Stability of EHI and regularity of MMD spaces

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August 12, 2020

Abstract

We show that elliptic Harnack inequality is stable under form-bounded perturbations for strongly local Dirichlet forms on complete locally compact separable metric spaces that satisfy metric doubling property (or equivalently, relative ball connectedness property).

Keywords: Elliptic Harnack inequality, Green function, quasisymmetry, relative ball connectedness, metric doubling, good doubling measure, relative capacity, time-change

Mathematical Subject Classification (2020): Primary 31B05, 31C25; Secondary: 47D07, 35J08

1 Introduction

Let $X = \{X_t, t \in [0,\infty); \mathbb{P}^x, x \in \mathcal{X}\}$ be a diffusion process on a metric space $(\mathcal{X}, d)$. A function h on a ball $B = B(x,r)$ is *harmonic* if $h(X_{t\wedge\tau_B})$ is a local martingale under $\mathbb{P}^x$ for every $x \in B$; here τ_B is the exit time from B by the process X and the filtration is the minimal augmented filtration generated by X. The (scale-invariant) *elliptic Harnack inequality* (EHI) holds for X if there exist constants $C \geq 1$ and $\delta \in (0,1)$ such that whenever h is non-negative and harmonic on a ball $B = B(x,r)$, then

$$\sup_{B(x,\delta r)} h \leq C \inf_{B(x,\delta r)} h. \tag{1.1}$$

If it holds, the EHI is a valuable tool for the study of the process X and its associated heat kernel. A well known theorem of Moser [Mo1] is that the EHI holds if X is the diffusion associated with a uniformly elliptic divergence form operator $\mathcal{A} = \mathrm{div}(A(x)\nabla)$. Associated with such a process is the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$, where

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) = \Big\{ f \in L^2(\mathbb{R}^d; dx) : \nabla f \in L^2(\mathbb{R}^d; dx) \Big\}$$

is the Sobolev space on $\mathbb{R}^d$ of order $(1,2)$ and

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x) \cdot A(x) \nabla f(x)\, dx, \quad f \in \mathcal{F}.$$

We say two Dirichlet forms on $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ on $L^2(\mathbb{R}^d; dx)$ are *comparable* if there exists C such that

$$C^{-1}\mathcal{E}^{(1)}(f,f) \leq \mathcal{E}^{(2)}(f,f) \leq C\mathcal{E}^{(1)}(f,f) \quad \text{for all } f \in \mathcal{F};$$

here $\mathcal{F}$ is the common domain of the two forms. Moser's result gives the stability of the EHI, in the sense that if $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are comparable Dirichlet forms on $L^2(\mathbb{R}^d; dx)$, associated with uniformly elliptic divergence form operators $\mathcal{A}_i$, then the EHI holds for $\mathcal{E}^{(2)}$ if and only if it holds for $\mathcal{E}^{(1)}$.

A few years later, Moser [Mo2, Mo3] proved a parabolic Harnack inequality PHI, which holds for non-negative solutions to the heat equation associated with a divergence form operator $\mathcal{A}$. In particular, if u is any non-negative solution to the heat equation $\frac{\partial u}{\partial t} = \mathcal{A}u$ in a time-space cylinder $Q = (0, r^2) \times B(x, r)$, then writing $T = R^2, Q_- = (T/4, T/2) \times B(x, \delta r), Q_+ = (3T/4, T) \times B(x, \delta r)$, we have

$$\operatorname*{ess\,sup}_{Q_-} u \leq C_P \operatorname*{ess\,inf}_{Q_+} u,$$

where constants $C_P > 1$ and $\delta \in (0, 1)$ do not depend on x, r or u. Subsequently Grigoryan and Saloff-Coste in [Gr0, Sal92] gave a characterization of the PHI, and the stability of the PHI follows immediately from this characterization. The methods of these papers are very robust, and this characterization of the PHI was extended to diffusions on locally compact separable metric spaces [St], and to random walks on graphs [De1].

For a number of years the stability of the apparently simpler EHI remained an open problem. Stability on a large class of unbounded spaces (including Riemannian manifolds and graphs) was proved by two of us recently in [BM1]. However, the result there relied on the metric space satisfying some strong local regularity conditions; one key use of this regularity was to ensure the existence of Green's functions.

The natural context for the study of the EHI is that of a metric measure space with a strongly local Dirichlet form, which we call MMD spaces. Examples include Riemannian manifolds, the cable systems of graphs [V], as well as various classes of fractals. Not only do MMD spaces provide a common framework for all these examples, but also certain transformations (change of measure, quasi-symmetric change of metric) which are not so natural for manifolds and graphs are natural in the MMD space context. These transformations are key to the argument in [BM1].

This paper has three main goals:

(i) We give a weak sufficient condition (a local Harnack inequality) for a MMD space to have Green's functions. This improves significantly the results of earlier papers, such as [BM1, BM2], which needed some parabolic regularity. In particular, it allows us to drop the Green function assumption ([BM1, Assumption 2.3]) made in [BM1].

(ii) We carry through the program of [BM1] in the context of a MMD space satisfying these weak regularity conditions. In particular, we drop the *bounded geometry* assumption (see [BM1, Assumption 2.5] for its definition) on the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, and relax the condition that $(\mathcal{X}, d)$ is a length (or geodesic) space; both are needed in [BM1]. We make the weaker assumption that $(\mathcal{X}, d)$ is 'relatively ball connected' – see Definition 1.1; this property has the advantage that is preserved by quasisymmetric changes of metric. Example 8.1 shows that some regularity of the metric is needed if we are to have stability of the EHI.

(iii) We cover metric spaces $(\mathcal{X}, d)$ not only of infinite diameters but also of bounded diameters.

For a metric sapce $(\mathcal{X}, d)$, we use $B(x, r)$ to denote the open ball centered at $x \in \mathcal{X}$ with radius ball, and $\overline{B(x, r)} = \{y \in \mathcal{X} : d(y, x) \leq r\}$ its closure. The following definition is adapted from [GH, Definition 5.5].

Definition 1.1. Let $K > 1$. A metric space $(\mathcal{X}, d)$ is *relatively K ball connected* if for each $\varepsilon \in (0, 1)$ there exists an integer $N = N_{\mathcal{X}}(\varepsilon)$ such that if $x_0 \in \mathcal{X}$, $R > 0$ and $x, y \in \overline{B(x_0, R)}$ then there exists a chain of balls $B(z_i, \varepsilon R)$ for $i = 0, \ldots, N$ such that $z_0 = x$, $z_N = y$, $B(z_i, \varepsilon R) \subset$

$B(x_0, KR)$ for each i and $d(z_{i-1}, z_i) < \varepsilon R$ for $1 \le i \le N$. We write $N_{\mathcal{X}}$ for the integer $N_{\mathcal{X}}(\varepsilon)$ with $\varepsilon = 1/4$. We say also that $(\mathcal{X}, d)$ satisfies the property RBC(K). We say that $(\mathcal{X}, d)$ is *relatively ball connected* if there exists $K > 1$ such that $(\mathcal{X}, d)$ is relatively K ball connected.

A relatively ball connected metric space $(\mathcal{X}, d)$ is topologically connected; see Lemma 5.2(a). Relatively ball connected property is equivalent to metric doubling property under the EHI; see Theorem 5.4. When $(\mathcal{X}, d)$ is a locally compact metric space, a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is said to be *strongly local* if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{F}$ have compact support with v being constant in an open neighborhood of supp$[u]$. See Proposition 2.2 below for its equivalent characterizations.

The main result of this paper is the following stability result on the (scale invariant) EHI. See Definition 4.1 for a precise definition of the EHI.

Theorem 1.2. *Let $(\mathcal{X}, d)$ be a complete, locally compact, relatively ball connected separable metric space, and let m be a Radon measure on $\mathcal{X}$ with full support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI. Let $(\mathcal{E}', \mathcal{F})$ be another strongly local Dirichlet form on $L^2(\mathcal{X}; m)$ such that*

$$C^{-1}\mathcal{E}(f, f) \le \mathcal{E}'(f, f) \le C\mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}.$$

Then $(\mathcal{X}, d, m, \mathcal{E}', \mathcal{F})$ satisfies the EHI.

Theorem 1.2 is established based on equivalent characterizations of the EHI given in Theorem 7.9. This stability result is further extended in Theorem 7.11 to strongly local MMD spaces that may have different symmetrizing measures.

The remaining of this paper is organized as follows. In Section 2, we present definitions and terminology associated with Dirichlet forms and some basic facts that will be used in this paper. Existence and regularity of Green functions are given in Section 3 for transient Dirichlet forms. The transience condition is removed in Section 4. It is shown there that any strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a connected locally compact metric space $\mathcal{X}$ that satisfies the local EHI is irreducible and has regular Green function. Various consequences of the EHI are presented in Section 5. In particular, it is shown that for a complete locally compact metric space $(\mathcal{X}, d)$, under the EHI, relatively ball connected, metric doubling and quasi-arc connected properties are all mutually equivalent. In Section 6, a good doubling measure μ is constructed on a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the EHI and is relative ball connected. This measure relates well with capacities and is a smooth measure with full quasi support on $\mathcal{X}$. It is shown in Section 7 that the Dirichlet form time-changed by the positive continuous additive functional generated by this doubling measure μ is a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^{\mu})$ that satisfies Poincare inequality PI(Ψ), the cutoff energy inequality CS(Ψ) and a capacity estimate cap(Ψ), where Ψ is a suitable regular scale function. From which we can obtain equivalent characterizations of the EHI in Theorem 7.9, and deduce the stability result of the EHI stated in Theorem 1.2. The aforementioned scale function Ψ varies both in space and in time; functions of this kind were considered in [Te] who first studied such location dependent scaling functions in detail. An extension of Theorem 1.2 is given at the end of Section 7 that the second Dirichlet form $\mathcal{E}'$ may have symmetrizing measure μ different from m; see Theorem 7.11. Three examples are given in Section 8. The first example shows that without certain regularity of the metric, the stability of the EHI may fail. The second example is a strongly local regular Dirichlet form that fails to satisfy non-scale-invariant Harnack inequality. The third one fits into the setting of this paper and has the EHI but fails to satisfy the local regularity required in [BM1].

2 Preliminaries

In this section, we give definitions of some terminology from Dirichlet form theory that are used in this paper and some basic facts. We refer the reader to [CF, FOT] for more details on the theory of symmetric Dirichlet forms. We use := as a way of definition.

Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be a measurable space and m a σ-finite measure on $\mathcal{X}$ with full support. A bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be a *symmetric Dirichlet form* if

(i) $\mathcal{F}$ is a dense linear subspace of $L^2(\mathcal{X}; m)$;

(ii) $\mathcal{E}$ is symmetric and bilinear on $\mathcal{F} \times \mathcal{F}$ such that $\mathcal{E}(f, f) \geq 0$ for every $f \in \mathcal{F}$;

(iii) $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \int_{\mathcal{X}} f(x)g(x)m(dx)$;

(iv) For every $f \in \mathcal{F}$, $g := (0 \vee f) \wedge 1$ is in $\mathcal{F}$ and $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$.

A bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ satisfying properties (i)-(iii) above is called a *symmetric closed form.* Any symmetric closed form is in one-to-one correspondence with a strongly continuous symmetric contraction semigroup $\{T_t; t \geq 0\}$ on $L^2(\mathcal{X}; m)$. Property (iv) above is called a Markovian property which is equivalent to the corresponding semigroup $\{T_t; t \geq 0\}$ being Markovian; that is, $0 \leq T_t f \leq 1$ for any $f \in L^2(\mathcal{X}; m)$ with $0 \leq f \leq 1$. A real-valued function f is said to be in the *extended Dirichlet space* $\mathcal{F}_e$ if there is an $\mathcal{E}$-Cauchy sequence $\{f_k; k \geq 1\} \subset \mathcal{F}$ so that $\lim_{k\to\infty} f_k = f$ m-a.e. on $\mathcal{X}$, and we define $\mathcal{E}(f, f) = \lim_{k\to\infty} \mathcal{E}(f_k, f_k)$. Clearly, $\mathcal{F} \subset \mathcal{F}_e$. It is known that $\mathcal{F} = \mathcal{F}_e \cap L^2(\mathcal{X}; m)$; see [CF, Theorem 1.1.5(iii)].

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *transient* if there exists a bounded $L^1(\mathcal{X}; m)$-integrable function g that is strictly positive on $\mathcal{X}$ so that

$$\int_{\mathcal{X}} |u(x)|g(x)m(dx) \leq \mathcal{E}(u, u) \quad \text{for every } u \in \mathcal{F}.$$

Clearly, if $(\mathcal{E}, \mathcal{F})$ is transient, then $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *recurrent* if $1 \in \mathcal{F}_e$ and $\mathcal{E}(1, 1) = 0$. Denote by $\{T_t; t \geq 0\}$ the semigroup on $L^2(\mathcal{X}; m)$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. By Theorem 2.1.5 and Theorem 2.18 0f [CF], $(\mathcal{E}, \mathcal{F})$ is transient if and only if there is some $L^1(\mathcal{X}; m)$-integrable function g that is strictly positive on $\mathcal{X}$ so that $Gg := \int_0^\infty T_t g\, dt < \infty$ m-a.e. on $\mathcal{X}$; and $(\mathcal{E}, \mathcal{F})$ is recurrent if and only if for any non-negative g on $\mathcal{X}$ with $0 < \int_{\mathcal{X}} g(x)m(dx) < \infty$, $Gg = \infty$ m-a.e. on $\mathcal{X}$.

Denote by $\mathcal{B}^*(\mathcal{X})$ the completion of the field $\mathcal{B}(\mathcal{X})$ under the measure m. A set $A \in \mathcal{B}^*(\mathcal{X})$ is said to be $\{T_t\}_{t\geq 0}$*-invariant* if $T_t(1_{A^c} f) = 0$ m-a.e. on A for all $t > 0$ and $f \in L^2(\mathcal{X}; m)$. By [CF, Proposition 2.1.6], $A \in \mathcal{B}^*(\mathcal{X})$ is $\{T_t\}_{t\geq 0}$-invariant if and only if $1_A u \in \mathcal{F}$ for every $u \in \mathcal{F}$ and

$$\mathcal{E}(u, v) = \mathcal{E}(1_A u, 1_A v) + \mathcal{E}(1_{A^c} u, 1_{A^c} v) \quad \text{for every } u, v \in \mathcal{F}. \tag{2.1}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *irreducible* if for any $\{T_t\}_{t\geq 0}$-invariant set A, either $m(A) = 0$ or $m(A^c) = 0$. An irreducible Dirichlet form is either transient or recurrent; see [CF, Proposition 2.1.3(iii)].

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *regular* if

(i) $(\mathcal{X}, d)$ is a locally compact separable metric space and m is a Radon measure on $\mathcal{X}$ with full support;

(ii) $\mathcal{F}\cap C_c(\mathcal{X})$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where $C_c(\mathcal{X})$ is the space of continuous functions on $\mathcal{X}$ having compact support;

(ii) $\mathcal{F}\cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$ with respect to the uniform norm $\|f\|_\infty = \sup_{x\in\mathcal{X}}|f(x)|$.

An increasing sequence $\{F_k; k\geq 1\}$ of compact subsets of $\mathcal{X}$ is said to be an $\mathcal{E}$*-nest* if $\cup_{k\geq 1}\mathcal{F}_{F_k}$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_k} := \{f\in\mathcal{F} : f = 0\ m\text{-a.e. on } \mathcal{X}\setminus F_k\}$. A set $N\subset\mathcal{X}$ is said to be $\mathcal{E}$-polar if there is an $\mathcal{E}$-nest $\{F_k; k\geq 1\}$ so that $N\subset\mathcal{X}\setminus\cup_{k\geq 1}\mathcal{F}_k$. An $\mathcal{E}$-polar set A always has $m(A)=0$. $\mathcal{E}$-polar sets can also be characterized by using capacity. Given a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$, we can define 1-capacity Cap_1 as follows. For any open subset $U\subset\mathcal{X}$,

$$\mathrm{Cap}_1(U) := \inf\{\mathcal{E}_1(f,f) : f\in\mathcal{F},\ f\geq 1\ m\text{-a.e. on } U\} \tag{2.2}$$

with the convention that $\inf\emptyset := \infty$, and for any subset $A\subset\mathcal{X}$,

$$\mathrm{Cap}_1(A) := \inf\{\mathrm{Cap}_1(U) : U\supset A\}. \tag{2.3}$$

It is known (see [CF, Theorem 1.3.14]) that for a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$, $A\subset\mathcal{X}$ is $\mathcal{E}$-polar if and only if it has zero 1-capacity. A statement depending on $x\in A$ is said to hold $\mathcal{E}$*-quasi-everywhere* ($\mathcal{E}$-q.e. in abbreviation) if there is an $\mathcal{E}$-polar set $N\subset A$ so that the statement is true for every $x\in A\setminus N$. A function f is said to be $\mathcal{E}$*-quasi-continuous* on $\mathcal{X}$ if there is an $\mathcal{E}$-nest $\{F_k; k\geq 1\}$ so that $f\in C(F_k)$ for every $k\geq 1$. When there is no possible ambiguity, we often drop $\mathcal{E}$- from $\mathcal{E}$-quasi-everywhere and $\mathcal{E}$-quasi-continuous. For a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$, every $f\in\mathcal{F}_e$ has an m-version that is quasi-continuous on $\mathcal{X}$, which is unique up to an $\mathcal{E}$-polar set; see [CF, Theorem 2.3.4] or [FOT, Theorem 2.1.7]. We always represent $f\in\mathcal{F}_e$ by its quasi-continuous version.

