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Some Boundary Harnack Principles With Uniform Constants

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Abstract We prove two versions of a boundary Harnack principle in which the constants do not depend on the domain by using probabilistic methods.

Keywords Boundary Harnack principle · Harmonic functions · Brownian motion · Harnack inequality

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1 Intoduction

A boundary Harnack principle (BHP) gives a result of the following general type. Let D be a domain in $\mathbb{R}^d$, and $\xi \in \partial D$, satisfying suitable properties. Let $r > 0$, $a_0 \geq 2$, $B_1 = B(\xi, a_0 r)$ and $B_2 = B(\xi, r)$; here $B(.,.)$ denote the usual Euclidean balls. Then there exists a constant C_D such that if u, v are positive harmonic functions on $B_1 \cap D$ vanishing on $\partial D \cap B_1$, one has

$$\frac{u(x)/v(x)}{u(y)/v(y)} \leq C_D \text{ for } x, y \in D \cap B_2. \quad (1)$$

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A BHP of this kind is called in [1] a *uniform* BHP, and in [14] a *scale invariant* BHP. Here 'uniform' or 'scale invariant' refers to the fact that the constant C_D does not depend on r. For Lipschitz domains D the scale invariant BHP was proved independently by Ancona, Dahlberg and Wu in [4,10,19]. This was extended to NTA domains by Jerison and Kenig [12]. Bass, Burdzy and Banuelos [7,8] used probabilistic methods to obtain a BHP for Hölder domains, but their BHP is not uniform. In [1] a scale invariant BHP is proved for uniform domains in $\mathbb{R}^d$, and in [3] this is extended to John domains. See the papers [1,2,14] for a further discussion on the history of the BHP, and the various different kinds of BHP.

In the above 'harmonic function' refers to functions which are harmonic with respect to the usual Laplacian operator in $\mathbb{R}^d$. (These functions are harmonic with respect to the infinitesimal generator of the semigroup of standard Brownian motion in $\mathbb{R}^d$.) Recent papers have studied functions which are harmonic with respect to the generators of more general diffusion processes – see [14,15] and the references therein.

In all these results the constant C_D depends on the domain D. For the standard Laplacian it is clear that such dependence is necessary, since the BHP does not hold for all domains $D \subset \mathbb{R}^d$. (See however [9] where a BHP with constants independent of the domain is proved for harmonic functions with respect to fractional Laplacians.)

This paper originates in the work of Masson [16], where a boundary estimate with a constant C_D *not depending on* D was needed – see [16, Proposition 3.5]. Masson's work was in the context of discrete potential theory for $\mathbb{Z}^2$. Let $S^x = (S^x_k, k \geq 0)$ be the simple random walk on $\mathbb{Z}^2$, started at x, and write $S = S^0$. Write $\mathbb{Z}^2_- = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 \leq 0\}$, and let $Q(x, n) = \{y \in \mathbb{Z}^2 : |x - y| \leq n\}$. Let $N \geq 1$, $K \subset Q(0, N) \cap \mathbb{Z}^2_-$, and $D = Q(0, N) - K$. (The case of interest is when $0 \in K$.) Let $\tau^+ = \min\{k \geq 1 : S_k \notin D\}$, and $F = \{S_{\tau^+} \in \mathbb{Z}^2 - K\}$, so that F is the event that S leaves $Q(0, N)$ before hitting K. Let $W = \{(x_1, x_2) : 0 \leq |x_2| \leq x_1\}$, so that W is a cone with vertex $(0, 0)$ and angle $\pi/4$. Massons's theorem is that there exists $p_0 > 0$, independent of N and K, such that $\mathbb{P}(S_{\tau^+} \in W|F) \geq p_0$. This result extends to give also

$$\mathbb{P}(S^x_{\tau^+} \in W|F) \geq p_1 \text{ for } x = (x_1, x_2) \in Q(0, N/16) \text{ with } x_1 \geq 0 \ . \qquad (2)$$

The fact that the constants p_i do not depend on the structure of K is essential in the context of [16], since K is a random path (actually a loop erased random walk), and the estimate (2) was needed for all possible K.

Although the connection with BHP is not made in [16], this result is clearly of BHP type. For $x \in \mathbb{Z}^2$ let $\tau = \min\{k \geq 0 : S_k \notin D\}$, and define the functions

$$v(x) = \mathbb{P}(S^x_\tau \in K^c), \quad u(x) = \mathbb{P}(S^x_\tau \in K^c \cap W).$$

These are (discrete) harmonic in D, and $\mathbb{P}(S^x_\tau \in W|F) = u(x)/v(x)$. Since $u \leq v$ it is immediate from (2) that

$$\frac{u(x)/v(x)}{u(y)/v(y)} \leq p_1^{-1}, \text{ for } x, y \in Q(0, N/16) \cap (\mathbb{Z}^2 - \mathbb{Z}^2_-). \qquad (3)$$

Thus we have a BHP for the specific functions u, v in which the constant $C_D = p_1^{-1}$ does not depend on K; the price is that the inequality only holds for those $x \in Q(0, N/16)$ with $x_1 > 0$.

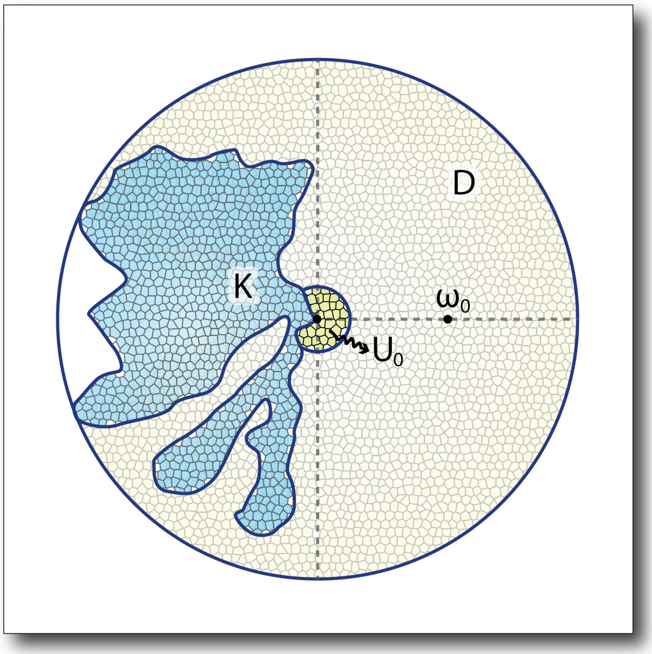

Fig. 1: The sets K and U_0.

Our first result is a BHP in two dimensions which holds with a constant independent of the domain. We write $B(x,r) = \{y : |x-y| < r\}$ for Euclidean balls with center at x and radius r, and set

$$\mathbb{H}_- = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 < 0\},$$

for the open left-halfspace, and define the open right-halfspace $\mathbb{H}_+$ analogously.

