = 1$. Write $x_i, 1 \le i \le m$ for the first m cluster seeds in generation T_m, R^i for the associated multitype branching process, and $G_\varepsilon(i)$ for the event defined by (5.6) for R^i. Then $\mathbb{P}(\cup G_\varepsilon(i) | T_m < \infty) = 1 - q^m$. On the event $\cup G_\varepsilon(i)$ we have $\lim_{n\to\infty} \frac{\ln R_n(0)}{n} \ge \theta - \varepsilon$, which implies that $\mathbb{P}(G_\varepsilon) \ge \mathbb{P}(F) - q^m$.

Consequently, for any $\varepsilon > 0$ we have $\mathbb{P}(G_\varepsilon) \ge \mathbb{P}(F)$, and (5.5) follows. □

Remark 5.3. Suppose in addition that $\sum k^2 p_k < \infty$. Let

$$u_R(k) = e^-\theta \sum_{k=n}^{\infty} e^{-(k-n)\theta} v_n.$$

Then u_R is a left eigenvector for M with eigenvalue e^θ, and $\sum_k u_R(k) u_L(k) < \infty$. By Criteriion III of [12] the matrix M satisfies Case II of [9], and it follows that

$$\lim_n e^{-\theta n} R_n \to v_R W \text{ in } L^2,$$

where W is a random variable with $\mathbb{E}(W) = 1$ and $\mathbb{P}(W > 0) = p_F$.

Theorem 5.4. *Let (X_n) be the cluster seed process, and let F be the event that (X_n) survives. On F we have, a.s.*

$$\lim_{n\to\infty} \frac{\ln Z_n^{CT}}{n} = \theta. \tag{5.7}$$

Proof. We recall the construction of A_n^* from Section 2. We consider first the case $b = 0$. Let $M_n = |A_n^* \cap \Lambda_n|$. Note that

$$\begin{aligned} R_n(0) &= \{x \in A_n^* \cap \Lambda_n : \eta_T(x) = 0\}, \\ Z_n^U &= \{x \in A_n^* \cap \Lambda_n : \eta_T(x) = \eta_D(x) 0\}, \end{aligned}$$

and that $Z_n^{CT} \le M_n$.

Conditional on M_n we have that $R_n \sim \texttt{Bin}(M_n, 1-\alpha)$. Hence (see [7]),

$$\mathbb{P}(|R_n - (1-\alpha)M_n| > t | M_n) \le 2\exp(-t^2/3M_n).$$

Let $\varepsilon > 0$, and set $m_n = e^{n(\theta+\varepsilon)}$, $a = 2/(1-\alpha)$. Then if $n \ge c_0(\alpha, p, \varepsilon)$,

$$\begin{aligned} \mathbb{P}(M_n \ge 2(1-\alpha)^{-1} m_n) &\le \mathbb{P}(M_n \ge 2(1-\alpha)^{-1} m_n), R_n < m_n) + \mathbb{P}(R_n \ge m_n) \\ &\le \mathbb{P}(|(1-\alpha)M_n - R_n| > m_n, M_n \ge 2(1-\alpha)^{-1} m_n) + e^{-\varepsilon n} \\ &\le \exp(-m_n^2/6(1-\alpha)^{-1} m_n) + e^{-\varepsilon n} \le 2e^{-\varepsilon n}. \end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} \frac{\ln M_n}{n} \le \theta, \quad a.s.,$$

and the upper bound for $\ln Z_n^{CT}/n$ follows immediately.