Recall that a Hunt process $X=\{X_t, t\geq 0; \mathbb{P}^x, x\in\mathcal{X}\}$ on a locally compact separable metric space $\mathcal{X}$ is a strong Markov process that is right continuous and quasi-left continuous on the one-point compactification $\mathcal{X}_\partial := \mathcal{X}\cup\{\partial\}$ of $\mathcal{X}$. A set $C\subset\mathcal{X}$ is said to be *nearly Borel measurable* if for any probability measure μ on $\mathcal{X}$ there are Borel sets A_1, A_2 such that $A_1\subset C\subset A_2$ and

$$\mathbb{P}^\mu\,(\text{there is some } t\geq 0 \text{ such that } X_t\in A_2\setminus A_1) = 0.$$

Let m be a Radon measure with full support on $\mathcal{X}$. A Hunt process X is said to be m*-symmetric* if the transition semigroup is symmetric on $L^2(\mathcal{X};m)$. For an m-symmetric Hunt process X on $\mathcal{X}$, a set $\mathcal{N}\subset\mathcal{X}$ is said to be *properly exceptional* for X if $\mathcal{N}$ is nearly Borel measurable, $m(\mathcal{N})=0$ and

$$\mathbb{P}^x(X_t\in\mathcal{X}_\partial\setminus\mathcal{N} \text{ and } X_{t-}\in\mathcal{X}_\partial\setminus\mathcal{N} \text{ for all } t>0) = 1 \quad \text{for every } x\in\mathcal{X}\setminus\mathcal{N}.$$

In 1971, Fukushima showed that any symmetric regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$ has an m-symmetric Hunt process process $X=\{X_t, t\geq 0; \mathbb{P}^x, x\in\mathcal{X}\}$ on $\mathcal{X}$ associated with it in the sense that the transition semigroup of X is a version of the strongly continuous semigroup $\{T_t; t\geq 0\}$ on $L^2(\mathcal{X};m)$ corresponding to $(\mathcal{E},\mathcal{F})$, see [FOT, Theorem 7.2.1]. Furthermore, for any non-negative Borel measurable $f\in L^2(\mathcal{X};m)$ and $t>0$,

$$P_tf(x) := \mathbb{E}^x[f(X_t)]$$

is a quasi-continuous version of T_tf on $\mathcal{X}$. The Hunt process X associated with a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$ is unique in the following sense (see [FOT, Theorem 4.2.8]):

if X' is another Hunt process associated with the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$, then there is a common properly exceptional set outside which these two Hunt processes have the same transition functions. We say the m-symmetric Hunt process X on $\mathcal{X}$ is *transient*, *recurrent*, and *irreducible* if so does its associated Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$.

In the remaining of this section, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathcal{X};m)$ and $X = \{X_t, t\geq 0; \mathbb{P}^x, x\in\mathcal{X}\}$ is the Hunt process associated with it. Let ζ denote the lifetime of X, and $\{\mathcal{F}_t; t\geq 0\}$ be the minumun augmented filtration generated by X.

A subset $N\subset\mathcal{X}$ is said to be *m-polar* if there is a nearly Borel set $\mathcal{N}_1\supset\mathcal{N}$ so that $\mathbb{P}^x(\sigma_{N_1}<\infty)=0$ for m-a.e. $x\in\mathcal{X}$, where $\sigma_{N_1}=\inf\{t>0 : X_t\in N_1\}$. It is known that a subset $\mathcal{N}\subset\mathcal{X}$ is $\mathcal{E}$-polar if and only if it is m-polar, and any $\mathcal{E}$-polar set is contained in a Borel properly exceptional set for X; see [CF, Theorems 3.1.3 and 3.1.5].

If $(\mathcal{E},\mathcal{F})$ is irreducible, then (see [CF, Theorem 3.5.6]) for any non-$\mathcal{E}$-polar nearly Borel measurable set A,

$$\mathbb{P}^x(\sigma_A<\infty)>0 \quad \text{for } \mathcal{E}\text{-q.e. } x\in\mathcal{X}. \tag{2.4}$$

Let D be an open subset of $\mathcal{X}$. The part process X^D of X killed upon exiting D is a Hunt process on D whose associated Dirichlet form $(\mathcal{E},\mathcal{F}^D)$ on $L^2(D;m|_D)$ is regular. Here $m|_D$ is the measure m restricted to the open set D and

$$\mathcal{F}^D=\{f\in\mathcal{F} : f=0\ \mathcal{E}\text{-q.e. on } D^c\}; \tag{2.5}$$

see, e.g., Exercise 3.37 and Theorem 3.3.9 of [CF]. Property (2.4) combined with [CF, Proposition 2.1.10] yields the following.

Proposition 2.1. *If $(\mathcal{E},\mathcal{F})$ is irreducible and D^c is not $\mathcal{E}$-polar, then the regular Dirichlet form $(\mathcal{E},\mathcal{F}^D)$ on $L^2(D;m|_D)$ is transient.*

For $u\in\mathcal{F}_e$, the following Fukushima decomposition holds (see [CF, Theorem 4.2.6] or [FOT, Theorem 5.2.2]):

$$u(X_t)-u(X_0)=M_t^u+N_t^u, \quad t\geq 0, \tag{2.6}$$

where M^u is a martingale additive functional of X having finite energy and N^u is a continuous additive functional of X having zero energy. The predictable quadratic variation $\langle M^u\rangle$ of the square-integrable martingale M^u is a positive continuous additive functional of X, whose corresponding Revuz measure is denoted by $\mu_{\langle u\rangle}$. We call $\mu_{\langle u\rangle}$ the energy measure of $u\in\mathcal{F}_e$. It is known that

$$\frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X})\leq\mathcal{E}(u,u)\leq\mu_{\langle u\rangle}(\mathcal{X}) \quad \text{for } u\in\mathcal{F}_e.$$

When $(\mathcal{E},\mathcal{F})$ admits no killings inside $\mathcal{X}$, which is equivalent to the Hunt process admits no killings inside $\mathcal{X}$ (that is, $\mathbb{P}^x(X_{\zeta-}\in\mathcal{X},\zeta<\infty)=0$ for $\mathcal{E}$-q.e. $x\in\mathcal{X}$), we have

$$\mathcal{E}(u,u)=\frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X}) \quad \text{for } u\in\mathcal{F}_e. \tag{2.7}$$

When $u\in\mathcal{F}_e$ is bounded, its energy measure $\mu_{\langle u\rangle}$ can be computed by the formula

$$\int_{\mathcal{X}} v(x)\mu_{\langle u\rangle}(dx)=2\mathcal{E}(u,uv)-\mathcal{E}(u^2,v) \qquad \text{for all bounded } v\in\mathcal{F}. \tag{2.8}$$

For general $u\in\mathcal{F}_e$, $\mu_{\langle u\rangle}$ is the increasing limit of $\mu_{\langle u_n\rangle}$ as $n\to\infty$, where $u_n:=(-n)\vee(u\wedge n)\in\mathcal{F}_e$. See (4.3.12)-(4.3.13), and Theorems 4.3.10 and 4.3.11 of [CF] for the above stated properties of $\mu_{\langle u\rangle}$.

The following is taken from Theorem 2.4.3 and Theorem 4.3.4 of [CF].

Proposition 2.2. *The following are equivalent.*

(i) $(\mathcal{E}, \mathcal{F})$ *is strongly local;*

(ii) $\mathcal{E}(u, v) = 0$ *whenever* $u, v \in \mathcal{F}$ *with* $u(v - c) = 0$ *m-a.e. on* $\mathcal{X}$ *for some constant* c*;*

(iii) *The associated Hunt process* X *is a diffusion with no killings inside* $\mathcal{X}$*; that is, there is a Borel properly exceptional set* $\mathcal{N}_0 \subset \mathcal{X}$ *so that for every* $x \in \mathcal{X} \setminus \mathcal{N}_0$*,*

$$\mathbb{P}^x(X_t \text{ is continuous in } t \in [0, \zeta)) = 1 \text{ and } \quad \mathbb{P}^x(X_{\zeta-} \in \mathcal{X}, \zeta < \infty) = 0. \tag{2.9}$$

In Theorem 4.5 below, a new irreducible criteria will be given for strong local regular Dirichlet forms.

We use notation $V \Subset D$ for V being a relatively compact open subset of D. For any open set U, we define

$$\mathcal{F}^U_{\text{loc}} := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable} \\ \text{functions on } \mathcal{X} \text{ such that for each } V \Subset U, \text{ there is some} \\ g \in \mathcal{F} \text{ so that } f = g \ m\text{-a.e. on } V. \end{array} \right\}. \tag{2.10}$$

Note that each $f \in \mathcal{F}^U_{\text{loc}}$ admits a m-version that is $\mathcal{E}$-quasi-continuous on U, which is unique modulo an $\mathcal{E}$-polar set. We always represent a function in $\mathcal{F}^U_{\text{loc}}$ by its quasi-continuous version. When $U = \mathcal{X}$, we simply write $\mathcal{F}_{\text{loc}}$ for $\mathcal{F}^{\mathcal{X}}_{\text{loc}}$.

When the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local, the energy measure $\mu_{\langle u \rangle}$ has the following strong local property; see [CF, Proposition 4.3.1 and Theorem 4.3.10(i)].

Proposition 2.3. *Suppose the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *is strongly local and* D *is an open subset of* $\mathcal{X}$*. Then*

(i) $\mu_{\langle u \rangle}(D) = 0$ *if* $u \in \mathcal{F}_e$ *and* u *is constant* $\mathcal{E}$*-q.e. on* D*;*

(ii) $\mu_{\langle u \rangle} = \mu_{\langle v \rangle}$ *on* D *for every* $u, v \in \mathcal{F}$ *so that* $u - v$ *is a constant* $\mathcal{E}$*-q.e. on* D*.*

Let $\{U_k; k \geq 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$. For $u \in \mathcal{F}_{\text{loc}}$, there is some $u_k \in \mathcal{F}$ so that $u_k = u$ m-a.e. on U_k. Define $\mu_{\langle u \rangle} = \mu_{\langle u_k \rangle}$ on U_k. Since $(\mathcal{E}, \mathcal{F})$ is strongly local, $\mu_{\langle u \rangle}$ is uniquely defined by Proposition 2.3(ii). In view of (2.7), this allows us to extend the definition of $\mathcal{E}$ to $\mathcal{F}_{\text{loc}}$ by setting

$$\mathcal{E}(u, u) := \frac{1}{2} \mu_{\langle u \rangle}(\mathcal{X}), \quad u \in \mathcal{F}_{\text{loc}}. \tag{2.11}$$

In this paper, we will use time change of Dirichlet form and its associated Hunt process so we need the notion of smooth measure. The following definition is from [CF, Definition 2.3.13].

Definition 2.4. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet space on $L^2(\mathcal{X}; m)$. A (positive) measure μ on $\mathcal{X}$ is *smooth* if it satisfies the following conditions:

(a) μ charges no $\mathcal{E}$-polar set;

(b) there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $\mu(F_k) < \infty$ for all $k \geq 1$.

By [CF, Theorem 1.2.14], the above definition of smooth measure is equivalent to that defined in [FOT, p.83]. Cleary every positive Radon measure charging no $\mathcal{E}$-polar set is smooth, as in this case we can take $\mathcal{E}$-nest $\{F_k\}$ to be the closure of an increasing sequence of relatively compact open sets whose union is $\mathcal{X}$. We say $D \subset \mathcal{X}$ is *quasi open* if there exists an $\mathcal{E}$-nest $\{F_n\}$ such that $D \cap F_n$ is an open subset of F_n in the relative topology for each $n \in \mathbb{N}$. The complement of a quasi open set is called *quasi closed*.

Definition 2.5. (See [CF, Definition 3.3.4] or [FOT, p.190].) Let μ be a Borel smooth measure. A set $F \subset \mathcal{X}$ is called a quasi support of μ if it satisfies the following:
(a) F is quasi closed and $\mu(\mathcal{X} \setminus F) = 0$.
(b) If $\tilde{F}$ is another set that satisfies (a), then $F \setminus \tilde{F}$ is $\mathcal{E}$-polar.
We say that μ has *full quasi support* if $\mathcal{X}$ is a quasi support of μ.

Except in Remark 3.12, we assume in the remaining of this paper that $(\mathcal{E}, \mathcal{F})$ is a symmetric strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. We call $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ a metric measure Dirichlet (MMD) space. Sometimes, to emphasize its dependence on the symmetrizing measure, we write $\mathcal{F}^m$ for $\mathcal{F}$. Let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ be the diffusion process associated with $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, whose lifetime is denoted as ζ. The one-point compactification of the locally compact metric space $(\mathcal{X}, d)$ is denoted as $\mathcal{X}_\partial := \mathcal{X} \cup \{\partial\}$.

3 Local regularity for transient spaces

Since $(\mathcal{E}, \mathcal{F})$ is strongly local, by Proposition 2.2, its corresponding Hunt process X is a diffusion that admits no killings inside $\mathcal{X}$. Thus there exists a Borel properly exceptional set $\mathcal{N}_0$ so that the Hunt process X, whose lifetime is denoted by ζ, can start from every point in $\mathcal{X} \setminus \mathcal{N}_0$ and that

$$\mathbb{P}^x(X_t \text{ is continuous in } t \in [0, \zeta) \text{ and } X_{\zeta-} = \partial) = 1 \qquad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0. \tag{3.1}$$

Here we used the convention that $X_{\infty-} := X_\infty := \partial$. It follows that

$$\mathbb{P}^x(X_t = x \text{ for all } t \in [0, \zeta) \text{ and } \zeta < \infty) = 0 \qquad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0. \tag{3.2}$$

In this section we assume in addition that $(\mathcal{E}, \mathcal{F})$ is transient. In view of [CF, Theorem 3.5.2], by enlarging the Borel properly exceptional set $\mathcal{N}_0$ if needed, we may and do assume that

$$\mathbb{P}^x\left(\zeta = \infty \text{ and } \lim_{t\to\infty} X_t = \partial\right) = \mathbb{P}^x\left(\zeta = \infty\right) \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0. \tag{3.3}$$

For a nearly Borel measurable set $A \subset \mathcal{X}$, define the stopping times

$$\sigma_A = \inf\{t > 0 : X_t \in A\}, \quad \tau_A = \sigma_{A^c} = \inf\{t > 0 : X_t \notin A\},$$

and write τ_x for $\tau_{\{x\}}$.