Theorem 1 *Let $d = 2$, let $K \subset B(0,1) \cap \overline{\mathbb{H}_-}$ be connected and relatively closed in $B(0,1)$, and let $D = B(0,1) - K$. Let U_0 be the connected component of $B(0, \frac{1}{16}) \cap D$ which contains $(\frac{1}{32}, 0)$. There is a positive constant C_0, independent of K, such that if u and v are positive and harmonic on D, are continuous on $\overline{D}$, and vanish on K then*

$$\frac{u(x)/v(x)}{u(y)/v(y)} < C_0 \quad \textit{for } x, y \in U_0. \tag{4}$$

A key estimate for the proof is the Carleson estimate Lemma 2, which is proved by a path-crossing argument. This estimate relies on the fact that K is connected, and does not generalize to $d \geq 3$. Examples at the end of Section 2 show that Theorem 1 does not hold in general if $d \geq 3$, or if K is not connected, and that the inequality (4) cannot be extended to $x, y \in B(0, \frac{1}{16}) \cap D$.

In Section 3 we extend Masson's result to Euclidean space, and prove it for $d \geq 2$. (The result is trivial for $d = 1$).

Throughout this paper we write $X = (X_t, t \in [0, \infty), \mathbb{P}^x, x \in \mathbb{R}^d)$ for Brownian motion in $\mathbb{R}^d$; $\mathbb{P}^x$ is the law of X started at x. For a set $A \subset \mathbb{R}^d$ we define

$$T_A = \inf\{t \geq 0 : X_t \in A\}, \quad \tau_A = T_{A^c} = \inf\{t \geq 0 : X_t \notin A\}.$$

We will use the classic and probabilistic definitions of harmonic functions interchangeably. We call a function h *harmonic* in a domain A if h is locally integrable and for all $x \in A$ and all $r < \text{dist}(x, \partial A)$,

$$h(x) = \frac{1}{|B(0,r)|} \int_{B(x,r)} h(y)dy.$$

Equivalently, h is harmonic in A if $h(X_{t \wedge \tau_A})$ is a martingale.

When we use notation such as $C = C(\alpha, d)$ this will mean that the (positive) constant C depends only on the parameters α and d.

2 BHP for positive harmonic functions in $d = 2$

In this section we prove Theorem 1. We begin with a general lemma concerning harmonic functions in a bounded domain in $\mathbb{R}^d$.

Lemma 1 *Let D be a bounded path connected domain in $\mathbb{R}^d$, with $d \geq 1$, and f be a non-negative harmonic function in D which is continuous on $\overline{D}$. Let $c_0 > 0$. If there exists $x_0 \in D$ such that $f(x_0) > c_0$ then the set*

$$\partial D_0 := \{x \in \partial D : f(x) > c_0\} \neq \emptyset.$$

Moreover, there is a path γ from x_0 to a point $x^* \in \partial D_0$ such that $\gamma \setminus \{x^*\} \subseteq D$ and $f(x) > c_0$ for all $x \in \gamma$.

Proof Since f is a non-negative harmonic function in D and it is continuous on $\overline{D}$, by the Maximum Principle for harmonic functions, there exists a point $x_1 \in \partial D$ such that $f(x_1) \geq f(x_0) > c_0$. Hence $\partial D_0 \neq \emptyset$. Define Γ to be the collection of all paths γ from x_0 to a point in ∂D_0 so that $\gamma \subseteq D$ and $f(y) > c_0$ for all $y \in \gamma$. We will show that $\Gamma \neq \emptyset$. Denote Brownian motion starting at the point x_0 by $(X_t, \mathbb{P}^{x_0})$, and stopping times

$$T_0 = \inf\{t \geq 0 : f(X_t) \leq c_0\} \quad \text{and} \quad T = T_0 \wedge \tau_D$$

where τ_D is the first exit time of X_t from the domain D. Then

$$c_0 < f(x_0) = \mathbb{E}^{x_0}(f(X_T)) = c_0 \, \mathbb{P}^{x_0}(T_0 \leq \tau_D) + \mathbb{E}^{x_0}(f(X_{\tau_D}); \tau_D < T_0).$$

If $\mathbb{P}^{x_0}(T_0 \leq \tau_D) = 1$ then $f(x_0) = c_0$ which contradicts the assumption. Hence $\mathbb{P}^{x_0}(T_0 \leq \tau_D) < 1$ and so $\mathbb{P}^{x_0}(T_0 > \tau_D) > 0$. Notice that on the event $\{T_0 > \tau_D\}$ the path of the Brownian motion is in Γ; since this event has non-zero probability, $\Gamma \neq \emptyset$. □

Now let $K \subset B(0,1)$ satisfy the hypotheses of the Theorem. First, we will create a domain enclosing the region U_0 inside the domain D. For this purpose, we let $r_1 = 1/4$ and $r_2 = 3/4$. We begin by assuming that

$$K \cap \partial B(0, r_j) \neq \emptyset$$

for $j = 1, 2$. (See Fig. 2.) We also write $r_0 = 1/16$; this is the radius of the smaller balls used in Carleson estimate below.

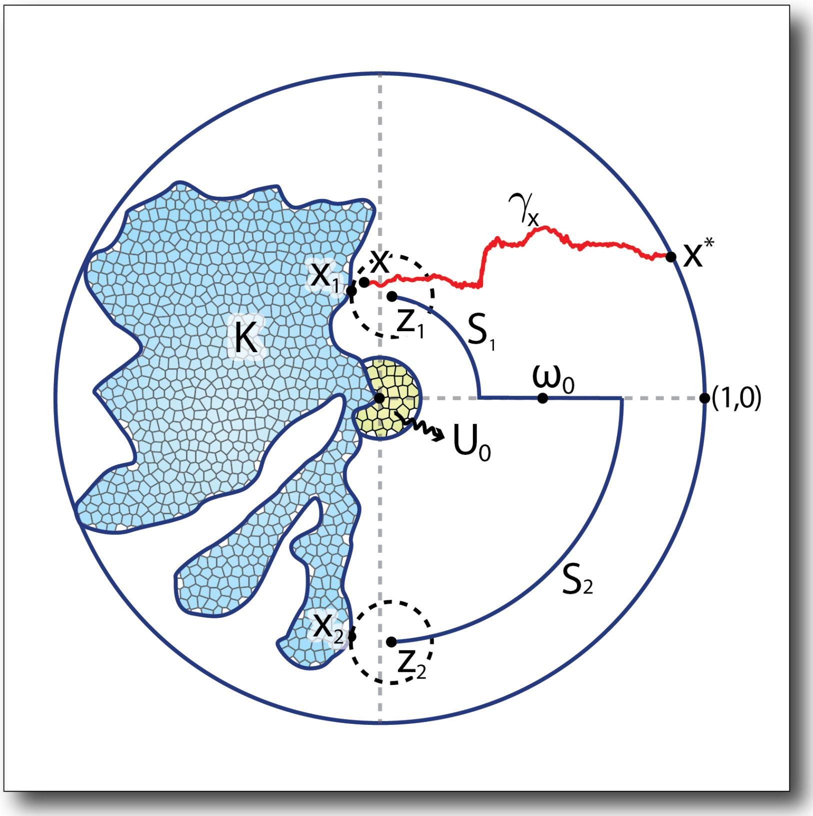

Fig. 2: The points x_j, z_j and curves S_j

We write $w_0 = (\frac{1}{2}, 0)$ and define two hitting angles θ_1 and θ_1 as follows:

$$\theta_1 = \inf\{\theta \in (0, 2\pi) : \overline{B(r_1 e^{i\theta}, r_0)} \cap K \neq \emptyset\},$$
$$\theta_2 = \sup\{\theta \in (-2\pi, 0) : \overline{B(r_2 e^{i\theta}, r_0)} \cap K \neq \emptyset\}.$$

Write $z_j = r_j e^{i\theta_j}$ and let $x_j \in K \cap \overline{B(z_j, r_0)}$ for $j = 1, 2$. Note that the balls $B(x_j, r_0)$, $j = 1, 2$ are disjoint and the distance between the points x_j and z_j is r_0, Let x_j' be the midpoint on the line segment $[x_j, z_j]$, L_j be the the line segment $[x'_j, z_j]$, and x''_j be the midpoint of L_j. We define the annulus $A_j = B(x_j, r_0) - \overline{B(x_j, \frac{r_0}{2})}$. Let U_j be the connected component of $B(x_j, \frac{r_0}{2}) - K$ which contains the open line segment between x_j and x'_j. As U_j is an open connected subset of $\mathbb{R}^2$ it is also path-connected.