As $M_n \ge R_n(0)$, it is immediate that on F

$$\liminf_{n\to\infty} \frac{\ln M_n}{n} \ge \theta, \quad a.s.$$

The estimate

$$\mathbb{P}(|Z_n^U - (1-\alpha)(1-p)M_n| > t | M_n) \le 2\exp(-t^2/3M_n)$$

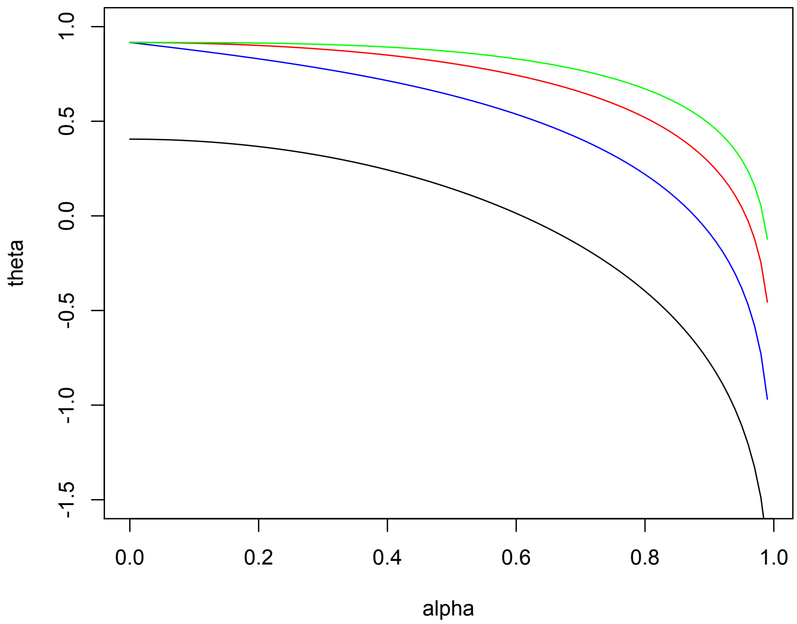

Figure 3: The Malthusian function θ as a function of α, for a Poisson mean 2.5 offspring distribution, and $p = 0.4$

and a similar argument to that above gives that

$$\liminf_{n\to\infty} \frac{\ln M_n}{n} \geq \theta, \qquad \text{a.s. on } F,$$

and as $Z_n^{CT} \geq Z_n^U$, the lower bound in (5.7) follows. □

Question. Let (p_k) and (p'_k) be offspring distributions, and $\mathcal{E}$ and $\mathcal{E}'$ be the corresponding parameter sets for extinction. Suppose that (p'_k) stochastically dominates (p_k). It follows that the branching process Z' stochastically dominates Z. Is it the case that $\mathcal{E}' \subset \mathcal{E}$?

Acknowledgment. This work was stimulated by a seminar in the B.C. COVID-19 group by Nils Bruin on contact tracing – see [4] for the related paper.

References

[1] K.B. Athreya, P. Ney. *Branching processes.* Springer. 1972.

[2] F. G. Ball, E S. Knock, P. D. O'Neill. Threshold behaviour of emerging epidemics featuring contact tracing. *Adv. Applied Prob.* **43**, No. 4, (2011) pp. 1048–1065.

[3] F. G. Ball, E S. Knock, P. D. O'Neill. Stochastic epidemic models featuring contact tracing with delays. *Math. Biosciences* **266** 23–35 (2015)

[4] N. Bruin, C. Colijn, A. van der Waall. Modelling the effects of epidemiological contact tracing, (in preparation).

[5] G.R. Grimmett. *Percolation.* (2nd edition). Springer 1999.

[6] T.E. Harris. Branching Processes. *Ann. Math. Statist.* **19**, Number 4 (1948), 474–494.

[7] S. Janson Large deviation inequalities for sums of indicator variables. arXiv:1609.00533. (2016)

[8] D. Klinkenberg, C. Fraser, H. Heesterbeek. The Effectiveness of Contact Tracing in Emerging Epidemics. PLoS One. 2006; 1(1): e12. doi: 10.1371/journal.pone.0000012

[9] S.-T. C. Moy. Extensions of a limit theorem of Everett, Ulam and Harris on multitype branching processes to a branching process with countably many types. *Ann. Math. Statist.* **38**, Number 4 (1967), 992–999.

[10] J. Müller, M. Kretzschmar, K. Dietz. Contact tracing in stochastic and deterministic epidemic models. *Math. Biosciences* **164**, Issue 1, March 2000, Pages 39–64.

[11] J. Müller, B. Koopmanna. The effect of delay on contact tracing. *Math. Biosciences* **282** (2016) 204–214.

[12] D. Vere-Jones. Ergodic properties of nonnegative matrices. I. *Pacific J. Math.* **22**, (1967), 361–386.

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