Lemma 3.1. *For each fixed $x \in \mathcal{X}$, $\mathbb{P}^x(\tau_x > 0) = 0$ for every $x \in \mathcal{X} \setminus \mathcal{N}_0$.*

Proof. We have by (3.2) and (3.3) that $\tau_x < \zeta$ $\mathbb{P}^x$-a.s. for every $x \in \mathcal{X} \setminus \mathcal{N}_0$. Clearly $X_{\tau_x} = x$ on $\{\tau_x < \zeta\}$ since X is a diffusion. Let $A_x := \{\tau_x > 0\}$. Since $A_x = A_x^c \circ \theta_{\tau_x}$ on $\{0 < \tau_x < \zeta\}$, we have by the strong Markov property of X that for $x \in \mathcal{X} \setminus \mathcal{N}_0$,

$$\mathbb{P}^x(A_x) = \mathbb{E}^x\left[\mathbb{P}^{X_{\tau_x}}(A_x^c); 0 < \tau_x < \zeta\right] = \mathbb{P}^x(A_x^c)\,\mathbb{P}^x(0 < \tau_x < \zeta) = (1 - \mathbb{P}^x(A_x))\mathbb{P}^x(A_x).$$

It follows that $\mathbb{P}^x(A_x) = 0$. $\square$

Denote by $\{P_t; t \geq 0\}$ the transition semigroup of the process X; that is,

$$P_t f(x) = \mathbb{E}^x[f(X_t)], \qquad x \in \mathcal{X} \setminus \mathcal{N}_0,\ t > 0,\ f \in \mathcal{B}_+(\mathcal{X}),$$

with the convention that $f(\partial) := 0$. Define the Green operator G by

$$Gf(x) := \mathbb{E}^x \int_0^\infty f(X_t)dt = \int_0^\infty \mathbb{E}^x[f(X_t)]dt = \int_0^\infty P_t f(x)dt, \quad x \in \mathcal{X} \setminus \mathcal{N}_0,\ f \in \mathcal{B}_+(\mathcal{X}).$$

Lemma 3.2. *By enlarging the Borel properly exceptional set $\mathcal{N}_0$ if necessary, there is an $L^1(\mathcal{X}; m)$-integrable function g_0 that takes values in $(0, 1]$ on $\mathcal{X}$ such that*

$$Gg_0(x) \leq 1 \textit{ for } x \in \mathcal{X} \setminus \mathcal{N}_0, \quad Gg_0 \in \mathcal{F}_e \quad \textit{and} \quad \mathcal{E}(Gg_0, Gg_0) \leq 1. \tag{3.4}$$

Proof. By [CF, Theorem 2.1.5(i)], there is an $L^1(\mathcal{X}; m)$-integrable function g_1 bounded by 1, strictly positive on $\mathcal{X}$, such that $Gg_1 < \infty$ m-a.e. on $\mathcal{X}$ and $Gg_1 \in \mathcal{F}_e$ with $\mathcal{E}(Gg_1, Gg_1) \leq 1$. Since Gg_1 is excessive and hence finely continuous, by enlarging the properly exceptional set $\mathcal{N}_0$ if necessary, we may and do assume that $Gg_1(x) < \infty$ for every $x \in \mathcal{X} \setminus \mathcal{N}_0$. Let $g_0 = \sum_{k=1}^\infty k^{-1} 2^{-k} 1_{\{Gg_1 \leq k\}} g_1 + 1_{\mathcal{N}_0}$. Then g_0 is strictly positive on $\mathcal{X}$,

$$Gg_0(x) \leq 1 \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0,$$

$Gg_0 \in \mathcal{F}_e$ and $\mathcal{E}(Gg_0, Gg_0) \leq \mathcal{E}(Gg_1, Gg_1) \leq 1$. $\square$

It follows from (3.4) that for every $x \in \mathcal{X} \setminus \mathcal{N}_0$, $G(x, dy)$, defined by $G(x, A) = G1_A(x)$, is a σ-finite measure on $\mathcal{X}$. By the symmetry of the process X, each P_t is a symmetric operator in $L^2(\mathcal{X}; m)$. Hence

$$\int_{\mathcal{X}} g(x)Gf(x)m(dx) = \int_{\mathcal{X}} f(x)Gg(x)m(dx) \qquad \text{for } f, g \in \mathcal{B}_+(\mathcal{X}). \tag{3.5}$$

Definition 3.3. For a nearly Borel measurable non-negative function u on $\mathcal{X}$, we say it is harmonic in a ball $B(x_0, r_0)$ if there is a Borel properly exceptional set $\mathcal{N} \supset \mathcal{N}_0$ such that for every $r \in (0, r_0)$,

$$\mathbb{E}^x \left[|u(X_{\tau_{B(x_0,r)}}|) \right] < \infty \quad \text{ and } \quad u(x) = \mathbb{E}^x \left[u(X_{\tau_{B(x_0,r)}}) \right] \quad \text{for every } x \in B(x_0, r) \setminus \mathcal{N}.$$

We say u is harmonic in an open subset $D \subset \mathcal{X}$ if for every $x_0 \in D$, u is harmonic in an open ball $B(x_0, r) \subset D$ centered at x_0.

The equivalence of the above probabilistic definition of harmonic functions with the analytic characterization of harmonic functions can be found in [Che]. Clearly, for every bounded $0 \leq f \leq cg_0$ for some $c > 0$, $u(x) := Gf(x)$ is harmonic in $\mathcal{X} \setminus \text{supp}[f]$.

Definition 3.4. *We say condition (HC) holds if there is an $\mathcal{E}$-nest $\{F_n; 1 \geq 1\}$ consisting of an increasing sequence of compact subsets with $\mathcal{N}_0 \subset \mathcal{X} \setminus \cup_n F_n$ such that if $x_0 \in \mathcal{X}$ and $r \in (0, 1]$, and f has compact support in $B(x_0, 2r)^c$, and satisfies $0 \leq f \leq cg_0$ for some $c > 0$, then $Gf(x)$ is continuous in $B(x_0, r) \cap F_n$ for every $n \geq 1$.*

Note that it follows from the definition of $\mathcal{E}$-nest in Section 2, if $\{F_n; n \geq 1\}$ is an $\mathcal{E}$-nest, then so is $\{K_n; n \geq 1\}$, where $K_n = \text{supp}[1_{F_n} m]$. Thus without loss of generality, in this paper we always assume that the $\mathcal{E}$-nest in (HC) has the property that $F_n = \text{supp}[1_{F_n} m]$ for every $n \geq 1$. For an $\mathcal{E}$-nest $\{F_n\}$, $\mathcal{N} = \mathcal{X} \setminus \cup_n F_n$ is $\mathcal{E}$-polar and, in particular, has zero m-measure.

Theorem 3.5. *Assume that condition (HC) holds with $\mathcal{E}$-nest $\{F_n\}$. Let $\mathcal{N}$ be a Borel properly exceptional set that contains $\mathcal{X} \setminus \cup_n F_n \supset \mathcal{N}_0$. Then for every $x \in \mathcal{X} \setminus \mathcal{N}$, $G(x, dy)$ is absolutely continuous with respect to m. Consequently, for every $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, $P_t(x, dy) := \mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to m.*

Proof. It follows from (3.5) that $G(x, A) = 0$ m-a.e. on $\mathcal{X}$ for every $A \subset \mathcal{X}$ with $m(A) = 0$ (by taking $f = 1_A$ and $g = 1$). Let g_0 be the strictly positive function from Lemma 3.2. Fix $x_0 \in \mathcal{X} \setminus \mathcal{N}$ and $r > 0$. For $j \geq 1$, let $E_j = \{x \in \mathcal{X} : 2^{-j} < g_0(x) \leq 2^{1-j}\}$. Then the E_j form a partition of $\mathcal{X} \setminus \mathcal{N}_0$. Let $A \subset B(x_0, r)^c$ with $m(A) = 0$. Since $1_{A \cap E_j} \leq 2^j g_0$, we have by condition (HC) that for each $k \geq 1$ and $i \geq 1$ the function $x \mapsto G(x, A \cap F_i \cap E_j)$ is continuous and therefore zero on $B(x_0, r) \cap F_k$. Thus $G(x, A \cap F_i \cap E_j) = 0$ for every $x \in B(x_0, r) \setminus \mathcal{N}$. Consequently, $G(x, A) = \sum_{i,j=1}^{\infty} G(x, A \cap F_j \cap E_j) = 0$ for every $x \in B(x_0, r) \setminus \mathcal{N}$. In particular, this shows that for every $x_0 \in \mathcal{X} \setminus \mathcal{N}$,

$$G(x_0, dy) \text{ is absolutely continuous with respect to } m(dy) \text{ on } \mathcal{X} \setminus \{x_0\}. \tag{3.6}$$

We claim that $G(x_0, dy)$ is absolutely continuous with respect to $m(dy)$ on $\mathcal{X}$. This is clearly true if $m(\{x_0\}) > 0$. We thus assume $m(\{x_0\}) = 0$ and set

$$h(x) := (G 1_{\{x_0\}})(x) = \mathbb{E}^x \int_0^\zeta 1_{\{x_0\}}(X_s) ds.$$

Then h is a harmonic function on $\mathcal{X} \setminus \{x_0\}$ and since $m(\{x_0\}) = 0$, we have by (3.5) that $h = 0$ m-a.e. on $\mathcal{X}$. Further, by condition (HC), $h(x) = 0$ on $\mathcal{X} \setminus (\mathcal{N} \cup \{x_0\})$. Thus if $A = \{y : h(y) > 0\}$ then $A \subset \mathcal{N} \cup \{x_0\}$. Since $\mathcal{N}$ is properly exceptional and $x_0 \notin \mathcal{N}$, $\mathbb{P}^{x_0}(T_{\mathcal{N}} < \infty) = 0$. Let $\delta > 0$. Let $F = \{(t, \omega) : 0 < t < \delta, X_t(\omega) \in \mathcal{X} \setminus (\mathcal{N} \cup \{x_0\})\}$, and $D_F(\omega) = \inf\{t : (t, \omega) \in F\}$ be the 'debut' of F. By Lemma 3.1 we have $\mathbb{P}^{x_0}(D_F < \infty) = 1$. So by the section theorem [DM, Theorem 44] there exists a stopping time T such that $\mathbb{P}^{x_0}(T < \infty) = 1$ and $(T(\omega), \omega) \in F$ for all ω. Hence $h(X_T(\omega)) = 0$ for all ω such that $T(\omega) < \infty$, and so

$$h(x_0) \leq \delta + \mathbb{E}^{x_0} h(X_T) = \delta.$$

As δ is arbitrary we deduce that $h(x_0) = 0$. This together with (3.6) shows that $G(x, dy)$ is absolutely continuous with respect to $m(dy)$ on $\mathcal{X}$ for every $x \in \mathcal{X} \setminus \mathcal{N}$. That $\mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to m for every $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$ follows immediately from [CF, Proposition 3.1.11] or [FOT, Theorem 4.2.4]. □

With Theorem 3.5 at hand, we can deduce the following. Denote by $\mathcal{B}^*(\mathcal{X})$ and $\mathcal{B}^*(\mathcal{X} \times \mathcal{X})$ the completion of Borel σ-fields $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{X} \times \mathcal{X})$ under m and $m \times m$, respectively.

Theorem 3.6. *Assume that condition (HC) holds with $\mathcal{E}$-nest $\{F_n\}$. Let $\mathcal{N}$ be a Borel properly exceptional set that contains $\mathcal{X} \setminus \cup_n F_n \supset \mathcal{N}_0$. Then there exists a non-negative jointly $\mathcal{B}^*(0, \infty) \times \mathcal{B}^*(\mathcal{X} \times \mathcal{X})$-measurable function $p(t, x, y)$ on $(0, \infty) \times (\mathcal{X} \setminus \mathcal{N}) \times (\mathcal{X} \setminus \mathcal{N})$ such that*

(i) *for every $f \in \mathcal{B}_+(\mathcal{X})$, $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, $\mathbb{E}^x f(X_t) = \int_{\mathcal{X}} p(t, x, y) f(y) m(dy)$;*

(ii) $p(t,x,y) = p(t,y,x)$ *for every* $x, y \in \mathcal{X} \setminus \mathcal{N}$ *and* $t > 0$;

(iii) *For every* $t, s > 0$ *and* $x, y \in \mathcal{X} \setminus \mathcal{N}$, $p(t+s,x,y) = \int_{\mathcal{X}} p(t,x,z)p(s,z,y)m(dy)$.

Consequently, $g(x,y) := \int_0^\infty p(t,x,y)dt$, $x, y \in \mathcal{X} \setminus \mathcal{N}$, *is a non-negative jointly* $\mathcal{B}^*(\mathcal{X} \times \mathcal{X})$*-measurable function on* $(\mathcal{X} \setminus \mathcal{N}) \times (\mathcal{X} \setminus \mathcal{N})$ *such that*

(iv) $Gf(x) = \int_{\mathcal{X}} g(x,y)f(y)m(dy)$ *for every* $x \in \mathcal{X} \setminus \mathcal{N}$ *and* $f \in \mathcal{B}_+(\mathcal{X})$;

(v) $g(x,y) = g(y,x)$ *for every* $x, y \in \mathcal{X} \setminus \mathcal{N}$, *and* $x \mapsto g(x,y)$ *is excessive for every* $y \in \mathcal{X} \setminus \mathcal{N}$.

(vi) *For every* $y_0 \in \mathcal{X} \setminus \mathcal{N}$, $x \mapsto g(x,y_0)$ *is harmonic in* $\mathcal{X} \setminus \{y_0\}$.

Proof. We first show that for each $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, X has a pointwisely defined transition density function $p(t,x,y)$. This part is almost the same as that for [BBCK, Theorem 3.1]. For reader's convenience, we spell out the details here.

By Theorem 3.5, for every $t > 0$ and $x \in \mathcal{X} \setminus \mathcal{N}$ there is an integrable kernel $y \mapsto p_0(t,x,y)$ defined on $\mathcal{X}$ such that

$$\mathbb{E}^x\left[f(X_t)\right] = P_t f(x) = \int_{\mathcal{X}} p_0(t,x,y)f(y)dy \qquad \text{for every } f \in \mathcal{B}_b(\mathcal{X}). \tag{3.7}$$

From the semigroup property $P_{t+s} = P_t P_s$, we have for every $t, s > 0$ and $x \in \mathcal{X} \setminus \mathcal{N}$,

$$p_0(t+s,x,y) = \int_{\mathcal{X}} p_0(t,x,z)p_0(s,z,y)m(dz) \qquad \text{for } m\text{-a.e. } y \in \mathcal{X}. \tag{3.8}$$

Note that since P_t is symmetric, we have for each fixed $t > 0$,

$$p_0(t,x,y) = p_0(t,y,x) \qquad \text{for } m\text{-a.e. } (x,y) \in \mathcal{X} \times \mathcal{X}. \tag{3.9}$$

For every $t > 0$ and $x, y \in \mathcal{X} \setminus \mathcal{N}$, let $s \in (0, t/3)$ and define

$$p(t,x,y) := \int_{\mathcal{X}} p_0(s,x,w)\left(\int_{\mathcal{X}} p_0(t-2s,w,z)p_0(s,y,z)m(dz)\right)m(dw). \tag{3.10}$$

By (3.8) and (3.9), the above definition is independent of the choice of $s \in (0, t/3)$. Clearly by (3.9) with $t - 2s$ in place of t and (w, z) in place of (x, y), we see that

$$p(t,x,y) = p(t,y,x) \quad \text{for every } x, y \in \mathcal{X} \setminus \mathcal{N}. \tag{3.11}$$

By the semigroup property, (3.7) and (3.9), we have for any $\phi \geq 0$ on $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{N}$,

$$\begin{aligned}
&\mathbb{E}^x\left[\phi(X_t)\right] \\
&= \int_{\mathcal{X}}\left(\int_{\mathcal{X}} p_0(s,x,w)\left(\int_{\mathcal{X}} p_0(t-2s,w,z)p_0(s,z,y)m(dz)\right)m(dw)\right)\phi(y)m(dy) \\
&= \int_{\mathcal{X}}\left(\int_{\mathcal{X}} p_0(s,x,w)\left(\int_{\mathcal{X}} p_0(t-2s,w,z)p_0(s,y,z)m(dz)\right)m(dw)\right)\phi(y)m(dy) \\
&= \int_{\mathcal{X}} p(t,x,y)\phi(y)m(dy).
\end{aligned} \tag{3.12}$$

Thus for each $x \in \mathcal{X} \setminus \mathcal{N}$, $p(t,x,y)$ coincides with $p_0(t,x,y)$ for m-a.e. $y \in \mathcal{X}$. For $t, s > 0$ and $x, y \in \mathcal{X} \setminus \mathcal{N}$, take $s_0 \in (0, (t \wedge s)/3)$. We have by (3.8)-(3.10)

$$
\begin{aligned}
&p(t+s,x,y) \\
&= \int_{\mathcal{X}} p_0(s_0,x,w) \left(\int_{\mathcal{X}} p_0(t+s-2s_0,w,z) p_0(s_0,y,z) m(dz) \right) m(dw) \\
&= \int_{\mathcal{X}^5} p_0(s_0,x,w) p_0(t-2s_0,w,u_1) p_0(s_0,u_1,u_2) p_0(s_0,u_2,v) p_0(s-2s_0,v,z) \\
&\qquad p_0(s_0,y,z) m(dw) m(du_1) m(du_2) m(dz) m(dv) \\
&= \int_{\mathcal{X}} p(t,x,v) p(s,v,y) m(dv).
\end{aligned} \tag{3.13}
$$

Note that for each $t > 0$, $\|P_t\|_{L^2 \to L^2} \leq 1$. Since

$$
\int_{\mathcal{X}} f(x) P_t g(x) m(dx) = \int_{\mathcal{X}} f(x) \left(\int_{\mathcal{X}} p(t,x,y) g(y) m(dy) \right) m(dx), \quad f, g \in L^2(\mathcal{X}; m),
$$

we conclude from [FOT, Lemma 1.4.1(i)] that $p(t,x,y)$ is a $\mathcal{B}^*(\mathcal{X} \times \mathcal{X})$-measurable function in (x,y) on $\mathcal{X} \times \mathcal{X}$. As P_t is a strongly continuous semigroup in $L^2(\mathcal{X}; m)$, we have by [DS, Theorem III.11.17] that $p(t,x,y)$ is jointly $\mathcal{B}^*([0,\infty) \times \mathcal{B}^*(\mathcal{X} \times \mathcal{X})$-measurable on $[0,\infty) \times (\mathcal{X} \times \mathcal{X})$.

Define $g(x,y) := \int_0^\infty p(t,x,y) dt$ for $x, y \in \mathcal{X} \setminus \mathcal{N}$. It follows from (3.12) and Fubini theorem that for every $f \in \mathcal{B}_+(\mathcal{X})$,

$$
Gf(x) := \mathbb{E}^x \int_0^\infty f(X_s) ds = \int_{\mathcal{X}} g(x,y) f(y) m(dy) \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}.
$$

Clearly by (3.11), $g(x,y) = g(y,x)$ for every $x, y \in \mathcal{X} \setminus \mathcal{N}$. Note that for each fixed $y \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, by Fubini theorem and (3.13),

$$
P_t G(\cdot, y)(x) = \int_{\mathcal{X}} p(t,x,z) g(z,y) dz = \int_t^\infty p(s,x,y) ds \leq g(x,y) \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N},
$$

and $\lim_{t \downarrow 0} P_t g(\cdot, y)(x) = g(x,y)$. This shows that for each fixed $y \in \mathcal{X} \setminus \mathcal{N}$, $x \mapsto g(x,y)$ is an excessive function of X.