We begin by proving two Lemmas; the first ensures existence of certain paths in the domain, while the second is a rooted local Carleson estimate. Recall from the statement of Theorem 1 that U_0 is the connected component of $B(0, \frac{1}{16}) \cap D$ which contains $(\frac{1}{32}, 0)$.

Lemma 2 (Carleson estimate) *Let K, D, U_1, U_2 be as above. Let $z \in U_0$ and u be as in Theorem 1. There exists a constant C, independent of K, z and u, such that*

$$u(y) \le Cu(w_0) \quad \textit{and} \quad G_D(z, y) \le C\, G_D(z, w_0), \qquad y \in U_j.$$

Proof We will prove this for U_1; the same argument also applies to U_2.

The set $\partial B(z_1, r_0) \cap A_1$ consists of the union of two connected arcs; denote these γ_2 and γ_3, labelled so that going anticlockwise round A_1 we meet the arcs L_1, γ_2, γ_3 in order. (See Fig. 3.)

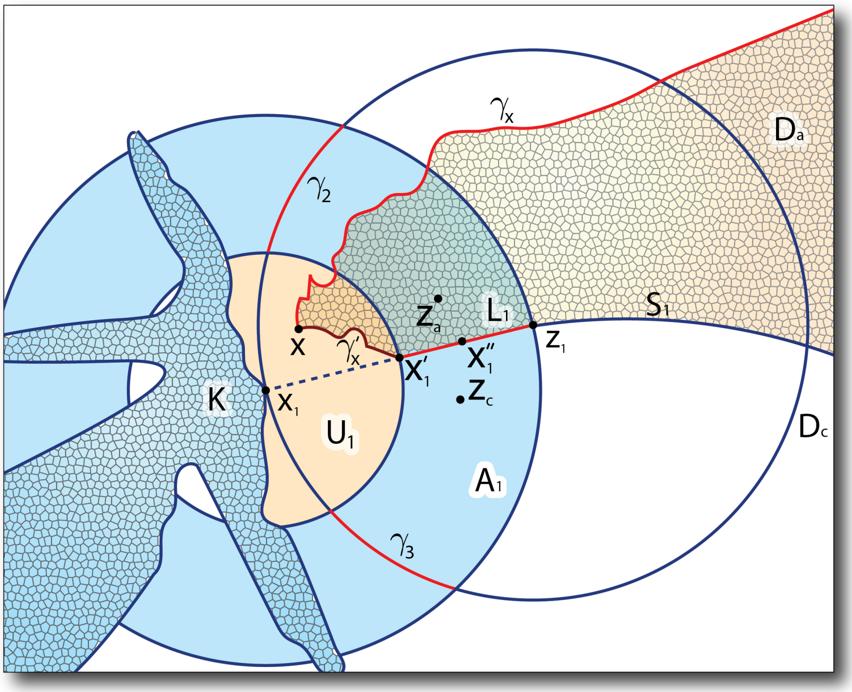

Fig. 3: We zoom into the region around the points x_1 and z_1. A similar region exists around x_2 and z_2.

Let H_2 be the event that a Brownian motion X, started on the line L_1 stays inside A_1 until after it has hit in order the sets γ_2, γ_3, and then L_1.

More precisely if we set

$$T_{23} = \inf\{t \geq 0 : X_t \in \gamma_2 \cup \gamma_3\},$$
$$T_{31} = \inf\{t \geq T_{23} : X_t \in \gamma_3 \cup L_1\},$$
$$T_{12} = \inf\{t \geq T_{31} : X_t \in L_1 \cup \gamma_2\},$$

then

$$H_2 = \{T_{23} < T_{31} < T_{12} < \tau_{A_1}, X_{T_{23}} \in \gamma_2, X_{T_{31}} \in \gamma_3, X_{T_{12}} \in L_1\}.$$

Let H_3 be the similar event with the roles of γ_2 and γ_3 interchanged. As the sets L_1, γ_2, γ_3 are separated by a distance cr_0, and using the symmetry of the set, there exists $p_1 > 0$ such that

$$\mathbb{P}^{x_1''}(H_2) = \mathbb{P}^{x_1''}(H_3) = p_1.$$

By the Harnack inequality there exists a constant C_2 such that if h is non-negative and harmonic in $B(z_1, r_0)$ then

$$h(y) \leq C_2\, h(x_1'') \text{ for all } y \in B(x_1'', \frac{r_0}{3}). \tag{5}$$

Now let $C_1 = \max\{2/p_1, C_2\}$. We consider first the case when $f = u$. It is enough to prove

$$f(x) \leq C_1 f(x_1''), \qquad \text{for } x \in U_1, \tag{6}$$

since then by using the Harnack inequality in a chain of balls on the arc $\{r_1 e^{i\theta}, 0 \leq \theta \leq \theta_1\}$ and the line $\{(t, 0), r_1 \leq t \leq \frac{1}{2}\}$ we have $f(x_1'') \leq cf(w_0)$.

If $f(x) \leq C_1 f(x_1'')$ for all $x \in U_1$ then we are done. So suppose there exists $x \in U_1$ with $f(x) > C_1 f(x_1'')$. As f is harmonic and non-negative in D, by Lemma 1, there exists a path γ_x from x to a point $x^* \in \partial B(0, 1)$ such that $f(y) > C_1 f(x_1'')$ for all $y \in \gamma_x$.

Suppose that there exists $y \in \gamma_x \cap B(x_1'', \frac{r_0}{3})$. Then using (5) we have $f(x_1'') \geq C_2^{-1} f(y) > C_2^{-1} C_1 f(x_1'') > f(x_1'')$, a contradiction. So we have that $\gamma_x \cap B(x_1'', \frac{r_0}{3}) = \emptyset$.

We now define a path γ in $\overline{D}$ between x^* and $(1, 0)$ as follows, which includes the points $x^*, x, x_1', z_1, (r_1, 0), (1, 0)$. By the definition of U_1, there is a path γ_x' in $B(x_1, \frac{r_0}{2}) - K$ connecting x and x_1'. Let S_1 be the path $(r_1 e^{i\theta}, 0 \leq \theta \leq \theta_1)$ which connects $(r_1, 0)$ and z_1, and S_2 be the line segment between $(r_1, 0)$ and $(1, 0)$. (See Fig. 2 and Fig. 3.) Then γ consists of the concatenation (with appropriate orientations) of $\gamma_x, \gamma_x', L_1, S_1, S_2$. (The path γ is not necessarily a simple curve – it may have multiple points.) We also write γ for the set of points in this path. By the construction of γ we have that $\gamma \cap K = \emptyset$. Let z_a and z_c be two points close to x_1'' on opposite side of the line segment L_1, in the anticlockwise and clockwise directions respectively, and let D_a and D_c be the connected components of $B(0, 1) - \gamma$ which contain z_a and z_c respectively. Since the path γ inside $B(x_1'', r_0/3) \cap A_1$ just consists of the line segment L_1 without its endpoints, the components D_a and D_c are distinct. (See Remark 1 below for more details.) Hence, as $K \cap \gamma = \emptyset$ and K is connected, at most one of $K \cap D_a$, $K \cap D_c$ is non-empty.