The proof of (vi) is similar to that for [KW, Proposition 6.2]. Let $y_0 \in \mathcal{X} \setminus \mathcal{N}$. For any $x_0 \in (\mathcal{X} \setminus \mathcal{N}) \setminus \{y_0\}$, take $0 < r < d(x_0, y_0)$ and $0 < r_1 < d(x, y_0) - r$. For any non-negative $f \in C_c(B(y_0, r_1))$, by Fubini theorem and the strong Markov property of X, for each $x \in B(x_0, r)$,

$$
\begin{aligned}
\int_{B(y_0,r_1)} \mathbb{E}^x \left[g(X_{\tau_{B(x_0,r)}}, y) \right] f(y) m(dy) &= \mathbb{E}^x \left[(Gf)(X_{\tau_{B(x_0,r)}}) \right] \\
&= Gf(x) = \int_{B(y_0,r_1)} g(x,y) f(y) m(dy).
\end{aligned}
$$

Hence for each $x \in B(x_0, r)$,

$$
\mathbb{E}^x \left[g(X_{\tau_{B(x_0,r)}}, y) \right] = g(x,y) \quad \text{for } m\text{-a.e. } y \in B(y_0, r_1).
$$

Since $y \mapsto g(z,y)$ is excessive, it follows from the monotone convergence theorem, Fubini theorem, the fine continuity of $y \mapsto g(z,y)$ and Fatou's lemma that

$$\begin{aligned}
\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y_0)\right] &= \lim_{t\downarrow 0}\left(P_t\mathbb{E}^x g(X_{\tau_{B(x_0,r)}},\cdot)\right)(y_0) \\
&\geq \limsup_{t\downarrow 0}\left(P_t g(x,\cdot)1_{B(y_0,r_1)}\right)(y_0) \\
&\geq \mathbb{E}^{y_0}\left[\liminf_{t\downarrow 0} g(x,X_t)1_{B(y_0,r_1)}(X_t)\right] \\
&= g(x,y_0).
\end{aligned}$$

On there other hand, clearly $\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y_0)\right] \leq g(x,y_0)$ as $x \mapsto g(x,y_0)$ is excessive for X. Thus we have $\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y_0)\right] = g(x,y_0)$ for every $x \in B(x_0,r)$. This proves that $x \mapsto g(x,y_0)$ is harmonic in $\mathcal{X} \setminus \{y_0\}$. □

Remark 3.7. There are gaps in the proofs of the existence of a Green function in [BBK] and [GH, Lemma 5.2]. For details of the gap in [GH], see [BM2, Remark 4.19]. The gap in [BBK] is that it is not proven that the Green's function is an integral kernel of the Green operator (cf. Theorem 3.6 (iv)).

We next give a sufficient condition for (HC).

Definition 3.8. (i) A positive function $\rho_0(x)$ on $\mathcal{X}$ is said to be *distance to the boundary like function on* $\mathcal{X}$ if for any $x \in \mathcal{X}$ and $y \in B(x,\rho_0(x))$, $\rho_0(y) \geq \rho_0(x) - d(x,y)$.

(ii) We say that the *(non-scale-invariant) elliptic Harnack inequality* (Ha) holds on $\mathcal{X}$ if there is a positive distance to the boundary like function ρ_0 on $\mathcal{X}$ bounded by 1 so that for any ball $B = B(x_0,r)$ in $\mathcal{X}$ with $0 < r < \rho_0(x)$, there is a constant $C_B > 1$ and $\delta_B \in (0,1)$ such that for any non-negative $u \in \mathcal{F}$ that is harmonic in $B(x_0,r)$,

$$\operatorname{esssup}_{B(x_0,\delta_B r)} u \leq C_B \operatorname{essinf}_{B(x_0,\delta_B r)} u. \tag{3.14}$$

Remark 3.9. (i) Any positive constant function is a distance to the boundary like function on $\mathcal{X}$. If ρ_1 and ρ_2 are two distance to the boundary like functions on $\mathcal{X}$, then so is $\rho_1(x) \wedge \rho_2(x)$. If $(\mathcal{X},d)$ is an open subset of another metric space $(\mathcal{Y},d)$, then clearly

$$\rho_0(x) := \inf\{x : d(x,y) : y \in \mathcal{Y} \setminus \mathcal{X}\}$$

is a distance to the boundary like function on $\mathcal{X}$.

(ii) Note that if property (3.14) holds for a ball $B = B(x,r)$ with constants C_B and δ_B then it holds for any larger ball $B(x,R)$ with constants C_B and $\delta_B r/R$.

Proposition 3.10. *Assume that (Ha) holds and that*

$$\lambda_{\mathcal{X}} := \inf\left\{\mathcal{E}(f,f) : f \in \mathcal{F} \text{ with } \|f\|_{L^2(\mathcal{X};m)} = 1\right\} > 0.$$

Then (HC) holds.

Proof. Since $\lambda_{\mathcal{X}} > 0$, $Gf \in L^2(\mathcal{X}; m)$ for every $f \in L^2(\mathcal{X}; m)$ with $\|Gf\|_{L^2} \leq \lambda_{\mathcal{X}}^{-1} \|f\|_{L^2}$. Under $\lambda_{\mathcal{X}} > 0$ and (Ha), for every $x_0 \in \mathcal{X}$, $r \in (0, 1]$, and any ball $B(y_0, R) \subset \mathcal{X} \setminus B(x_0, r)$ with $R \in (0, 1]$, by the same argument as that for [GH, (5.10)], we have for any $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(y_0, \delta_{B(y_0,R)}R))^c$,

$$\text{esssup}_{B(x_0,\delta r)} |Gf| \leq \frac{C_{B(x_0,r)} C_{B(y_0,R)}}{\lambda_{\mathcal{X}} \sqrt{m(B(x_0, \delta_{B(x_0,r)} r)) m(B(y_0, \delta_{B(y_0,R)} R))}} \|f\|_{L^1(\mathcal{X};m)}. \tag{3.15}$$

Let $\{x_k; k \geq 1\} \subset \mathcal{X}$ be a dense sequence of points in $\mathcal{X}$, and

$$\begin{aligned} \Lambda \ := \ & \big\{ \eta = (x_i, x_j, r_k, r_l) : i, j, k, l \geq 1, r_k \in \mathbb{Q} \cap (0, \rho_0(x_i)), r_l \in \mathbb{Q} \cap (0, \rho_0(x_j)) \\ & \qquad \text{with } B(x_i, r_k) \cap B(x_j, r_l) = \emptyset \big\}. \end{aligned}$$

Let $C_{i,k} > 1$ and $\delta_{i,k} \in (0, 1)$ be the constants in (Ha) for Harnack inequality in the ball $B(x_i, r_k)$. Note that Λ is a countable set. For each $\eta = (x_i, x_j, r_k, r_l) \in \Lambda$, let $\{f_p, p \geq 1\} \subset C_c(B(x_j, \delta_{j,l} r_l))$ be dense in $L^1((B(x_j, \delta_{j,l} r_l); m)$. Since $f_k \in L^2(\mathcal{X}; m)$, $Gf_k \in \mathcal{F}$ and it is quasi-continuous by the 0-order version of [CF, Proposition 3.1.9] or [FOT, Theorem 4.2.3]. Thus there is an $\mathcal{E}$-nest $\{F_n^{(\eta)}, n \geq 1\}$ consisting of an increasing sequence of compact sets such that Gf_p is continuous on each $F_n^{(\eta)}$ for every integer $p \geq 1$; see [CF, Lemma 1.3.1]. Let $\mathcal{N}_\eta := \mathcal{X} \setminus \cup_{n=1}^\infty F_n^{(\eta)}$, which is $\mathcal{E}$-polar and in particular has zero m-measure, and $C_\eta := \frac{C_{i,k} C_{j,l}}{\lambda_{\mathcal{X}} \sqrt{m(B(x_i, \delta_{i,k} r_k)) m(B(x_j, \delta_{j,l} r_l))}}$. Inequality(3.15) yields that for every $n \geq 1$,

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \cap F_n^{(\eta)}} |Gf_{k_1}(x) - Gf_{k_2}(x)| \leq C_\eta \|f_{k_1} - f_{k_2}\|_{L^1(B(x_j, \delta_{j,l} r_l); m)}.$$

Since $\{f_p, p \geq 1\} \subset C_c(B(x_j, \delta_j r_j)$ is dense in $L^1(B(x_j, \delta_{j,l} r_l); m)$, it follows that Gf is continuous on each $B(x_i, \delta_{i,k} r_k) \cap F_n^{(\eta)}$ and

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} |Gf(x)| \leq C_\eta \|f\|_{L^1(B(x_j, \delta_{j,l} r_l); m)},$$

for every $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(x_j, \delta_{j,l} r_l)^c$. By [CF, Lemma 1.3.1] and its proof, by taking suitable intersections of $F_{n_k}^{(\eta)}$'s, there is an $\mathcal{E}$-nest $\{F_n, n \geq 1\}$ consisting of an increasing sequence of compact subsets of $\mathcal{X}$ such that for every $\eta \in \Lambda$, Gf is continuous on each $B(x_i, \delta_{i,k} r_k) \cap F_n$ and

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}} |Gf(x)| \leq C_\eta \|f\|_{L^1(B(x_j, \delta_{j,l} r_l); m)} \tag{3.16}$$

for every $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(x_j, \delta_{j,l} r_l)^c$, where $\mathcal{N} := \mathcal{X} \setminus \cup_n F_n$ which is $\mathcal{E}$-polar.

As $\{x_i\}$ is dense in $\mathcal{X}$, For any compact subset K of $\mathcal{X}$, one can deduce from (3.16) by finite covering that for any $f \in L^1(\mathcal{X}; m)$ that vanishes outside K, Gf is continuous on each $K^c \cap F_n$, and for any $x_0 \in K^c$, there is some $r_0 > 0$ with $B(x_0, r_0) \subset K^c$ so that

$$\sup_{x \in B(x_0, r) \setminus \mathcal{N}} |Gf(x)| \leq C(K, B) \|f\|_{L^1(K; m)} \tag{3.17}$$

□

Under the assumption of Theorem 3.10, we have by (3.17) that for every compact subset $K \subset \mathcal{X}$, $G(x, K) < \infty$ for $x \in (\mathcal{X} \setminus \mathcal{N}) \setminus K$.

Using a time change argument, we can remove the assumption of $\lambda_{\mathcal{X}} > 0$ in Proposition 3.10.

Theorem 3.11. *Assume that (Ha) holds. Then (HC) holds. Consequently, the conclusions of Theorems 3.5 and 3.6 hold.*

Proof. Recall that our running assumption is that the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{X};m)$ (or equivalently, its associated Hunt process X) is transient. Let g_0 be as in Lemma 3.2, and $\mu(dx)=g_0(x)m(dx)$. We now make a time change of X via the inverse of the positive continuous additive functional $A_t:=\int_0^t g_0(X_s)ds$. That is, let $Y_t=X_{\tau_t}$, where $\tau_t:=\inf\{s>0: A_s>t\}$. Then Y is μ-symmetric and transient, and its extended Dirichet form is the same as that of X; see [CF, FOT] (since μ and m are mutually absolutely continuous). So the Dirichlet form of Y is $(\mathcal{E},\mathcal{F}_e\cap L^2(\mathcal{X};\mu))$ on $L^2(\mathcal{X};\mu)$. Since Y and X share the same family of harmonic functions, (Ha) holds for Y. We claim that

$$\lambda_{\mathcal{X}}^Y:=\inf\{\mathcal{E}(f,f): f\in\mathcal{F}_e\cap L^2(\mathcal{X};\mu) \text{ with } \|f\|_{L^2(\mathcal{X};\mu)}=1\}\geq 1. \tag{3.18}$$

Denote by $\widetilde{G}$ the Green potential of Y, that is, for $f\geq 0$ on $\mathcal{X}$,

$$\widetilde{G}f(x):=\mathbb{E}^x\int_0^\infty f(Y_t)dt=\mathbb{E}^x\int_0^\infty f(X_{\tau_t})dt.$$

Using the time change, we see that $\widetilde{G}f(x)=\mathbb{E}^x\int_0^\infty (fg_0)(X_t)dt=G(fg_0)(x)$. In particular, we have $\widetilde{G}1=Gg_0\leq 1$. Thus for $u\in L^2(\mathcal{X};\mu)$, by Cauchy-Schwarz and the symmetry of $\widetilde{G}$ with respect to μ,

$$\begin{aligned}\int_{\mathcal{X}}(\widetilde{G}u)^2(x)\mu(dx) &\leq \int_{\mathcal{X}}\widetilde{G}(u^2)(x)\widetilde{G}1(x)\mu(dx)\leq\int_{\mathcal{X}}\widetilde{G}(u^2)(x)\mu(dx)\\ &\leq \int_{\mathcal{X}}u(x)^2\widetilde{G}1(x)\mu(dx)\leq\int_{\mathcal{X}}u(x)^2\mu(dx).\end{aligned} \tag{3.19}$$

Since the spectrum of $\widetilde{G}$ as a symmetric operator from $L^2(\mathcal{X};\mu)$ into itself is the reciprocal of that of the infinitesimal generator of Y, we conclude from (3.19) that $\lambda_{\mathcal{X}}^Y\geq 1$. Alternatively, for any $u\in L^2(\mathcal{X};\mu)$, $\int_{\mathcal{X}}u(x)\widetilde{G}u(x)\mu(dx)\leq\int_{\mathcal{X}}u(x)^2\mu(dx)<\infty$ by (3.19). It follows (cf. [CF, Theorem 2.1.12] or [FOT]) that $\widetilde{G}u\in\mathcal{F}_e\cap L^2(\mathcal{X};\mu)$ with $\mathcal{E}(\widetilde{G}u,\widetilde{G}u)\leq\int_{\mathcal{X}}u(x)\widetilde{G}u(x)\mu(dx)$. Hence for $u\in\mathcal{F}_e\cap L^2(\mathcal{X};\mu)$, we have by (3.19) and the Cauchy-Schwarz,

$$\int_{\mathcal{X}}u^2(x)\mu(dx)=\mathcal{E}(\widetilde{G}u,u)\leq\mathcal{E}(u,u)^{1/2}\mathcal{E}(\widetilde{G}u,\widetilde{G}u)^{1/2}\leq\mathcal{E}(u,u)^{1/2}\|u\|_{L^2(\mathcal{X};\mu)}.$$

Consequently,

$$\|u\|_{L^2(\mathcal{X};\mu)}\leq\mathcal{E}(u,u)^{1/2}\quad\text{for every } u\in\mathcal{F}_e\cap L^2(\mathcal{X};\mu).$$

This again proves the claim (3.18).

For process Y, we can take $g_0^Y=1$ in the role of g_0 for X in (3.4) as $\widetilde{G}1=Gg_0\leq 1$. By Theorem 3.10, (HC) holds for process Y. Since $\widetilde{G}f=G(fg_0)$, we conclude that (HC) holds for process X. □

Remark 3.12. All the results in this section in fact hold for any transient strong local quasi-regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on a metrizable Lusin space $(\mathcal{X},d,m)$ that has the property that $m(B)<\infty$ for any finite ball B. This is because any such Dirichlet form is quasi-homeomorphic to a transient strongly local regular Dirichlet form on a locally compact separable metric space;

see, e.g., [CF, Chapter 1] for this and related terminologies. Thus Lemma 3.1 holds for a transient strong local quasi-regular Dirichlet form on a Lusin space through this quasi-homeomorphism. The remaining results in this section do not need to require the strongly local Dirichlet form is regular except that $m(B) < \infty$ for every finite ball. It is shown in [BG] that the non-scale-invariant Harnack inequality fails for infinite-dimensional Ornstein-Uhlenbeck process but any of its bounded harmonic functions are Lipschitz continuous. This gives us an example of a strongly local quasi-regular Dirichlet form on a (non-locally-compact) infinite-dimensional Hilbert space that such that (Ha) fails but (HC) holds. In Example 8.2, we will given an example of an irreducible strongly local Dirichlet form for which (Ha) fails. Moreover, (HC) fails for its part Dirichlet form on an open ball.

4 Green functions

We now drop the hypothesis that $(\mathcal{E}, \mathcal{F})$ is transient.

Definition 4.1. For a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, we say

(i) the *(scale invariant) elliptic Harnack inequality* (EHI) holds if there exist constants $\delta_H \in (0, 1)$ and $C_H \in (1, \infty)$ so that for any $x \in \mathcal{X}$, $R > 0$, and for any nonnegative harmonic function h on a ball $B(x, R)$, one has

$$\operatorname*{ess\,sup}_{B(x,\delta_H R)} h \le C_H \operatorname*{ess\,inf}_{B(x,\delta_H R)} h; \tag{4.1}$$

(ii) the $\mathrm{EHI}_{\le 1}$ holds if (4.1) holds for nonnegative harmonic function on balls $B(x, R)$ with $0 < R \le 1$;

(iii) the *(scale invariant) local elliptic Harnack inequality* $\mathrm{EHI}_{\mathrm{loc}}$ if there is a distance to the boundary like function $\rho_0(x)$ on $\mathcal{X}$ bounded by 1 (see Definition 3.8(i)) so that (4.1) holds for nonnegative harmonic function on balls $B(x, R)$ with $0 < R < \rho_0(x)$.

Remark 4.2. (i) Clearly, the EHI implies the $\mathrm{EHI}_{\le 1}$, the $\mathrm{EHI}_{\le 1}$ implies the $\mathrm{EHI}_{\mathrm{loc}}$, and the $\mathrm{EHI}_{\mathrm{loc}}$ implies (Ha).

(ii) If $(\mathcal{X}, d)$ is a geodesic metric space and inequality (4.1) holds for some value of δ, then it holds for any other $\delta' \in (0, 1)$ with a constant $C_H(\delta')$.

(iii) If the $\mathrm{EHI}_{\mathrm{loc}}$ holds, then iterating the condition (4.1) gives a.e. Hölder continuity of harmonic functions, and it follows that any harmonic function has a continuous modification.

Let D be an open set of $\mathcal{X}$. Note that if $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the $\mathrm{EHI}_{\mathrm{loc}}$, then so does $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$, where $(\mathcal{E}, \mathcal{F}^D)$ is the Dirichlet form for the part process X^D of X killed upon leaving D (see (2.5)). Let D_{diag} denote the diagonal in $D \times D$. For a subset $A \subset \mathcal{X}$, we use $\overline{A}$ to denote its closure and ∂A its boundary.

Definition 4.3. Let D be a non-empty open subset of $\mathcal{X}$ such that D^c is not $\mathcal{E}$-polar. We say that $(\mathcal{E}, \mathcal{F})$ has a *regular Green function* on D if there exists a Green function $g_D(x, y)$ on $D \times D \setminus D_{\mathrm{diag}}$ with the following properties:

(i) (Symmetry) $g_D(x, y) = g_D(y, x)$ for all $(x, y) \in D \times D \setminus D_{\mathrm{diag}}$;

(ii) (Continuity) $g_D(x,y)$ is jointly continuous in $(x,y) \in D \times D \setminus D_{\text{diag}}$;

(iii) (Occupation density) There is a Borel properly exceptional set $\mathcal{N}$ of X such that

$$\mathbb{E}^x \int_0^{\tau_D} f(X_s)ds = \int_D g_D(x,y)f(y)\,m(dy), \quad x \in D \setminus \mathcal{N},$$

for any $f \in C_c(D)$.

(iv) (Harmonicity) For any fixed $x \in D$, the function $y \mapsto g_D(x,y)$ is in $\mathcal{F}^{D\setminus\{x\}}_{\text{loc}}$ and is harmonic in $D \setminus \{x\}$.

(v) (Maximum principles) If $x_0 \in U \Subset D$, then

$$\inf_{U\setminus\{x_0\}} g_D(x_0,\cdot) = \inf_{\partial U} g_D(x_0,\cdot), \qquad \sup_{D\setminus U} g_D(x_0,\cdot) = \sup_{\partial U} g_D(x_0,\cdot). \tag{4.2}$$

We say that $(\mathcal{E},\mathcal{F})$ has *regular Green functions* if for any bounded, non-empty open set $D \subset \mathcal{X}$ whose complement D^c is not $\mathcal{E}$-polar, $(\mathcal{E},\mathcal{F})$ has a regular Green function on D.

Theorem 4.4. *Suppose that the MMD space $(\mathcal{X},d,m,\mathcal{E},\mathcal{F})$ is irreducible and D is an open subset of $\mathcal{X}$ such that D^c is not $\mathcal{E}$-polar.*

(i) *Assume that $(D,d,m|_D,\mathcal{E},\mathcal{F}^D)$ satisfies (Ha). Then $(\mathcal{E},\mathcal{F}^D)$ has a Green function $g_D(x,y)$ in the sense that*

(i.a) *$g_D(x,y)$ is a non-negative jointly $\mathcal{B}^*(D\times D)$-measurable function and there is a Borel properly exceptional set $\mathcal{N}_D$ of X^D such that*

$$\mathbb{E}^x \int_0^{\tau_D} f(X_s)ds = \int_D g_D(x,y)f(y)\,m(dy), \quad x \in D \setminus \mathcal{N}_D, \tag{4.3}$$

for any $f \in C_c(D)$;

(i.b) *$g_D(x,y) = g_D(y,x)$ for every $x,y \in D \setminus \mathcal{N}_D$;*

(i.c) *For every $y_0 \in D \setminus \mathcal{N}_D$, $x \mapsto g(x,y_0)$ is harmonic in $D \setminus \{y_0\}$.*

(ii) *If $(D,d,m|_D,\mathcal{E},\mathcal{F}^D)$ satisfies the EHI_{loc}, then $(\mathcal{E},\mathcal{F})$ has a regular Green function on D.*

Proof. Since $(\mathcal{E},\mathcal{F})$ is irreducible and D^c is not $\mathcal{E}$-polar, the regular Dirichlet form $(\mathcal{E},\mathcal{F}^D))$ on $L^2(D;m|_D)$ is transient by Proposition 2.1.

(i) The conclusion of this part follows directly from Theorem 3.11 by replacing $\mathcal{X}$ and X by D and X^D, respectively.

(ii) Suppose now that $(D,d,m|_D,\mathcal{E},\mathcal{F}^D)$ satisfies the EHI_{loc} in D. Let u be a bounded harmonic function in $B(x_0,2r) \subset D$. Iterating the condition (4.1) yields that there are constants $c_0 > 0$ and $\beta \in (0,1)$ that depend only $\delta_H(D)$ and $C_H(D)$ in (4.1) (with D in place of $\mathcal{X}$) such that

$$|u(x)-u(y)| \le c_0 \|u\|_{L^\infty(B(x_0,2r))} |x-y|^\beta \quad \text{for a.e. } x,y \in B(x_0,r). \tag{4.4}$$

Since (Ha) holds on D, we have by (i) a Green function $g_D(x,y)$ in D. For each fixed $y_0 \in D\setminus\mathcal{N}_D$, $x \mapsto g_D(x,y_0) < \infty$ m-a.e. and is harmonic in $D \setminus \{y_0\}$. It follows from the EHI_{loc} that $x \mapsto g_D(x,y_0)$ is (essentially) locally bounded in $D \setminus \{y_0\}$. By the characterization of harmonic functions in [Che], $g_D(\cdot, y_0) \in \mathcal{F}_{\rm lock}^{D\setminus\{y_0\}}$. So there is an $\mathcal{E}$-nest $\{F_k; k\geq 1\}$ for $D\setminus\{y_0\}$ consisting of an increasing sequence of compact subset of $D\setminus\{y_0\}$ such that $g_D(\cdot, y_0)\in C(F_k)$ for every $k\geq 1$. Hölder estimate (4.4) implies that there is a locally Hölder continuous function $\widetilde g_D(\cdot, y_0)$ on $D\setminus\{y_0\}$ such that $\widetilde g_D(x, y_0)=g_D(x,y_0)$ for every $x\in \cup_{k\geq 1}F_k$. Since $g_D(\cdot, y_0)$ is X^D-excessive, we have for every $x\in D\setminus(\mathcal{N}_D\cup\{y_0\})$, $\mathbb{P}^x$-a.s.,

$$g_D(x,y_0)=\lim_{t\to 0} g_D(X^D_t, y_0)=\lim_{t\to 0}\widetilde g_D(X^D_t, y_0)=\widetilde g_D(x,y_0).$$

This establishes that $g_D(x,y_0)=\widetilde g_D(x,y_0)$ for every $x\in D\setminus(\mathcal{N}_D\cup\{y_0\})$.

For $x_0\in D\setminus \mathcal{N}_D$, define $\widetilde g_D(x_0,y)=\widetilde g(y,x_0)$ for $y\in D$. Note that $y\mapsto \widetilde g_D(x_0,y)$ is continuous in $D\setminus\{x_0\}$. Clearly we have by (i.b) that $\widetilde g_D(x,y)=g_D(x,y)$ for every $x,y\in D\setminus\mathcal{N}_D$ with $x\neq y$. We next show that such defined $\widetilde g_D(x,y)$ on $(D\times D)\setminus(\mathcal{N}_D\times\mathcal{N}_D\cup D_{\rm diag})$ is locally jointly Hölder continuous and hence can be extended to $(D\times D)\setminus D_{\rm diag}$.

Let $x_0,y_0\in D\setminus\mathcal{N}_D$ with $x_0\neq y_0$. There is $r>0$ such that $B(x_0,2r)\cap B(y_0,2r)=\emptyset$. By the $\mathrm{EHI}_{\rm loc}$ in D, for every $x\in B(x_0,r)\setminus\mathcal{N}_D$ and $y\in B(y_0,r)\setminus\mathcal{N}_D$,

$$\widetilde g_D(x,y)\leq C_H(D)\widetilde g_D(x,y_0)\leq C_H(D)^2\widetilde g_D(x_0,y_0).$$

It follows from (4.4) that for $x_1,x_2\in B(x_0,r/2)\setminus\mathcal{N}_D$ and $y_1,y_2\in B(y_0,r/2)\setminus\mathcal{N}_D$,

$$\begin{aligned}|\widetilde g_D(x_1,y_1)-\widetilde g_D(x_2,y_2)| &\leq |\widetilde g_D(x_1,y_1)-\widetilde g_D(x_2,y_1)|+|\widetilde g_D(x_2,y_1)-\widetilde g_D(x_2,y_2)|\\ &\leq C_H(D)^2\widetilde g_D(x_0,y_0)c_0\left(|x_1-x_2|^\beta+|y_1-y_2|^\beta\right).\end{aligned}$$

Consequently $\widetilde g_D(x,y)$ can be extended continuously to $B(x_0,r)\times B(y_0,r)$ and hence to $D\times D\setminus D_{\rm diag}$ as a locally Hölder continuous function. Clearly, $\widetilde g_D(x,y)=\widetilde g_D(y,x)$ for $x,y\in D$ with $x\neq y$, and for each fixed y, $x\mapsto\widetilde g_D(x,y)$ is harmonic in $D\setminus\{y\}$. From now on, we take this jointly continuous version $\widetilde g_D(x,y)$ for the Green function $g_D(x,y)$ in D and drop the tilde from $\widetilde g_D(x,y)$.

Suppose U is a relatively compact open subset of D and $x_0\in U$. Let $r_0>0$ be such that $B(x_0,r_0)\subset U$. For every $y\in D\setminus U$ and for every $r\in(0,r_0)$, we have by the strong Markov property of X that

$$\begin{aligned}\int_{B(x_0,r)} g_D(x,y)m(dx) &= \mathbb{E}^y\int_0^{\tau_D}1_{B(x_0,r)}(X_s)ds=\mathbb{E}^y\mathbb{E}^{X^D_{\sigma_U}}\int_0^{\tau_D}1_{B(x_0,r)}(X_s)ds\\ &= \int_{B(x_0,r)}\mathbb{E}^y tg_D(x,X^D_{\sigma_U})m(dx).\end{aligned}$$

Dividing both sides by $m(B(x_0,r))$ and then taking $r\to 0$ yields that $g_D(x_0,y)=\mathbb{E}^y g_D(x_0,X^D_{\sigma_U})$. It follows then $\sup_{y\in D\setminus U}g_D(x_0,y)=\sup_{y\in\partial U}g_D(x_0,y)$.

For each fix $x\in U\setminus\mathcal{N}_D$ and any Borel measurable function $f\geq 0$ on D, by the strong Markov property of X,

$$\int_D g_D(x,y)f(y)m(dy)=\mathbb{E}^x\int_0^{\tau_D}f(X_s)ds\geq \mathbb{E}^xG_Df(X_{\tau_U})=\int_D\mathbb{E}^x[g_D(X_{\tau_U},y)]f(y)m(dy).$$

Hence $g_D(x,y)\geq\mathbb{E}^x[g_D(X_{\tau_U},y)]$ for m-a.e. $y\in D$. Since for every $z\in D$, $y\mapsto g_D(z,y)$ is continuous on $D\setminus\{z\}$, we have by Fatou's lemma that

$$g_D(x,y)\geq\mathbb{E}^x[g_D(X_{\tau_U},y)]\quad\text{for every } y\in U\setminus\{x\}.$$

Taking $y = x_0$ and by the symmetry of $g_D(x, y)$, we get for every $x \in U \setminus (\mathcal{N}_D \cup \{x_0\})$,

$$g_D(x_0, x) \geq \mathbb{E}^x[g_D(x_0, X_{\tau_U})] \geq \inf_{y \in \partial U} g_D(x_0, y).$$

In the last inequality, we used the fact that $\mathbb{P}^x(\tau_U < \infty) = 1$, which follows from the transience of X^D and compactness of ∂U in view of (3.1) and (3.3). By the continuity of $x \mapsto g_D(x_0, x)$ on $D \setminus \{x_0\}$, one deduces

$$\inf_{y \in U \setminus \{x_0\}} g_D(x_0, y) = \inf_{y \in \partial U} g_D(x_0, y).$$

This shows that $g_D(x, y)$ is a regular Green function on D. □

We next give a sufficient condition for a strongly local MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ to be irreducible. First we present a characterization of irreducibility for such a Dirichlet form, which in fact holds also for any strongly local quasi-regular Dirichlet forms by using quasi-homeomorphism. See [CF, Theorem 5.2.16] for an irreducible characterization for recurrent Dirichlet forms.

Theorem 4.5. *Let $\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. Then the following are equivalent.*

(i) $(\mathcal{E}, \mathcal{F})$ *is irreducible;*

(ii) *If $u \in \mathcal{F}_{\rm loc}$ having $\mathcal{E}(u, u) = 0$, then u is constant $\mathcal{E}$-q.e. on $\mathcal{X}$.*

Proof. (i) ⇒ (ii): Suppose $u \in \mathcal{F}_{\rm loc}$ and $\mathcal{E}(u, u) = 0$. Let $\{U_k; k \geq 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$. Then for each $k \geq 1$, there is some $u_k \in \mathcal{F}$ so that $u_k = u$ m-a.e. on U_k. By Fukushima's decomposition,

$$u_k(X_t) - u_k(X_0) = M_t^{u_k} + N_t^{u_k}, \quad t \geq 0,$$

where M^{u_k} is a martingale additive functional of X having finite energy and N^{u_k} is a continuous additive functional of X having zero energy. Since $\mu_{\langle u_k \rangle}(U_k) = \mu_{\langle u \rangle}(U_k) = 0$, we have $M_t^{u_k} = 0$ for every $t \in [0, \tau_{U_k}]$ and

$$\mathcal{E}(u_k, \varphi) = 0 \quad \text{for every } \varphi \in \mathcal{F} \cap C_c(U_k).$$

The last display implies by [FOT, Theorem 5.4.1] that $N_t^{u_k} = 0$ for $t \in [0, \tau_{U_k}]$. Consequently, we have for each $k \geq 1$ that almost surely

$$u(X_t) - u(X_0) = u_k(X_t) - u_k(X_0) = 0 \quad \text{for } t \in [0, \tau_{U_k}].$$

As $\lim_{k \to \infty} \tau_{U_k} = \zeta$, we have for quasi-every $x \in \mathcal{X}$, $\mathbb{P}^x$-a.s.,

$$u(X_t) = u(X_0) \quad \text{for every } t \in [0, \zeta). \tag{4.5}$$

For $a \in \mathbb{R}$, define $A_a = \{x \in \mathcal{X} : u(x) > a\}$, which is quasi open. In view of (4.5), $P_t 1_{A_a} \leq 1_{A_a}$ m-a.e. on $\mathcal{X}$. Hence by the irreducibility of $(\mathcal{E}, \mathcal{F})$, either $m(A_a) = 0$ or $m(\mathcal{X} \setminus A_a) = 0$. This proves that u is constant m-a.e. and hence $\mathcal{E}$-q.e. on $\mathcal{X}$.