We suppose that $K \cap D_a = \emptyset$. By the construction of γ, if the event H_2 holds then the process X hits γ_x before it exits D. So we have

$$f(x_1'') \geq \mathbb{E}^{x_1''}(f(X_{T_{\gamma_x} \wedge \tau_D}); H_2) \geq p_1 \inf_{y \in \gamma_x} f(y) \geq p_1 C_1 f(x_1'') \geq 2f(x_1''),$$

a contradiction. (If $K \cap D_c = \emptyset$, we use the event H_3.)

For the second part, let us first fix any point $x_0 \in U_0$ and prove the argument for $f = G_D(x_0, \cdot)$. The argument for the case $f = G_D(x_0, \cdot)$ is similar; the main difference is in the definition of the path γ. In this case f is harmonic in $D - \{x_0\}$ (see [17, Theorem 3.35]), and so the path γ_x on which we have $f(y) > C_1 f(x_1'')$ goes from x to x_0. As $x_0 \in U_0$, there exists a path $\gamma_0 \in D \cap B(0, \frac{1}{16})$ from x_0 to $(\frac{1}{32}, 0)$. Let S_3 be the line segment between $(\frac{1}{32}, 0)$ and $(r_1, 0)$. We then obtain a loop γ in D which contains the points $x, x_0, (\frac{1}{32}, 0), (r_1, 0), z_1, x_1', x$ by concatenating $\gamma_x, \gamma_0, S_3, S_1, L_1, \gamma_x'$. The construction of γ gives that $\gamma \cap K = \emptyset$, and the remainder of the argument is the same as for the case $f = u$. □

Remark 1 One can prove formally that the domains D_a and D_c are distinct using winding numbers. We just give a sketch for the case of the function f. Let γ be the path between x^* and $(1,0)$ constructed in the Lemma above, and γ_1 be a path in $\overline{B}(0,1)^c$ connecting x^* and $(1,0)$. Let γ_0 be the closed path obtained by combining γ and γ_0.

Consider the contour integrals $\oint_{\gamma_0} dz/(z - z_b)$ for $b = a, c$. Let $\delta > 0$ and assume that $|z_a - x_1''| = |z_c - x_1''| = \delta$. As z_a and z_c are on opposite sides of L_1, if δ is small enough, then

$$\Big| \int_{L_1} \frac{dz}{(z - z_a)} - \int_{L_1} \frac{dz}{(z - z_c)} \Big| \geq 1.$$

The path $\gamma' = \gamma_0 \setminus L_1$ is at least a distance $r_0/3$ from x_1'', and thus if δ is small enough the integrals $\int_{\gamma'} dz/(z - z_b)$ for $b = a, c$ will differ by less than $c\delta$. It follows that z_a and z_c are in different components of $\mathbb{R}^2 \setminus \gamma_0$, and are therefore also in different components of $B(0,1) \setminus \gamma$.

The Lemma above controls the functions u and $G_D(x, \cdot)$ (for $x \in U_0$) in the set U_j, $j = 1, 2$. We are ready to prove the main result of this section.

Proof of Theorem 1. Let u, v satisfy the hypotheses of the theorem. Suppose first that $K \cap \partial B(0, \frac{1}{4})$ and $K \cap \partial B(0, \frac{3}{4})$ are non-empty. Assume x_i, z_i, θ_i for $i = 1, 2$ and w_0 are as in Lemma 2.

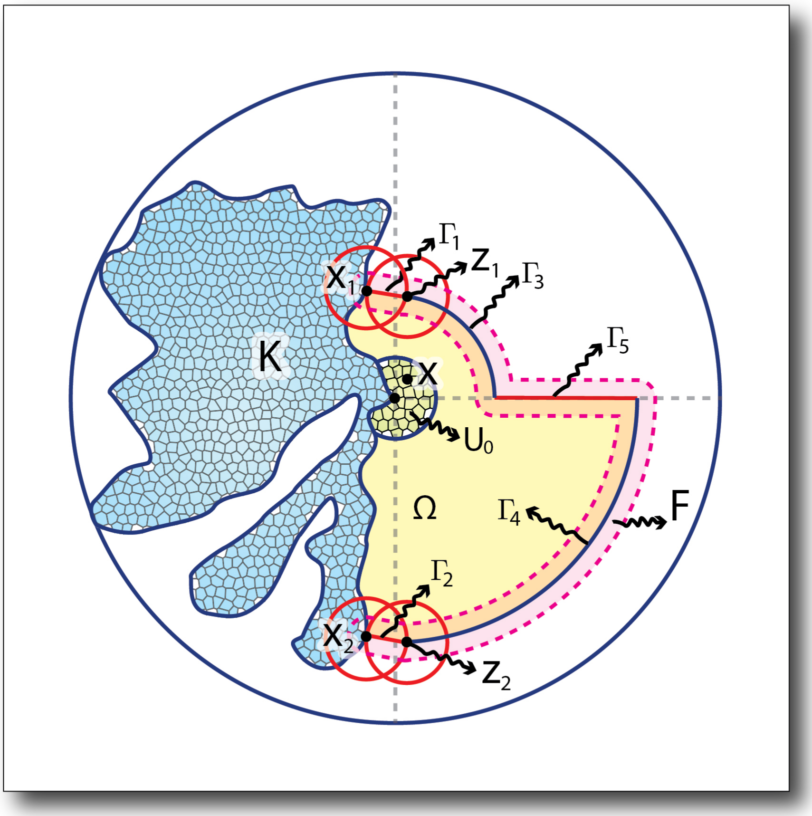

Fig. 4: The sets Ω and F

First we define the following paths which enclose the region we will work on:

$$
\begin{aligned}
\Gamma_1 &:= \{t\, x_1 + (1-t)\, z_1 : t \in (0,1)\},\\
\Gamma_2 &:= \{t\, x_2 + (1-t)\, z_2 : t \in (0,1)\},\\
\Gamma_3 &:= \{r_1 e^{i\theta} : 0 \le \theta \le \theta_1\},\\
\Gamma_4 &:= \{r_2 e^{i\theta} : \theta_2 \le \theta \le 0\},\\
\Gamma_5 &= \{(t,0), r_1 \le t \le r_2\}.
\end{aligned}
$$

We write Γ for the union of $\Gamma_1, \ldots \Gamma_5$. Let Ω be the connected domain enclosed by the curves Γ_j, $j = 1, \ldots 5$ and the set K and including the point $w_1 = (\frac{1}{8}, 0)$. Let $x \in U_0 \subset \Omega$. Since u is harmonic in D and zero on K,

$$
u(x) = \int_{\partial\Omega} u(y)\mathbb{P}^x(X_{\tau_\Omega} \in dy) = \int_{\Gamma} u(y)\mathbb{P}^x(X_{\tau_\Omega} \in dy).
$$