(ii) ⇒ (i): Were the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ not irreducible, there would exist a nearly Borel measurable set A with $m(A) > 0$ and $m(A^c) > 0$ such that for $1_A u \in \mathcal{F}$ for any

$u \in \mathcal{F}$ and (2.1) holds. In particular, both 1_A and 1_{A^c} are in $\mathcal{F}_{\rm loc}$. Since $(\mathcal{E}, \mathcal{F})$ is strong local, by Proposition 2.2,

$$\mathcal{E}(1_A u, 1_{A^c} v) = 0 \quad \text{for every } u, v \in \mathcal{F}.$$

This together with (2.8) gives that for any bounded $u, v \in \mathcal{F}$,

$$\begin{aligned} \int_{\mathcal{X}} v(x)\mu_{\langle 1_A u\rangle}(dx) &= 2\mathcal{E}(1_A u, 1_A uv) - \mathcal{E}(1_A u^2, v) \\ &= 2\mathcal{E}(u, (1_A v)u) - \mathcal{E}(u^2, 1_A v) \\ &= \int_{\mathcal{X}} (1_A v)(x)\mu_{\langle u\rangle}(dx). \end{aligned}$$

This yields

$$\mu_{\langle 1_A u\rangle}(dx) = 1_A(x)\mu_{\langle u\rangle}(dx). \tag{4.6}$$

Let $\{U_k; k \geq 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$ and $u_k \in \mathcal{F} \cap C_c(\mathcal{X})$ such that $u_k = 1$ on U_k. We have by (4.6) and Proposition 2.3(i) that

$$\mu_{\langle 1_A\rangle}(U_k) = \mu_{\langle 1_A u_k\rangle}(U_k) = \mu_{\langle u_k\rangle}(U_k \cap A) = 0 \quad \text{for each } k \geq 1,$$

and so

$$\mathcal{E}(1_E, 1_E) = \frac{1}{2}\mu_{\langle 1_A\rangle}(\mathcal{X}) = 0.$$

This contradiction, as 1_A is not constant m-a.e. on $\mathcal{X}$, establishes that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is irreducible. □

Theorem 4.5 in particular implies that irreducibility is invariant under form-bounded perturbations in the following sense. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and $(\mathcal{X}, d, \mu, \mathcal{E}', \mathcal{F})$ are two strongly local regular MMD spaces such that the Radon measure μ does not change $\mathcal{E}$-polar sets and full quasi support on $\mathcal{X}$, and there is a constant $C \geq 1$ so that

$$C^{-1}\mathcal{E}(u, u) \leq \mathcal{E}'(u, u) \leq C\mathcal{E}(u, u) \quad \text{for } u \in \mathcal{F},$$

then $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is irreducible if and only if so is $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, \mu)$. Note that, by [CF, Theorem 5.2.11 and Exercise 3.3.2(iii)], the condition that μ is a smooth measure of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ with full quasi support on $\mathcal{X}$ ensures that two extended Dirichlet spaces coincide.

Theorem 4.6. *Suppose that $(\mathcal{X}, d)$ is connected. If a strongly local $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies (Ha) and any function that is harmonic in a ball is continuous there, then $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible.*

Proof. Suppose $u \in \mathcal{F}_{\rm loc}$ and $\mathcal{E}(u, u) = 0$. By the proof of (i) $\Rightarrow$ (ii) part of Theorem 4.5, we know u is harmonic on $\mathcal{X}$ and

$$u(X_t) = u(X_0) \quad \text{for all } t \in [0, \zeta) \tag{4.7}$$

$\mathbb{P}^x$-a.s. for $\mathcal{E}$-q.e. $x \in \mathcal{X}$. By assumption, u has a continuous version; we will use this continuous version and still denote it by u. Since $(\mathcal{E}, \mathcal{F})$ is strongly local, $1 \in \mathcal{F}_{\rm loc}$ and $\mathcal{E}(1, 1) = 0$. Let x_0 be an arbitrary point in $\mathcal{X}$ and denote $u(x_0)$ by a_0. Then $u - a_0 \in \mathcal{F}_{\rm loc}$ and $\mu_{\langle u-a_0\rangle} = \mu_{\langle u\rangle}$ by Proposition 2.3(i). Thus

$$\mathcal{E}(u - a_0, u - a_0) = \frac{1}{2}\mu_{\langle u-a_0\rangle}(\mathcal{X}) = \frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X}) = \mathcal{E}(u, u) = 0.$$

Let $v = |u - a_0|$. By [CF, Theorem 4.3.10], $v \in \mathcal{F}_{\rm loc}$ and $\mathcal{E}(v, v) = 0$. By the same reasoning as that for u in the above, v is harmonic on $\mathcal{X}$. Since $v(x_0) = 0$, by (Ha) $v(x) = 0$ on $B(x_0, r)$ for some $r > 0$; that is, $u(x) = u(x_0)$ on $B(x_0, r)$ for some $r > 0$. This shows that for any constant $a \in \mathbb{R}$, both $A_a := \{x \in \mathcal{X} : u(x) > a\}$ and its complement $A_a^c = \{x \in \mathcal{X} : u(x) \le a\}$ are open subsets of $\mathcal{X}$. If u is not a constant, then there is a constant a so that neither A_a nor A_a^c are empty sets. This would contradict to the assumption that $(\mathcal{X}, d)$ is connected. So u must be constant. This establishes the irreducibility of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ by Theorem 4.5. □

Theorem 4.7. *Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible and satisfies the* $\mathrm{EHI}_{\rm loc}$. *Then $(\mathcal{E}, \mathcal{F})$ has regular Green functions.*

Proof. Since $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the $\mathrm{EHI}_{\rm loc}$, every locally bounded harmonic function is locally Hölder continuous, and $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$ satisfies the $\mathrm{EHI}_{\rm loc}$ for every non-empty open subset D of $\mathcal{X}$ whose complement D^c is not $\mathcal{E}$-polar. The conclusion of this corollary follows directly from Proposition 2.1 and Theorem 4.4(ii). □

Combining Theorem 4.6 with Theorem 4.7 shows that if $(\mathcal{X}, d)$ is connected and $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the $\mathrm{EHI}_{\rm loc}$, then $(\mathcal{E}, \mathcal{F})$ has regular Green functions.

5 Implications of EHI

Recall the definition of relative K ball connected from Definition 1.1. We introduce a few related properties.

Definition 5.1. (i) A metric space $(\mathcal{X}, d)$ is said to be *metric doubling* (MD) if there exists $N \ge 2$ such that given $x \in \mathcal{X}$, $R > 0$ there exist $z_1, \dots, z_N$ such that $B(x, R) \subset \cup_{i=1}^N B(z_i, R/2)$.

(ii) A metric space $(\mathcal{X}, d)$ is said to be *uniformly perfect*, if there exists $C > 1$ such that for all $x \in \mathcal{X}, r > 0$ with $B(x, r)^c \neq \emptyset$ satisfies $B(x, r) \setminus B(x, r/C) \neq \emptyset$.

(iii) A metric space $(\mathcal{X}, d)$ is said be *L-linearly connected* (for some $L > 1$), if for all $x, y \in \mathcal{X}$, there exists a connected compact set J such that $x, y \in J$ and $\mathrm{diam}(J) \le L d(x, y)$.

(iv) A *distortion function* is a homeomorphism of $[0, \infty)$ onto itself. Let η be a distortion function. A map $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ between metric spaces is said to be *η-quasisymmetric*, if f is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \le \eta\left(\frac{d_1(x, a)}{d_1(x, b)}\right)$$

for all triples of points $x, a, b \in \mathcal{X}_1$, $x \neq b$. We say f is a *quasisymmetry* if it is η-quasisymmetric for some distortion function η. We say that metric spaces $(\mathcal{X}_1, d_1)$ and $(\mathcal{X}_2, d_2)$ are *quasisymmetric*, if there exists a quasisymmetry $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$. We say that metrics d_1 and d_2 on $\mathcal{X}$ are *quasisymmetric* (or, d_1 is *quasisymmetric* to d_2), if the identity map $\mathrm{Id} : (\mathcal{X}, d_1) \to (\mathcal{X}, d_2)$ is a quasisymmetry.

(v) We say a metric space $(\mathcal{X}, d)$ is *quasi-arc connected*, if there exists a distortion function $\eta : [0, \infty) \to [0, \infty)$ such that for all pairs of distinct points $x, y \in X$, there exists a subset $J \subset \mathcal{X}$ and a η-quasisymmetry $\gamma : [0, 1] \to J$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here J is endowed with the metric d and $[0, 1]$ has the Euclidean metric.

The following lemma clarifies some relationships between these conditions.

Lemma 5.2. Let $(\mathcal{X}, d)$ be a complete metric space.

(a) Assume that $(\mathcal{X}, d)$ is relatively K ball connected metric space. Then exists $L > 1$, such that for all $x, y \in \mathcal{X}$, there exists a curve $\gamma : [0,1] \to \mathcal{X}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{diam}(\gamma[0,1]) \leq Ld(x,y)$. In particular, $(\mathcal{X}, d)$ is L-linearly connected.

(b) If $(\mathcal{X}, d)$ is L-linearly connected then it is uniformly perfect.

(c) If $(\mathcal{X}, d)$ is relatively ball connected and satisfies metric doubling, then $(\mathcal{X}, d)$ is quasi-arc connected.

(d) If $(\mathcal{X}, d)$ is quasi-arc connected, then $(\mathcal{X}, d)$ is relatively ball connected.

(e) Assume that $(\mathcal{X}, d)$ is relatively K ball connected metric space. If ρ is quasisymmetric to d, then $(\mathcal{X}, \rho)$ is also relatively ball connected metric space. In other words, the property of being relatively ball connected is a quasisymmetry invariant.

Proof. (a) Fix $\varepsilon \in (0,1)$ and let $K, N = N_{\mathcal{X}}(\varepsilon)$ be the constants of relative ball connectivity.

Let $x, y \in \mathcal{X}$ a pair of distinct points. For each $k \in \mathbb{N}$, we define $\gamma_k : [0,1] \to \mathcal{X}$ as follows. Let $z_0^{(1)}, z_1^{(1)}, \ldots, z_N^{(1)}$ be a sequence of points in $B(x, Kd(x,y))$ such that $d(z_i^{(1)}, z_{i+1}^{(1)}) < \varepsilon d(x,y)$, with $z_0^{(1)} = x, z_N^{(1)} = y$. Let $\gamma_1 : [0,1] \to \mathcal{X}$ be a piecewise constant function on intervals defined by

$$\gamma_1(t) = z_i^{(1)}, \quad \text{for all } i = 0, 1, \ldots, N \text{ and for all } i/(N+1) \leq t < (i+1)/(N+1)$$

and $\gamma_1(1) = y$. Similarly, for all $i = 0, \ldots, N$, we chose $z_j^{(2)}, j = i(N+1), i(N+1)+1, \ldots, i(N+1)+N$ such that $z_{i(N+1)}^{(2)} = z_i^{(1)}, z_{i(N+1)+N}^{(2)} = z_{i+1}^{(1)}, d(z_j^{(2)}, z_{j+1}^{(2)}) < \varepsilon^2 d(x,y)$ and set

$$\gamma_2(t) = z_j^{(2)}, \quad \text{for all } i = 0, 1, \ldots, (N+1)^2 - 1 \text{ and for all } i/(N+1)^2 \leq t < (i+1)/(N+1)^2,$$

with $\gamma_2(1) = y$. We similarly define $\gamma_k : [0,1] \to \mathcal{X}$ a piecewise constant function on intervals $[j/(N+1)^k, (j+1)/(N+1)^k)$, $j = 0, 1, \ldots, (N+1)^k - 1$. Since for all $t \in [0,1]$, $d(\gamma_k(t), \gamma_{k+1}(t)) < K\varepsilon^k d(x,y)$, the sequence $\{\gamma_k(t), k \in \mathbb{N}\}$ is Cauchy, and hence converges to say $\gamma(t) \in \mathcal{X}$. This defines a function $\gamma : [0,1] \to \mathcal{X}$. Note that

$$d(x, \gamma(t)) \leq \sum_{k=0}^{\infty} K\varepsilon^k d(x,y) = Kd(x,y)/(1-\varepsilon).$$

If $|t_1 - t_2| \leq \frac{1}{N^k}$ for some $k \in \mathbb{N}$, we have

$$\begin{aligned} d(\gamma(t_1), \gamma(t_2)) &\leq d(\gamma_k(t_1), \gamma(t_1)) + d(\gamma_k(t_2), \gamma(t_2)) + d(\gamma_k(t_1), \gamma_k(t_2)) \\ &\leq 2\left(\sum_{l=k}^{\infty} K\varepsilon^l d(x,y)\right) + 2(K+1)\varepsilon^k d(x,y) \\ &\leq 2(K+1)(1 + (1-\varepsilon)^{-1})\varepsilon^k d(x,y), \end{aligned}$$

which implies the continuity of γ.

This shows that the image $J = \gamma([0,1])$ is a compact, connected set with $x, y \in J$ with $\operatorname{diam}(J) \leq Ld(x,y)$, where $L = 2K/(1-\varepsilon)$. Therefore $(\mathcal{X}, d)$ is L-linearly connected.

(b) Let $B(x,r)$ be a ball such that $B(x,r)^c \neq \emptyset$. Let $y \in B(x,r)^c$ and let J be a compact connected set containing x and y. Let $r' \leq r$. By the continuity of the map $z \mapsto d(x,z)$, for all $z \in J$, there exists z' such that $d(x,z') = r'$. Therefore $B(x,r) \setminus B(x,r/C) \neq \emptyset$ for all $C > 1$.
(c) By part (a), $(\mathcal{X}, d)$ is linearly connected. By Tukia's theorem ([Mac, Corollary 1.2] and [TV, Theorem 4.9]), $(\mathcal{X}, d)$ is quasi-arc connected.
(d) Let η be the distortion function corresponding to quasi-arc connectedness. Define $K = 1 + \eta(1)$. Let $x, y \in B(x_{,0}, R)$ and $\gamma : [0,1] \to J$ be a η-quasisymmetry such that $\gamma(0) = x, \gamma(1) = y$. For all $t \in [0,1]$,

$$d(x, \gamma(t)) \leq \eta(t) d(x,y) \leq \eta(1) d(x,y),$$

and hence $B(\gamma(t), \varepsilon d(x,y)) \leq K d(x,y)$.

Let $\varepsilon \in (0,1)$ be arbitrary. Let $N \in \mathbb{N}$ be such that $2\eta(2/N)\eta(1) < \varepsilon$ and define $z_i = \gamma(i/N)$ for $i = 0, 1, \ldots, N$. By η-quasisymmetry, we have

$$d(z_i, z_{i+1}) \leq \eta(2/N) d(z_i, w) \leq \eta(2/N)\eta(1) d(x,y) \leq 2\eta(2/N)\eta(1) R < \varepsilon R,$$

where $w = x$ if $i \leq N/2$ and $w = y$ otherwise. This implies that $(\mathcal{X}, d)$ is relatively K ball connected, where $K = 1 + \eta(1)$.
(e) Let $\mathrm{Id} : (\mathcal{X}, \rho) \to (\mathcal{X}, d)$ be a η-quasisymmetry, where $(\mathcal{X}, d)$ is relatively K ball connected. Let $\varepsilon \in (0,1)$ and let $x, y \in \mathcal{X}$ be arbitrary. Chose $\varepsilon' \in (0,1)$ such that

$$\eta(2\varepsilon')\,(\eta(K) + 1) < \varepsilon$$

Choose points $z_0, z_1, \ldots, z_N$ such that $d(z_i, z_{i+1}) < \varepsilon d(x,y)$, where $N = N_{(\mathcal{X},d)}(\varepsilon')$ is the constant associated with the relative ball connected property of $(\mathcal{X}, d)$. For any $i = 0, 1, \ldots, N-1$, let $w \in \{x, y\}$ be such that $d(z_i, w) = \max(d(z_i, x), d(z_i, y))$. Since $d(x,y)/2 \leq d(z_i, w)$, we obtain

$$\begin{aligned}\rho(z_i, z_{i+1}) &\leq \eta(d(z_i, z_{i+1})/d(z_i, w))\rho(z_i, w) \leq \eta(2\varepsilon')(\rho(x, z_i) + \rho(x,y)) \\ &\leq \eta(2\varepsilon')\,(\eta(K) + 1)\,\rho(x,y) < \varepsilon \rho(x,y).\end{aligned}$$

Since $\rho(x, z_i) \leq \eta(K)\rho(x,y)$, (X, ρ) is relatively K_ρ ball connected, where $K_\rho = 2 + \eta(K)$. □

Remark 5.3. See [GH] for the definition of *relatively* (ε, K) *ball connected*. It is immediate that if $(\mathcal{X}, d)$ is relatively K ball connected then it is relatively (ε, K) ball connected for any $\varepsilon \in (0,1)$. Conversely it is straightforward to show that if for some $\varepsilon \in (0,1)$, $K > 1$ $(\mathcal{X}, d)$ is relatively (ε, K) ball connected then it is relatively K' ball connected with $K' = (1+K)/(1-\varepsilon)$.

The main result of this section is the following.

Theorem 5.4. *Let $(\mathcal{X}, d)$ be a complete locally compact metric space. Assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI. The following are equivalent:*
(a) *$(\mathcal{X}, d)$ is relatively K ball connected for some $K > 1$.*
(b) *$(\mathcal{X}, d)$ satisfies metric doubling.*
(c) *$(\mathcal{X}, d)$ is quasi-arc connected.*

Proof. (b) ⇒ (a). This follows by the argument in [GH, Proposition 5.6]: for any $K > 1 + \delta_H^{-1}$ we obtain relative K-ball connectedness. (The hypothesis of volume doubling there is only used to obtain metric doubling.)

(c) ⇒ (a), (a) + (b) ⇒ (c) (and so (b) ⇒ (c)) are proved in Lemma 5.2.

The proof of (a) $\Rightarrow$ (b) needs more preparation and will be given after Lemma 5.16. $\square$

For the proof of (a) $\Rightarrow$ (b) in Theorem 5.4, we follow [BM1, Section 3]; however it was assumed there that the metric d was geodesic, and some changes are needed to handle the case when we only have that $(\mathcal{X}, d)$ is relatively K ball connected. We now outline these changes.