By the Carleson estimate Lemma 2, u is bounded above by $c_1\, u(w_0)$ on $\Gamma_1 \cup \Gamma_2$. By using a Harnack chain as in the proof of Lemma 2 and the regular Harnack inequality, we also obtain that $u(y) \leq c_2 u(w_0)$ for $y \in \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$. Hence, by setting $c_3 = \max\{c_1, c_2\}$,

$$u(x) \leq c_3\, u(w_0)\, \mathbb{P}^x(X_{\tau_\Omega} \in \Gamma). \tag{7}$$

Next, we define a tube F around Γ by

$$F = \bigcup_{y \in \Gamma} B(y, \frac{r_0}{4}).$$

By Urysohn's Lemma, there exists a smooth function ψ with compact support in F such that $\psi = 1$ on Γ. We can choose ψ so that $|\Delta\psi| \leq C_1$.
For $x \in U_0$ we have

$$\int_{\partial\Omega} \psi(y)\, \mathbb{P}^x(X_{\tau_\Omega} \in dy) = \psi(x) + \int_{\Omega \cap \mathrm{supp}(\psi)} \Delta\psi(y) G_\Omega(x, y)\, dy.$$

(The functions on each side are harmonic in Ω and have the same boundary conditions – see [2, Lemma 1] for details.) This equation yields

$$\mathbb{P}^x(X_{\tau_\Omega} \in \Gamma) \leq \int_{\Omega \cap F} |\Delta\psi(y)| G_D(x, y)\, dy \leq C_2 \int_{\Omega \cap F} G_D(x, y)\, dy. \tag{8}$$

Now fix $x_0 \in U_0$. Thus $G_D(x_0, \cdot)$ is a positive harmonic function in the domain $D - U_0$, which contains $\Omega \cap F$. We have

$$\partial(\Omega \cap F) = [\partial(\Omega \cap F) \cap K] \cup \Gamma \cup [\Omega \cap \partial F]. \tag{9}$$

We now claim that

$$G_D(x_0, y) \leq c_4\, G_D(x_0, w_0) \text{ for } y \in \partial(\Omega \cap F). \tag{10}$$

We consider in turn the three parts of the boundary given by (9). If $y \in K$ then $G_D(x_0, y) = 0$ so (10) holds for any $x \in \partial(\Omega \cap F) \cap K$. If $y \in \Gamma \cap B(x_j, r_0/2)$ then Lemma 2 implies (10). If $y \in \Gamma - B(x_j, r_0/2)$ then y is at a positive distance from K and hence (10) holds by Harnack inequality applied on a chain of balls. So (10) holds for any $y \in \Gamma$.
For the final part of the boundary, let $y \in \Omega \cap \partial F$. If $y \in B(x_1, r_0/2)$ then $y \in U_1$ and we can we use Lemma 2 again. A similar argument gives that (10) holds if $y \in B(x_2, r_0/2)$. The remaining part of $\Omega \cap \partial F$ is a distance at least $c > 0$ away from K, so using the Harnack inequality on a Harnack chain we obtain (10), completing the proof of the claim (10).
Once we have (10) the maximum principle gives that

$$G_D(x_0, y) \leq c_4\, G_D(x_0, w_0), \quad y \in F \cap \Omega. \tag{11}$$

Combining this with (8), and using the fact that $|\Delta\psi| \leq C_2$ on F we obtain

$$\mathbb{P}^{x_0}(X_{\tau_\Omega} \in \Gamma) \leq c_6\, G_D(x_0, w_0). \tag{12}$$

Combining (12) with (7) gives that

$$u(x) \leq c u(w_0) G_D(x, w_0) \text{ for } x \in U_0.$$

For the final part of the proof, consider the circle $\partial B(w_0, r_0)$. By our assumption on the harmonic function v and the Harnack inequality

$$v(z) \geq c_7\, v(w_0), \qquad z \in \partial B(w_0, r_0).$$

Moreover

$$G_D(z, w_0) \leq c_8, \qquad z \in \partial B(w_0, r_0)$$

and so

$$G_D(z, w_0) \leq (c_8/c_7)\, \frac{v(z)}{v(w_0)}, \qquad z \in \partial(D - B(w_0, r_0))$$

since v is positive. By the maximum principle, the last inequality holds inside the domain $D - B(w_0, r_0)$ which includes U_0. Using this inequality, (7) and (12), we obtain

$$u(x) \leq c_9\, u(w_0)\, \frac{v(x)}{v(w_0)} \leq c_{10}\, u_n(w_0)\, \frac{v(x)}{v(w_0)}$$

where $c_9 = c_3 c_6 c_8 c'/c_7$ and we applied the Harnack inequality to have $v(w_0) \leq c'\, v(w_0)$.
Finally, if we switch the roles of u and v and the roles of x and y we also obtain

$$\frac{v(y)}{u(y)} \leq c_{10} \frac{v(w_0)}{u(w_0)}$$

which leads to the result

$$\frac{u(x)/v(x)}{u(y)/v(y)} \leq c_{10}^2 = C_0.$$

Now suppose that $K \cap \partial B(0, \frac{1}{4}) = K \cap \partial B(0, \frac{3}{4}) = \emptyset$. If $K \cap B(0, \frac{1}{4}) = \emptyset$ then as u, v are harmonic in $B(0, \frac{1}{4})$ the inequality (4) follows from the Harnack inequality. So it remains to consider the case when $K \subset B(0, \frac{1}{4})$.
Let $\Gamma = \partial B(0, \frac{1}{2})$. Then for $x \in B(0, \frac{1}{16}) - K$ we have

$$u(x) = \int_\Gamma u(y) \mathbb{P}^x(X_{T_\Gamma \wedge \tau_D} \in dy).$$

The Harnack inequality gives that

$$C^{-1} u(w_0) \leq u(y) \leq C u(w_0) \text{ for } y \in \Gamma,$$

and thus if $p(x) = \mathbb{P}^x(T_\Gamma < \tau_D)$ we have

$$C^{-1} u(w_0) p(x) \leq u(x) \leq C u(w_0) p(x).$$

A similar inequality holds for v, and (4) follows immediately □

Example 1 The following example shows that one cannot expect a similar uniform BHP in higher dimensions. Let $d \geq 3$, $B = B(0,1)$, $K_0 = B \cap \mathbb{H}_0$, where $\mathbb{H}_0 = \{x : \pi_1(x) = 0\}$. Let δ be small and positive. Set

$$K = K_0 - B(0,\delta), \quad D = B(0,1) - K.$$

Thus K is a $d-1$ dimensional plate with a small hole in the centre, and is connected. Let y be on the x_1 axis with $\pi_1(y) = 1/4$. Let u_- and u_+ be the harmonic functions in D with boundary condition 1 on $\partial B \cap \mathbb{H}_-$ and $\partial B \cap \mathbb{H}_+$ respectively, and zero boundary conditions elsewhere. Set $v = u_- + u_+$. So if $\tau = \tau_D$ then we can write

$$u_-(x) = \mathbb{P}^x(X_\tau \in \partial D \cap \mathbb{H}_-, \tau < T_K), \quad v(x) = \mathbb{P}^x(\tau < T_K).$$

By symmetry we have

$$\frac{u_-(0)}{v(0)} = 1/2.$$

On the other hand if $B' = B(0,\delta)$ then

$$\mathbb{P}^y(T_{B'} < \tau_D) \leq \mathbb{P}^y(T_{B'} < \tau_B) \leq c\delta^{d-2}.$$

So we have

$$v(y) \asymp 1, \quad u_-(y) \leq c\delta^{d-2}.$$

Thus

$$\frac{u_-(0)/v(0)}{u_-(y)/v(y)} \geq c\delta^{2-d}. \tag{13}$$

By continuity this inequality will also hold if 0 is replaced by a point x close to 0 with $\pi_1(x) > 0$.

Example 2 The same example, taking $K = \{(y,0) : \delta \leq |y| < 1\}$, shows that one cannot drop the hypothesis that K is connected from Theorem 1.

Example 3 Now let $K = \{(0,y) : |y| < 1 - \varepsilon\}$ and $u_\pm^{(\varepsilon)}$ be as in Example 1. Let $r = \frac{1}{20}$ and $x_- = (-r,0)$, $x_+ = (r,0)$. Then we have

$$\lim_{\varepsilon \to 0} u_-^{(\varepsilon)}(x_-) = \lim_{\varepsilon \to 0} u_+^{(\varepsilon)}(x_+) = p, \quad \lim_{\varepsilon \to 0} u_-^{(\varepsilon)}(x_+) = \lim_{\varepsilon \to 0} u_+^{(\varepsilon)}(x_-) = 0,$$

for some $p \in (0,1)$. Thus we cannot have a BHP which holds for all $x, y \in B(0, \frac{1}{16}) \cap D$.

3 Uniform BHP for a harmonic function associated with cones

In this section we will prove the Boundary Harnack Principle in $\mathbb{R}^d$ with $d \geq 2$ for two fundamental harmonic functions.
Let

$$\mathbb{H}_- = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 < 0\}, \quad B^- = B(0,1) \cap \mathbb{H}_-,$$

and define $\mathbb{H}_+$, B^+ analogously. We write $B = B(0,1)$. Let $K \subset \overline{\mathbb{H}}_-$ be closed and connected. Set

$$D = B(0,1) - K.$$

Let $\pi_1 : \mathbb{R}^d \to \mathbb{R}^d$ be projection onto the x_1-axis, so $\pi_1((x_1, x_2, \ldots, x_d)) = (x_1, 0, \ldots, 0)$. Let W_α be the cone

$$W_\alpha = \{z \in \mathbb{R}^d : |z - \pi_1(z)| < z_1 \tan(\alpha)\}.$$

Set

$$W_\alpha(r) = B(0,r) \cap W_\alpha.$$

Write $\tau = \tau_D$ for the exit time of X from D.
We define the functions

$$v(x) = \mathbb{P}^x(X_\tau \in \partial D \cap K^c), \quad u_\alpha(x) = \mathbb{P}^x(X_\tau \in \partial D \cap W_\alpha). \tag{14}$$

Thus u_α and v are bounded, positive harmonic functions which vanish on ∂K, and have boundary values 1 on $\partial D \cap K^c$ and $\partial D \cap W_\alpha$ respectively. It is clear that $u_\alpha \leq v$ on D. Both u_α and v are bounded, positive and harmonic inside the domain D. Hence they satisfy the usual Harnack Inequality (see [6, Theorem II.1.19]) in balls which are far enough from the boundary of D. The main result of this section is that these two functions satisfy a BHP with a constant which depends only on d. (Note that since the geometry of the boundary of K is not specified, classical results on the Boundary Harnack Principle such as [6, Theorem III.1.2], do not apply.)
The main result of this section is the following Theorem.

Theorem 2 *Let $\alpha \in (0, \pi/2]$. There is a constant $C = C(\alpha, d) > 0$ depending only on α and d, and independent of K, such that for any $x, y \in B(0, 1/2) \cap \mathbb{H}_+$,*

$$\frac{u_\alpha(x)/u_\alpha(y)}{v(x)/v(y)} \leq C. \tag{15}$$

Remark 2 (1) The result does not require $0 \in K$, though this is the most delicate case.
(2) Let $x = (\frac{1}{4}, 0)$. Then the usual Harnack inequality gives that $u_\alpha(x) \geq C'(\alpha, d)$, and so, since $v \leq 1$, (15) implies that

$$C'(\alpha, d) \leq u_\alpha(y)/v(y) \leq 1 \quad \text{ for } y \in B(0, 1/2) \cap \mathbb{H}_+. \tag{16}$$

On the other hand, given (16) the inequality (15) follows immediately.
(3) Suppose that (15) (and therefore (16)) holds for some $\beta \in (0, \pi/2)$, and let $\alpha \in (\beta, \pi/2]$. We have $u_\beta \leq u_\alpha \leq v \leq 1$. So (16) for u_β implies (16), and therefore Theorem 2, for u_α. It is therefore sufficient to prove that for each d there exists α_d such that for all $\beta \in (0, \alpha_d)$, Theorem 2 is true.

Let $\kappa_d = [4(d+2)]^{-1/2}$, and define $\alpha_d = \arcsin(\kappa_d)$ for $d \geq 2$. Set

$$A_{\alpha_d} = W_{\alpha_d} \cap \partial B(0,1).$$

Note that the proportion of this surface A_{α_d} to the surface of the unit ball decreases as dimension grows. Similarly, let $\beta \in (0, \alpha_d)$. We define $\kappa_\beta = \sin(\beta)$, write κ_d for $\sin(\alpha_d)$ in short and

$$A_\beta = W_\beta \cap \partial B(0,1).$$

We begin by observing that it is enough to prove the Theorem for points x, y on the x_1-axis.

Lemma 3 *If $x \in W_{\alpha_d}(1/2)$ then $\pi_1(x) \in W_{\alpha_d}(1/2)$, and there is $c = c(\alpha_d)$ such that $c^{-1}f(\pi_1(x)) \leq f(x) \leq cf(\pi_1(x))$ for any non-negative bounded harmonic function f in D.*

Proof By definition of the cone, it is clear that $\pi_1(x) \in W_{\alpha_d}(1/2)$. If $d \geq 3$ then $x \in B(\pi_1(x), x_1 \tan(\alpha_d)) \subset B(\pi_1(x), x_1) \subset B^+$, and applying Harnack's inequality, the result follows. If $d = 2$ then it is enough to consider the Harnack chain with two balls and to apply the Harnack inequality twice. □

We now consider the exit distribution of Brownian motion from the unit ball B, without considering the set K. Set

$$h_\beta(x) = \mathbb{P}^x(X_{\tau_B} \in W_\beta).$$

This is a harmonic function on B, and potential analysis gives that

$$h_\beta(x) = \int_{W_\beta \cap \partial B} P_d(x,y)\sigma_d(dy),$$

where P_d is the d-dimensional Poisson kernel

$$P_d(x,y) = w_{d-1}^{-1} \frac{1-|x|^2}{|x-y|^d}.$$

Here w_{d-1} is the surface area of the unit sphere with respect to the surface measure $\sigma_d(dy)$ on the d-dimensional unit sphere. (See [6, Theorem II.1.17]).

Our argument is based on two main steps: comparison of the values of h_β inside the ball and the connection of h_β with two functions u_β and v. For this purpose, we compare first the values of h_β on the left-half of the ball, B^-, with the values on the positive axes $\{(x_1, 0, ..., 0) : 0 < x_1 < 1\}$.