Definition 5.5. We say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the condition (HG) if $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ has regular Green functions and there exist constants C_G, K_G such that for any $x_0 \in \mathcal{X}$ and $R > 0$ and bounded domain D with $B(x_0, K_G R) \subset D$ with D^c non-$\mathcal{E}$-polar

$$\sup_{y_2 \in D \setminus B(x_0,R)} g_D(x_0, y_2) \le C_G \inf_{y_1 \in \overline{B(x_0,R)} \setminus \{x_0\}} g_D(x_0, y_1). \tag{5.1}$$

Assumption 5.6. Throughout the remainder of this section except for Lemma 5.15, we assume that $(\mathcal{X}, d)$ is a complete locally compact separable metric space, that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the (scale invariant) EHI with constants C_H, δ_H and is relatively K ball connected for some $K \ge 2$.

Recall that by Lemma 5.2(a), a complete metric space $(\mathcal{X}, d)$ that is relatively K ball connected is connected. Thus under Assumption 5.6, $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible by Theorem 4.6 and has regular Green functions by Theorem 4.7. By the maximum principle (4.2) for the regular Green function g_D in Theorem 4.4(ii), we have for any $B = B(x_0, R) \Subset D$,

$$\sup_{D \setminus B} g_D(x_0, \cdot) = \sup_{\partial B} g_D(x_0, \cdot), \qquad \inf_{y \in \overline{B} \setminus \{x_0\}} g_D(x_0, \cdot) = \inf_{\partial B} g_D(x_0, \cdot). \tag{5.2}$$

Proposition 5.7. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Then for any $K_G > K$, (HG) holds with constants C_G, K_G, where C_G depends only on C_H, δ_H, K_G, N_K.*

Proof. (a) This follows from the proof of [GH, Lemma 5.7]. (The statement of the result in [GH] has stronger hypotheses, but these are only used to obtain the existence and regularity of the Green function, and prove that $(\mathcal{X}, d)$ is relatively (ε, K) ball connected for some $\varepsilon \in (0, 1)$ and $K > 1$.) $\square$

Under Assumption 5.6, $(\mathcal{E}, \mathcal{F})$ has regular Green functions by Theorems 4.6 and 4.7, and (HG) holds with constant $K_G = K + 1$ and $C_G > 1$ by Proposition 5.7.

Corollary 5.8. *(See [BM1, Corollary 3.2].) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $K_1 = K + 1$. For any $\delta \in (0, 1/2]$, there exists a positive constant C that denotes only on δ and the constants in Assumption 5.6 such that the following holds: for any bounded domain D whose complement D^c is non-$\mathcal{E}$-polar and for any $B(x_0, K_1 R) \subset D$,*

$$g_D(x_0, x) \le C g_D(x_0, y) \quad \textit{for } x, y \in B(x_0, R) \setminus B(x_0, \delta R).$$

Proof. Let $x, y \in B(x_0, R) \setminus B(x_0, \delta R)$. If $d(x, x_0) \ge d(y, x_0)$ then the inequality is immediate from (HG). So suppose that $d(x, x_0) < d(y, x_0)$. We can assume that $\delta_H < 1/2$. Let $\varepsilon = \delta\delta_H/(1+\delta_H)$; we have $\varepsilon < \delta/2$. We connect y to x_0 by a chain of balls $B(z_i, \varepsilon R)$, $i = 0, 1, \dots N$, with the properties given in Definition 1.1 of relatively K ball connected. Let i_0 be the first integer such that $d(z_{i_0}, x_0) < \delta R$. With the definition of ε given above, $g_D(x_0, \cdot)$ is harmonic on $B(z_i, \varepsilon R/\delta_H)$ for $i = 0, \dots, i_0 - 1$, and so we can use the EHI to deduce that $g_D(x_0, z_{i_0}) \le C_H^N g_D(x_0, y)$. Finally by (HG) we have $g_D(x_0, x) \le C_G g_D(x_0, z_{i_0})$. $\square$

Lemma 5.9. *(See [BM1, Lemma 3.3].) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $K_2 > 2K + 1$. There exists $C_0 > 1$ that depends only on the constants in Assumption 5.6 such that the following holds: Let $x_0 \in \mathcal{X}$, $R > 0$ and let $B(x_0, (2K+1)R) \subset D$, where D is a bounded domain such that D^c is non-$\mathcal{E}$-polar. Then if $x_1, x_2, y_1, y_2 \in B(x_0, R)$ with $d(x_j, y_j) \geq R/4$, then*

$$g_D(x_1, y_1) \leq C_0 g_D(x_2, y_2). \tag{5.3}$$

Proof. Note that for any four numbers $a_i \in [0, R)$, $1 \leq i \leq 4$,

$$\left|(0, R) \setminus \cup_{i=1}^4 [a_i - (R/9), a_i + (R/9)]\right| \geq R - (8R/9) = R/9 > 0$$

so there is some $a_0 \in (0, R)$ so that $|a_0 - a_i| > R/9$ for all $1 \leq i \leq 4$. As $d(x_0, \cdot)$ is continuous and by the RBC(K) property and Lemma 5.2(a), $B(0, R)$ contains a connected path from x_0 to $B(0, R)^c$, we have $\{d(x_0, x) : x \in B(x_0, R)\} = [0, R)$. Thus there is $a_0 \in (0, R)$ so that $|a_0 - d(x_0, w)| > R/9$ for $w \in \{x_1, x_2, y_1, y_2\}$. Let $z \in B(0, R)$ having $d(x_0, z) = a_0$. Now applying Corollary 5.8 to the balls $B(x_1, 2R)$, $B(z, 2R)$ and $B(x_2, 2R)$ with $\delta = 1/18$ consecutively, we get by the symmetry of the Green function $g_D(x, y)$ that

$$g_D(x_1, y_1) \leq C g_D(x_1, z) \leq C^2 g_D(x_2, z) \leq C^3 g_D(x_2, y_2).$$

This establishes the lemma by taking $C_0 = C^3$. □

As in [BM1], we define for an open set $D \subset \mathcal{X}$ with non-$\mathcal{E}$-polar complement:

$$g_D(x, r) = \inf_{y: d(x,y)=r} g_D(x, y),$$
$$\mathrm{Cap}_D(A) = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}^D,\ f \geq 1\ \mathcal{E}\text{-q.e. on } A\}, \quad A \subset D.$$

We call $\mathrm{Cap}_D(A)$ the relative capacity of A in D. The maximum principle (4.2) implies that $g_D(x, r)$ is non-increasing in r, and an easy application of (HG) gives that if $d(x, y) = r$ and $B(x, K_G r) \cup B(y, K_G r) \subset D$ then

$$g_D(x, r) \leq C_G g_D(y, r). \tag{5.4}$$

The proof of the next Lemma is the same as in [BM1, Lemma 3.5].

Lemma 5.10. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There is a constant $C_G > 0$ depending only on the constants in Assumption 5.6 such that for any bounded open set D whose complement D^c is non-$\mathcal{E}$-polar and for any $B(x_0, K_G r) \subset D$ where $K_G > K$,*

$$g_D(x_0, r) \leq \mathrm{Cap}_D(B(x_0, r))^{-1} \leq C_G g_D(x_0, r), \tag{5.5}$$

Remark 5.11. For any $x \in \mathcal{X}$, $0 < R < \mathrm{diam}(\mathcal{X}, d)/2$, the ball $B(x, R)^c$ is non-$\mathcal{E}$-polar. This is because by the arc-connectedness of $(\mathcal{X}, d)$ and the triangle inequality, there exists $z \in \mathcal{X}$ and $0 < r < \mathrm{diam}(\mathcal{X}, d)/2 - R$ so that $B(z, r) \subset B(x, R)^c$. Since m has full support, $m\left(B(x, R)^c\right) \geq m(B(z, t)) > 0$ and thus $B(x, R)^c$ has positive capacity.

Lemma 5.12. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6 (with $K \geq 2$ there). Let $B = B(x_0, R) \subset \mathcal{X}$, and $B_1 = B(x_1, R/(8K))$ with $x_1 \in B(x_0, R/(4K))$. There exists $p_0 > 0$ depending only on the constants in Assumption 5.6 such that*

$$\mathbb{P}^y(T_{B_1} < \tau_B) \geq p_0 > 0 \quad \textit{for } y \in B(x_0, R/(2K)). \tag{5.6}$$

Proof. We consider two cases.

(i) Suppose $B(x,R)^c$ is non-$\mathcal{E}$-polar. By the maximum principle it is enough to prove this for $y \in \partial B(x_0, R/(2K))$. The argument, which uses Corollary 5.8, is the same as in [BM1, Lemma 3.7].

(ii) Now suppose $B(x,R)^c$ is $\mathcal{E}$-polar. By Remark 5.11, $R \geq \operatorname{diam}(\mathcal{X}, d)/2$ and $\operatorname{diam}(\mathcal{X}, d) < \infty$. If $R > 2\operatorname{diam}(\mathcal{X}, d)$, then $B_1 = \mathcal{X}$ and (5.6) is obviously true. Therefore, it suffices to consider the case when $R \leq 2\operatorname{diam}(\mathcal{X}, d) < \infty$.

Similar to the first case, it suffices to consider $x_0 \in \mathcal{X}, x_1 \in B(x_0, R/(4K)), y \in \partial B(x_0, R/(2K))$. Let $\varepsilon = 1/(33K)$ and let $B_i = B(z_i, \varepsilon R), 1 \leq i \leq N := N_{\mathcal{X}}(\varepsilon)$ be a chain of balls with $z_0 = y$, $z_N = x_1$, $B_i \subset B(x_0, R/4)$ for all each i and $d(z_{i-1}, z_i) < \varepsilon R/(4K)$ for $1 \leq i \leq N$ as in Definition 1.1 with $R/(4K)$ in place of R there. Since $8K\varepsilon R \leq 16K\varepsilon \operatorname{diam}(\mathcal{X}, d) < \operatorname{diam}(\mathcal{X}, d)/2$, by Case 1 and Remark 5.11, we have

$$\mathbb{P}^w(T_{B(z_i,\varepsilon R)} < \tau_{B(z_{i-1},8K\varepsilon R)}) > p_0 \quad \text{for } w \in B(z_{i-1}, 4\varepsilon R).$$

Since $B(z_i, 8K\varepsilon R) \subset B(x_0, R/2)$ for all i, using the strong Markov property, we conclude that

$$\mathbb{P}^y(T_{B_1} < \tau_B) \geq p_0^N > 0 \quad \text{for } y \in B(x_0, R/(2K)).$$

By replacing p_0 by p_0^N, we obtain (5.6) in the second case as well. $\square$

Remark 5.13. (i) In [BM1, Lemma 3.7], the corresponding result held for $y \in B(x_0, 7R/8)$; we cannot expect that here, since such a point y might not be connected to B_1 by a path inside B.

(ii) Let $B_i = B(z_i, \varepsilon R)$, $0 \leq i \leq n$ be a chain of balls as in Definition 1.1. Using this Lemma we have for each i

$$\mathbb{P}^y(T_{B(z_i,\varepsilon R)} < \tau_{B(z_{i-1},8K\varepsilon R)}) > p_0 \ \text{ for } y \in B(z_{i-1}, \varepsilon R).$$

Thus if

$$D = \cup_{i=0}^n B(z_i, 8K\varepsilon R),$$

then

$$\mathbb{P}^y(T_{B_n} < \tau_D) \geq p_0^n \ \text{ for } \ y \in B(z_0, \varepsilon R). \tag{5.7}$$

Corollary 5.14. *(See [BM1, Corollary 3.8]). Let $B(x,R) \subset D$, where D is a bounded domain and D^c is non-$\mathcal{E}$-polar. There exist positive constants c and θ that depend only on the constants in Assumption 5.6 such that if $0 < s < r < R/(K+1)$ then*

$$\frac{g_D(x,r)}{g_D(x,s)} \geq c\Big(\frac{s}{r}\Big)^\theta. \tag{5.8}$$

Proof. This follows easily from Corollary 5.8. $\square$

The following Lemma is used to regularize chains of balls obtained by the using the RBC(K) property.

Lemma 5.15. *Suppose that $(\mathcal{X}, d)$ satisfies the RBC(K) property. Let $d(x,y) < R$, $\varepsilon \in (0,1)$ and $\varepsilon R < r < R$. There exists a chain of balls $B(z_i, \varepsilon R)$, $0 \le i \le n$ with the following properties:*
(i) $z_0 = x$, $z_n = y$ *and* $d(z_{i-1}, z_i) < \varepsilon R$ *for* $1 \le i \le n$*;*
(ii) $B(z_i, \varepsilon R) \subset B(x, KR)$ *for* $0 \le i \le n$*;*
(iii) *If* $j = \max\{i : z_i \in B(x,r)\}$ *then* $B(z_i, \varepsilon R) \subset B(x, Kr)$ *for* $0 \le i \le j$*;*
(iv) $n \le N_{\mathcal{X}}(\varepsilon) + N_{\mathcal{X}}(\varepsilon R/r)$.

Proof. Using the RBC(K) property there exists a chain of balls $B(w_i, \varepsilon R)$, $0 \le i \le m_1$ connecting x and y and satisfying the conditions of Definition 1.1. Let $k = \max\{i : w_i \in B(x,r)\}$. Using the property again for x and w_k, and with ε replaced by $\varepsilon' = \varepsilon R/r$, there exists a chain $B(w'_i, \varepsilon R)$, $0 \le i \le m_2$ with $B(w'_i, \varepsilon R) \subset B(x, Kr)$. Joining the paths $w'_0, \dots, w'_{m_2}$ and $w_{k+1}, \dots w_{m_1}$ gives a path (z_i) which satisfies the conditions (i)–(iv). □

Lemma 5.16. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists an integer $N \ge 1$ that depends only on the constants in Assumption 5.6 such that if $x_0 \in \mathcal{X}$, $R > 0$ and $B(z_i, R/8)$, $1 \le i \le m$, are disjoint balls with $z_i \in B(x_0, R) \setminus B(x_0, R/2)$, then $m \le N$.*

Proof. This lemma corresponds to [BM1, Lemma 3.10]. The proof in [BM1], where the metric d on $\mathcal{X}$ is assumed to be geodesic distance, uses the geodesic property quite strongly. The proof here is much longer since we only have the weaker property that $(\mathcal{X}, d)$ is relatively K ball connected.

Let $(z_k, 1 \le k \le m)$ satisfy the hypotheses of the Lemma, and write $B_k = B(z_k, R/8)$. Choose $\varepsilon = 1/(120K^2)$, and let $n = N_{\mathcal{X}}(\varepsilon) + N_{\mathcal{X}}(8\varepsilon)$. For each k we use Lemma 5.15 with $r = R/12K$ to find a chain of balls $B(w_{ki}, \varepsilon R)$ with $0 \le i \le n$ connecting z_k to x_0. Note that by taking $w_{k,i} = x_0$ for large i if necessary, we can assume that all the chains have length n. We set $i_k = \max\{i : B(w_{ki} \in B(z_i, R/12K)\}$, and write $z'_k = w_{k,i_k}$.

We now find a subset I of the balls B_i such that the chain $(w_{ik}, 0 \le k \le n)$ associated with one ball does not hit any other ball with index in I.

For $1 \le i,j \le m$ set $a_{ij} = 1$ if $\{w_{ik}, 1 \le k \le n\} \cap B_j \ne \emptyset$, and let $a_{ij} = 0$ otherwise. Let $b_j = \sum_i a_{ij}$. Since each w_{ik} is in at most one ball B_j, we have $\sum_j a_{ij} \le n$, and hence $\sum_j b_j = \sum_i \sum_j a_{ij} \le mn$. Thus if $J = \{j : b_j \le 2n\}$ then $|J| \ge m/2$.

We now consider the collection of balls $(B_i, i \in J)$, and relabel them $B_1, \dots, B_{m_1}$ where $m_1 = |J| \ge m/2$. We now start with the ball B_1, and remove from the collection of balls $B_2, \dots, B_{m_1}$ any ball B_j such that either $a_{1j} = 1$ or $a_{j1} = 1$. Since $1 \in J$, at most $3n$ balls are removed. Set $j_1 = 1$. Let j_2 be the smallest label of a ball which has not been removed; we now repeat the procedure above with this ball, and remove any ball B_i such that $i > j_2$ and $a_{j_2,i} + a_{i,j_2} \ge 1$. We continue until there are no balls left, and write $I = \{j_k, 1 \le k \le m'\}$ for the set of balls which are retained. Since at each step we remove at most $3n$ balls, we have $3nm' \ge \frac{1}{2}m$.