Lemma 4 *For any $x = (x_1, 0, \ldots, 0)$ with $x_1 \in (0,1)$*

$$h_\beta(0) \leq h_\beta(x).$$

Proof Define the distance function

$$d(z, E) = \inf\{|z - y| : y \in E\}.$$

Set $S_1 = \partial B \cap \partial W_\beta$; thus S_1 is a $d-2$ dimensional sphere. Let $r_1 = d(x, S_1)$; by symmetry x is the same distance from all points in S_1. Set

$$B^x = B(x, r_1).$$

This is the ball with center at x whose surface crosses the surface of the unit ball through S_1. Then $A_\beta = \partial B \cap B^x$, and writing $\sigma_d'(\cdot)$ for the surface measure on ∂B^x,

$$\frac{\sigma_d'(\partial B^x - B)}{\sigma_d'(\partial B^x)} \geq \frac{\sigma_d(A_\beta)}{\sigma_d(\partial B)}.$$

Note that a Brownian motion X started at x and leaving the ball B^x through $\partial B^x - B$ must leave the unit ball through A_β. So

$$\begin{aligned} h_\beta(x) &= \mathbb{P}^x(X_{\tau_B} \in A_\beta) \geq \mathbb{P}^x(X_{\tau_{B^x}} \in \partial B^x - B) \\ &= \frac{\sigma_d'(\partial B^x - B)}{\sigma_d'(\partial B^x)} \geq \frac{\sigma_d(A_\beta)}{\sigma_d(\partial B)} = \mathbb{P}^0(X_{\tau_B} \in A_\beta) = h_\beta(0). \end{aligned}$$

□

The final piece of argument is based on the comparison of the values of h_β. For this purpose, we need the following two technical lemmas.

Lemma 5 *Let $d \geq 2$. For any $x \in [0, \kappa_d]$*

$$\frac{\kappa_d - x}{\kappa_d + x} \leq \left[\frac{1 + x^2 - 2x\kappa_d}{1 + x^2 + 2x\kappa_d}\right]^{1+\frac{d}{2}}.$$

Proof Denote the functions on the left hand side and the right hand side of the inequality by $f(x)$ and $g(x)$, respectively. It is clear that $f(0) = g(0)$. Hence it is enough to show that for every $x \in [0, \kappa_d]$ the derivatives satisfy $f'(x) \leq g'(x)$. Here

$$\begin{aligned} f'(x) &= -\frac{2\kappa_d}{(\kappa_d + x)^2} \quad , \\ g'(x) &= -\frac{2\kappa_d(d+2)(1-x^2)(1+x^2-2x\kappa_d)^{d/2}}{(1+x^2+2x\kappa_d)^{2+d/2}}. \end{aligned}$$

Now using $0 \leq x \leq \kappa_d$, it is not difficult to see that

$$-g'(x) \leq 2\kappa_d(d+2) = \frac{2\kappa_d}{(2\kappa_d)^2} \leq \frac{2\kappa_d}{(\kappa_d + x)^2} = -f'(x).$$

This inequality together with the Mean Value Theorem leads to our result. □

Lemma 6 *Suppose κ_d is as defined before. Then for any $x, z \in [0, \kappa_d]$*

$$f(x,z) := \frac{1-x^2}{(1+x^2-2xz)^{d/2}} + \frac{1-x^2}{(1+x^2+2xz)^{d/2}} \leq 2.$$

Proof The result follows immediately for $x = 0$. First we fix $x \in (0, \kappa_d]$ and consider the change in the direction of z. It is easy to show that $\frac{\partial}{\partial z} f(x,z) \geq 0$. Hence the maximum occurs at $z = \kappa_d$, that is, $f(x,z) \leq f(x,\kappa_d)$ for any $x, z \in [0, \kappa_d]$. Next we differentiate $f(x, \kappa_d)$ with respect to x.

$$\frac{\partial}{\partial x} f(x,\kappa_d) = \left[\frac{d(1-x^2)(\kappa_d - x)}{(1+x^2-2x\kappa_d)^{1+d/2}} - \frac{d(1-x^2)(\kappa_d + x)}{(1+x^2+2x\kappa_d)^{1+d/2}} \right] - \left[\frac{2x}{(1+x^2-2x\kappa_d)^{d/2}} + \frac{2x}{(1+x^2+2x\kappa_d)^{d/2}} \right].$$

The difference inside the first parenthesis is negative by Lemma 5. Therefore the derivative is negative and the function $f(\cdot, \kappa_d)$ reaches its maximum at $x = 0$ where $f(0, \kappa_d) \leq 2$. So $f(x,z) \leq f(x,\kappa_d) \leq f(0,\kappa_d) \leq 2$ for any $x, z \in [0, \kappa_d]$. □

The previous lemma helps us to prove the following statement.

Lemma 7 *Let $d \geq 2$ and $\beta \in (0, \alpha_d)$. Then for any $x \in B^-$, $h_\beta(x) \leq h_\beta(0)$.*

Proof Since h_β is harmonic in B and continuous on $\overline{B^-}$, it reaches its maximum on $\partial B^- = (\partial B \cap \partial B^-) \cup (\{(x_1, x_2, \ldots, x_d) : x_1 = 0\} \cap \partial B^-)$. Since h_β is zero on $(\partial B \cap \partial B^-)$, it is enough to find the maximum of h_β on the set $\{(x_1, x_2, \ldots, x_d) : x_1 = 0\} \cap \partial B^-$ and to show that this maximum value is bounded above by $h_\beta(0)$.
We can reduce the set of interest further. The rotational invariance of Brownian motion together with the symmetry of the domain give that $h_\beta(x) = h_\beta(y)$ for any $x = (0, x_2, \ldots, x_d)$ and $y = (0, y_2, \ldots, y_d)$ with $|x| = |y|$. Hence we only need to consider the points of the form $x = (0, x_2, 0, \ldots, 0)$ with $x_2 \in [0, 1)$.

First note that for any $z = (z_1, \ldots, z_d) \in A_\beta$, $|z_2| < \kappa_\beta < \kappa_d$. So if $\kappa_\beta \leq x_2 < 1$ then

$$\frac{\partial h_\beta}{\partial x_2} = \frac{1}{\omega_{d-1}} \int_{A_{\alpha_d}} \frac{-2x_2|x-z|^2 - d(x_2 - z_2)(1 - x_2^2)}{|x-z|^{d+2}} \sigma_d(dz) \leq 0.$$

Hence h_β is a decreasing function in x_2 whenever $x_2 \in (\kappa_\beta, 1)$ and

$$h_\beta((0, x_2, 0, \ldots, 0)) \leq h_\beta((0, \kappa_\beta, 0, \ldots, 0)) \tag{17}$$

for $\kappa_\beta \leq x_2 < 1$.

Assume that $0 \leq x_2 < \kappa_\beta < \kappa_d$. Split the surface A_β into $2n$ parts as follows: Take $n+1$ non-negative numbers $\{\zeta(i)\}_{i=0}^n$ such that $0 = \zeta(0) < \zeta(1) < \zeta(2) < \cdots < \zeta(n) = \kappa_\beta$ and define strips $\{S_i\}_{i=1}^n$ in a way that

$$S_i = \{z = (z_1, \ldots, z_d) \in A_\beta : z_2 \in [\zeta(i-1), \zeta(i))\}$$

and the measure of each strip, $\sigma_d(S_i)$, equals each other. Similarly, define n negative numbers $\{\zeta(i)\}_{i=-n}^{-1}$ by $\zeta(-i) = -\zeta(i)$ for $i = 1, ..., n$. Also define the strips, $\{S_{-i}\}_{i=0}^{n-1}$, on the lower-half of A_β the same way as above

$$S_{-i} = \{z = (z_1, \dots, z_d) \in A_\beta : z_2 \in [\zeta(-i-1), \zeta(-i))\}.$$

Then

$$A_\beta = \bigcup_{-n+1 \le i \le n} S_i \quad \text{and} \quad \sigma_d(S_i) = \sigma_d(S_j) = \frac{\sigma_d(A_\beta)}{2n} \quad i, j \in \{-n+1, \dots, n\}.$$

Note that if $z = (z_1, \dots, z_d) \in S_i$ then

$$|x - z|^2 = 1 + x_2^2 - 2x_2 z_2 \ge 1 + x_2^2 - 2x_2\zeta(i).$$

Using this partition

$$\begin{aligned}
h_\beta(x) &= \int_{A_\beta} P_d(x,z)\sigma(dz) = \sum_{i=-n+1}^{n} \int_{S_i} P_d(x,z)\sigma(dz) \\
&= \int_{S_n} P_d(x,z)\sigma(dz) + \int_{S_0} P_d(x,z)\sigma(dz) \\
&\quad + \sum_{i=1}^{n-1} \left[\int_{S_i} P_d(x,z)\sigma(dz) + \int_{S_{-i}} P_d(x,z)\sigma(dz) \right] \\
&\le \frac{1}{\omega_{d-1}} \left[\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\kappa_d)^{d/2}} \sigma_d(S_n) + \frac{1 - x_2^2}{(1 + x_2^2)^{d/2}} \sigma_d(S_0) \right] \\
&+ \frac{1}{\omega_{d-1}} \sum_{i=1}^{n-1} \left[\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^{d/2}} \sigma_d(S_i) + \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(-i))^{d/2}} \sigma_d(S_{-i}) \right] \\
&\le \frac{\sigma_d(A_\beta)}{\omega_{d-1} \cdot 2n} \left[\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\kappa_d)^{d/2}} + \frac{1 - x_2^2}{(1 + x_2^2)^{d/2}} \right] \\
&\quad + \frac{\sigma_d(A_\beta)}{\omega_{d-1} \cdot 2n} \sum_{i=1}^{n-1} \left[\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^{d/2}} + \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(-i))^{d/2}} \right]
\end{aligned}$$

By the restriction $0 \le x_2 < \kappa_\beta < \kappa_d$, the first term is bounded by $c_d/2n$ where $c_d = 2\sigma_d(A_{\alpha_d})/\omega_{d-1}(1 - 2\kappa_d^2)^{d/2}$. For the term inside the second bracket, Lemma 6 provides an upper bound and hence

$$\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^{d/2}} + \frac{1 - x_2^2}{(1 + x_2^2 + 2x_2\zeta(i))^{d/2}} \le 2.$$

Hence for any $n \in \mathbb{Z}^+$we obtain

$$h_\beta(x) \leq \frac{c_d}{2n} + \frac{\sigma_d(A_\beta)}{\omega_{d-1}} \frac{n-1}{n}.$$

Finally, if we take the limit as $n \to \infty$ then

$$h_\beta(x) \leq \frac{\sigma_d(A_\beta)}{\omega_{d-1}} = h_\beta(0). \tag{18}$$

for any $x = (0, x_2, 0..., 0)$ with $x_2 \in [0, \kappa_\beta)$.
By (17) and (18), we conclude that $h_\beta(x) \leq h_\beta(0)$ for any $x \in B^-$. □

Now, we are ready to prove the main result of this section.

Proof (Proof of Theorem 2)
Let $r \in (0, 1]$, $0 < \beta \leq \alpha_d$, and write h for h_β. Let u_β be the function defined in (14) with angle β.
First, let $x = (x_1, 0, \dots, 0)$, with $0 < x_1 \leq 1/2$. By the Markov property, Lemma 4 and Lemma 7,

$$\mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] = \mathbb{E}(\mathbb{E}^{X_{T_K}}[h(X_{\tau_B})]) = \mathbb{E}h(X_{T_K}) \leq \sup_{y \in K} h(y) \leq h(0) \leq h(x).$$

Thus

$$\mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] \leq h(x).$$

So

$$\begin{aligned} h(x) &= \mathbb{E}^x[h(X_{\tau_B})] \\ &= \mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] \cdot \mathbb{P}^x[T_K \leq \tau_B] + \mathbb{E}^x[h(X_{\tau_B}); \tau_B < T_K] \\ &\leq h(x)(1 - v(x)) + u_\beta(x). \end{aligned}$$

Hence

$$c_\beta = h(0) \leq h(x) \leq \frac{u_\beta(x)}{v(x)} = \mathbb{P}^x[X_{\tau_B} \in A_\beta | \tau_B < T_K]. \tag{19}$$

This proves (16) for x on the x_1-axis, and by Lemma 3 it then follows for $x \in W_\beta$.

Now let $x \in B(0, 1/4) \cap H_+$, and set $x' = x - \pi_1(x)$. Let $W' = x' + W_\beta$, $A' = W' \cap \partial B(x', 1/4)$, and write $\tau' = \tau_{B(x',1/4)}$. Then applying (19) to the ball $B(x', 1/4) - K$, we obtain

$$c_\beta \leq \mathbb{P}^x[X_{\tau'} \in A' | \tau' < T_K] = \frac{\mathbb{P}^x[X_{\tau'} \in A'; \tau' < T_K]}{\mathbb{P}^x[\tau' < T_K]}. \tag{20}$$

Since $\tau' < \tau = \tau_{B_1}$ this implies

$$\mathbb{P}^x[X_{\tau'} \in A'; \tau' < T_K] \geq c_\beta v(x).$$

Now by the standard Harnack inequality,

$$\mathbb{P}^y[X_\tau \in A_\beta, \tau < T_K] \geq c_1 \text{ for } y \in A'. \tag{21}$$

Then

$$\begin{aligned} u_\beta(x) &= \mathbb{P}^x[X_\tau \in A_\beta, \tau < T_K] \\ &\geq \mathbb{P}^x[X_\tau \in A_\beta, X_{\tau'} \in A', \tau' < T_K, \tau < T_K] \\ &= \mathbb{E}^x\Big[1_{(\tau' < T_K, X_{\tau'} \in A')} \mathbb{P}^{X_{\tau'}}[X_\tau \in A_\beta, \tau < T_K]\Big] \\ &\geq c_1 \mathbb{P}^x[\tau' < T_K, X_{\tau'} \in A'] \geq c_1 c_\beta v(x). \end{aligned}$$

Since we have $u_\beta(x) \leq v(x)$ everywhere, it follows that

$$C(\beta, d) \leq \frac{u_\beta(x)}{v(x)} \leq 1 \text{ for } x \in \mathbb{H}_+ \cap B(0, 1/2). \tag{22}$$

This proves (16) for β, and Theorem 2 then follows from Remark 2. □

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