By the construction above we have that

$$w_{ik} \notin \cup_{j \ne i} B_j \quad \text{ for } i \in I,\ k = 0, \dots, n.$$

For $i \in I$ set $B'_i = B(z_i, \varepsilon R)$, $A_i = B(z'_i, \varepsilon R)$, and let

$$D = B(x_0, 2KR) \setminus \cup_{i \in I} B'_i.$$

We now claim that

$$\mathbb{P}^y(T_{B_i'} < \tau_{B_i}) > p_0^n \quad \text{for } y \in A_i, \tag{5.9}$$

$$\mathbb{P}^{x_0}(T_{A_i} < \tau_D) > p_0^n. \tag{5.10}$$

Both these inequalities follow by chaining the bound in Lemma 5.12, as in Remark 5.13(2), along a sequence of balls. For (5.9) we use the sequence $B(w_{ij}, \varepsilon R), 0 \le j \le j_i$, and for (5.10) we use $B(w_{ij}, \varepsilon R), j_i \le j \le n$. (We start at $j = n$ and end at $j = j_1$.)

The remainder of the proof is as in [BM1, Lemma 3.10]. Let $F_i = \{T_{A_i} < \tau_D\}$, and

$$Y = \sum_{i \in I} 1_{F_i}$$

be the number of distinct balls A_i hit by $(X_t, 0 \le t \le \tau_D)$. The bound (5.9) implies that if X hits A_i then with probability at least p_0^n it leaves D before it hits any other ball A_j with $j \ne i$. Thus Y is stochastically dominated by a geometric r.v. with mean p_0^{-n}, and so

$$\mathbb{E}^{x_0} Y \le p_0^{-n}.$$

However, by (5.10) we also have

$$\mathbb{E}^{x_0} Y = \sum_{i \in I} \mathbb{P}^{x_0}(F_i) \ge |I| p_0^n = m' p_0^n.$$

Using the bound on m' given above, it follows that $m \le 6np_0^{-2n}$. □

Now we can finish the proof of Theorem 5.4 by giving the

Proof of (a) ⇒ (b) in Theorem 5.4. (i) Suppose that a metric space $(\mathcal{X}, d)$ has the property that there is an integer $N' \ge 1$, so that any ball $B(x, R)$ contains at most N' points that are at distance of at least $R/2$. Given any ball $B(x, R) \subset \mathcal{X}$, take $z_1 \in B(x, R)$, $z_2 \in B(x, R) \setminus B(z_1, R/2)$, and for $k \ge 3$, $z_k \in B(x, R) \setminus \cup_{j=1}^{k-1} B(z_j, R/2)$ if the set is non-empty. By the assumption, we can only proceed this procedure up to some number k_0 no larger than N'. Clearly $\cup_{j=1}^{k_0} B(z_j, R/2) \supset B(x, R)$. Thus $(\mathcal{X}, d)$ is metric doubling. Conversely, suppose $(\mathcal{X}, d)$ is metric doubling with positive integer $N \ge$ in Definition 5.1(i). For any ball $B(x, R)$, applying the definition of (MD) to $B(x, R)$ and to balls with radius $R/2$, there are N^2 number of points $x_1, \ldots, x_{N^2}$ in $B(x, R)$ so that $\cup_{j=1}^{N^2} B(x_j, R/4) \supset B(x, R)$. Suppose $\{z_1, \ldots, z_n\}$ are n points in $B(x, R)$ that are at distance of at least $R/2$, then each z_j can be in exactly one of the balls $\{B(x_k, R/4); 1 \le k \le N^2\}$. Thus $n \le N^2$. This proves that a metric space $(\mathcal{X}, d)$ is (MD) if and only if there is some constant N' so that any ball $B(x, R)$ contains at most N' points that are at distance of at least $R/2$ from each other.

(ii) Now let $N \ge 1$ be the integer in Lemma 5.16. Let $x_0 \in \mathcal{X}$, $R > 0$, and let $z_i \in B(x_0, R)$, $1 \le i \le n$, with the property that the balls $B(z_i, R/8)$ are disjoint. By Lemma 5.16 applied first to $B(x_0, R)$ and then to $B(x_0, R/2)$, there are at most $2N$ of the z_i in $B(x_0, R) \setminus B(x_0, R/4)$. Using the K-ball connectivity of $\mathcal{X}$, we can find x_1 such that $R/2 < d(x_0, x_1) < 3R/4$. Thus $B(x_0, R/4) \subset B(x_1, R) \setminus B(x_1, R/4)$. So by Lemma 5.16 applied to $B(x_1, R)$, there are at most $2N$ points z_i in $B(x_0, R/4)$. Consequently, $m \le 4N$. This proves that $(\mathcal{X}, d)$ is (MD) in view of its equivalent characterization given in (i). □

We need to compare the Green function in two domains.

Lemma 5.17. *(See [BM1, Lemma 3.12].) Let $(X,d,m,\mathcal{E},\mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists a constant C_1 that depends only on the constants in Assumption 5.6 such that if $B = B(x_0,R)$, $2B = B(x_0,2R)$ and $B(x_0,(2+1/(64K^2))R)^c$ is non-empty, then*

$$g_{2B}(x,y) \le C_1 g_B(x,y) \quad \text{for } x,y \in B(x_0,R/(16K)).$$

Proof. Let $a_1 = 1/(8K)$, $a_2 = 1/(4K)$, $a_3 = R/(2K)$, $\varepsilon = 1/(64K^2)$ and $B_i = B(x_0,a_iR)$. Let $p_1 > 0$. Suppose that for each $y \in B_1$ there exists a domain D with $B_3 \subset D \subset B$ and a set A such that

$$\mathbb{P}^x(X_{\tau_D} \in A) \ge p_1, \text{ for } x \in \overline{B}_2, \tag{5.11}$$
$$\mathbb{P}^w(\tau_{2B} < T_{B_2}) \ge p_1 \text{ for } w \in A. \tag{5.12}$$

Let $x_1 = x_1(y) \in \partial B_2$ be chosen to maximize $g_{2B}(x',y)$ for $x' \in \partial B_2$. Write $h(w) = \mathbb{P}^w(\tau_{2B} < T_{B_2})$. Then

$$g_{2B}(x_1,y) = g_D(x_1,y) + \mathbb{E}^{x_1} g_{2B}(X_{\tau_D},y) \le g_B(x_1,y) + \mathbb{E}^{x_1}(1-h(X_{\tau_D}))g_{2B}(x_1,y).$$

Using (5.11) and (5.12),

$$g_B(x_1,y) \ge g_{2B}(x_1,y)\mathbb{E}^{x_1} h(X_{\tau_D}) \ge g_{2B}(x_1,y)p_1^2.$$

Then if $x \in B_1$,

$$\begin{aligned} g_{2B}(x,y) &\le g_{B_2}(x,y) + \mathbb{E}^x g_{2B}(X_{\tau_{B_2}},y) \le g_B(x,y) + g_{2B}(x_1,y) \\ &\le g_B(x,y) + p_1^{-2} g_B(x_1,y). \end{aligned}$$

Let x_1' be the point in ∂B_2 which minimizes $g_B(x',y)$. By the maximum principle (4.2), $g_B(x,y) \ge g_B(x_1',y)$. We now apply Corollary 5.8 to the ball $B(y,a_1R + a_2R)$ to deduce that $g_B(x_1,y) \le c g_B(x_1',y)$. Combining this with the inequalities above we obtain the bound $g_{2B}(x,y) \le C g_B(x,y)$. (Note that the constant C only depends on p_1 and the constants in Corollary 5.8; it does not depend on y.)

It remains to find $p_1 > 0$ such that for each $y \in B_1$ there exist sets D and A satisfying (5.11) and (5.12). Let $y_0 \in \mathcal{X} \setminus B(x_0,(2+\varepsilon)R)$. By Lemma 5.15 there exists a sequence $x_0 = z_0,\dots,z_n = y_0$ such that if $j = \max\{i : z_i \in B_3\}$ then $B(z_i,\varepsilon R) \subset B(x_0,Ka_3R)$ for $0 \le i \le j$. Write $B_i' = B(z_i,\varepsilon R)$. Now let $D = B \setminus \overline{B}_j'$, and $A = \partial B_j'$. Using the EHI, it is sufficient to prove (5.11) for $x \in B_2$, and (5.12) for $w \in B_j'$.

If $i \ge j$ then $B(z_i,8K\varepsilon R) \cap B_2 = \emptyset$. So we can chain along the sequence of balls $B_j,\dots B_n$ to obtain (5.12) with $p_1 = p_0^n$.

If $0 \le i \le j$ then $d(x_0,z_i) \le Ka_3R$ and so $B(z_i,8\varepsilon KR) \subset B(x_0,Ka_3R + 8\varepsilon KR) \subset B$. Hence, chaining along this sequence we obtain

$$\mathbb{P}^x(T_{\tau_D} \in A) > p_0^j \text{ for } x \in B_0'.$$

To complete the proof of (5.11) we need to extend this estimate to $x \in \overline{B}_2$.

Let $x \in B_2$. Then there exists a chain of balls $B(w_j,\varepsilon R)$, $0 \le j \le k$ with $w_0 = x$, $w_k = x_0$ and with $B(w_j,\varepsilon R) \subset B(x_0,Ka_2R)$. Since then $B(w_j,8\varepsilon KR) \subset B$, we deduce that

$$\mathbb{P}^x(T_{B_0'} < \tau_B) \ge p_0^k.$$

It follows that

$$\mathbb{P}^x(T_{\tau_D} \in A) > p_0^{k+n}.$$

Since l and n only depend on the constants $N_{\mathcal{X}}(\varepsilon)$, this completes the proof of (5.11). $\square$

The following corollary is a direct consequence of Lemmas 5.10 and 5.17 and Remark 5.11.

Corollary 5.18. *(See [BM1, Corollary 3.13].) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists $C_1 = C(\delta_H, C_H, K)$ such that for all $A > 16K$ and for all $0 < r < \operatorname{diam}(\mathcal{X}, d)/6A$, $x \in \mathcal{X}$,*

$$\operatorname{Cap}_{B(x,2Ar)}(B(x,r)) \le \operatorname{Cap}_{B(x,Ar)}(B(x,r)) \le C_1 \operatorname{Cap}_{B(x,2Ar)}(B(x,r)). \tag{5.13}$$

In the following, notations $f \asymp g$, $f \lesssim g$ and $f \gtrsim g$ mean that there are positive constants c_1, c_2 so that $c_1 g \le f \le c_2 g$, $f \le c_2 g$ and $f \ge c_1 g$, respectively, on the common domain of definition of f and g.

Lemma 5.19. *(See [BM1, Lemma 3.14].) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6 with $K \ge 2$.*

(a) *Let D be a bounded domain in $\mathcal{X}$ such that D^c is non-$\mathcal{E}$-polar. Let $x \in \mathcal{X}$ and $r > 0$ be such that $B(x, C_0 r) \subset D$, where $C_0 > 2K + 1$. There exists a constant $C_1 > 0$ such that*

$$\operatorname{Cap}_D(B(x,r)) \le C_1 \operatorname{Cap}_D(B(y,r)) \quad \textit{for } y \in B(x,r).$$

(b) *Let $A > 16K$. There exists a constant $C_2 > 0$ such that*

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \le C_2 \operatorname{Cap}_{B(y,Ar)}(B(y,r)) \quad \textit{for } y \in B(x,r),\ 0 < r < \operatorname{diam}(\mathcal{X}, d)/(6A).$$

(c) *Let $A > 16K$ and $A_1 > 0$. There exists a constant $C_3 > 0$ such that*

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \le C_3 \operatorname{Cap}_{B(y,Ar)}(B(y,r)) \quad \textit{for } y \in B(x, A_1 r),\ 0 < r < \operatorname{diam}(\mathcal{X}, d)/(6A).$$

Here the constants C_1, C_2, C_3 depend only on A, A_1 and the constants in Assumption 5.6.

Proof. (a) As in the proof of Lemma 5.9, choose $z \in B(x,r)$ be such that $\min(d(z,x), d(z,y)) \ge r/4$. By Corollary 5.8 and Lemma 5.10, $\operatorname{Cap}_D(B(x,r)) \asymp g_D(x,z)^{-1}$ and $\operatorname{Cap}_D(B(y,r)) \asymp g_D(x,z)^{-1}$. The conclusion now follows from Lemma 5.9.
(b) By Corollary 5.18 and part(a), we have

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \lesssim \operatorname{Cap}_{B(x,2Ar)}(B(x,r)) \asymp \operatorname{Cap}_{B(x,2Ar)}(B(y,r)).$$

Since $B(y, Ar) \subset B(x, 2Ar)$, we have $\operatorname{Cap}_{B(x,2Ar)}(B(y,r)) \le \operatorname{Cap}_{B(y,Ar)}(B(y,r))$.
(c) The case $A_1 \le 1$ follows from (b). For $A_1 \ge 1$, by the RBC(K) conditition there exists N such that for $x, y \in \mathcal{X}$ with $d(x,y) < A_1 r$, can be connected by a sequence of points $x_0 = x, x_1, \ldots, z_N = y$, where N depends only on A_1 and the constants in RBC(K) condition. By applying (b) repeatedly, we obtain (c) with $C_3 = C_2^N$, where C_2 is the constant in (b). $\square$

Proposition 5.20. *(See [BM1, Proposition 3.15]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6 with $K \geq 2$. Let $D \subset \mathcal{X}$ be a bounded open set such that D^c is non-$\mathcal{E}$-polar and let $B(x_0, 2KR) \subset D$. Let $b \geq 24$. Suppose there exist disjoint Borel subsets $\{Q_i, 1 \leq i \leq n\}$ of $\mathcal{X}$ with $n \geq 2$ such that*

$$F = \cup_{i=1}^n Q_i$$

and for each i, there exists $z_i \in \mathcal{X}$ so that $B(z_i, R/b) \subset Q_i$. Then there exists $\delta = \delta(\delta_H, b, C_H, K) > 0$ such that

$$\mathrm{Cap}_D(F) \leq (1-\delta) \sum_{i=1}^n \mathrm{Cap}_D(Q_i).$$

Proof. The proof is similar to that of [BM1, Proposition 3.15]. The only difference is that we use RBC(K) condition, a chaining argument using the EHI along with Lemma 5.12 to obtain the lower bound on the equilibrium potentials h_i for $\mathrm{Cap}_D(Q_i)$. □

The following lemma is an extension of Corollary 5.18.

Lemma 5.21. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6 with $K \geq 2$. Let $1 < A_1 \leq A_2 < \infty$. There exist positive constants C_1 and C_2 that depend only on the constants in Assumption 5.6 such that for all $0 < r < \mathrm{diam}(\mathcal{X}, d)/C_1$,*

$$\mathrm{Cap}_{B(x,A_2 r)}(B(x,r)) \leq \mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \leq C_2 \,\mathrm{Cap}_{B(x,A_2 r)}(B(x,r)).$$

Proof. The estimate $\mathrm{Cap}_{B(x,A_2 r)}(B(x,r)) \leq \mathrm{Cap}_{B(x,A_1 r)}(B(x,r))$ follows from domain monotonicity. For the other estimate, by domain monotonicity we may assume $A_2 > 16K$.

Choose $A_3 > 16K$ so that $A_2/A_3 < A_1 - 1$. Then $B(y, A_2 r/A_3) \subset B(x, A_1 r)$ for all $y \in B(x,r)$. By the metric doubling property, there exists $N \in \mathbb{N}$ (depending only on A_3 and the constant associated with metric doubling) such that $y_1, \ldots, y_N \in B(x,r)$ and $\cup_{i=1}^N B(y, r/A_3) \supset B(x,r)$. By considering the function $e = \max_{1 \leq i \leq N} e_i$ where e_i is the equilibrium potential corresponding to $\mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3))$, we obtain

$$\mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \leq \sum_{i=1}^N \mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3)).$$

By connecting the points x and y_i using a r/A_3 chain and using Lemma 5.19(b), we obtain

$$\mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3)) \asymp \mathrm{Cap}_{B(x, A_2 r/A_3)}(B(x, r/A_3)),$$

for all $x \in \mathcal{X}$, $r \lesssim \mathrm{diam}(\mathcal{X}, d)$, and $i = 1, \ldots, N$. By Corollary 5.18 and domain monotonicity, we have

$$\mathrm{Cap}_{B(x,A_2 r/A_3)}(B(x, r/A_3)) \asymp \mathrm{Cap}_{B(x,A_2 r)}(B(x, r/A_3)) \leq \mathrm{Cap}_{B(x,A_2 r)}(B(x,r)),$$

for all $x \in \mathcal{X}, r \lesssim \mathrm{diam}(\mathcal{X}, d)$. We obtain the desired bound

$$\mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \lesssim \mathrm{Cap}_{B(x,A_2 r)}(B(x,r))$$

by combining the above three estimates. □

The following lemma is used to compare capacities at different scales.

Lemma 5.22. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $A_1 > 1$. There exist constants $C_1, C_2 > 1$ and $\gamma > 0$ that depend only on the constants in Assumption 5.6 such that for all $x, y \in \mathcal{X}$ and $0 < s \le r < \operatorname{diam}(\mathcal{X}, d)/C_1$, we have*

$$C_2^{-1} \left(\frac{r}{s}\right)^{-\gamma} \operatorname{Cap}_{B(x,A_1 s)}(B(x,s)) \le \operatorname{Cap}_{B(x,A_1 r)}(B(x,r)) \le C_2 \left(\frac{r}{s}\right)^{\gamma} \operatorname{Cap}_{B(x,A_1 s)}(B(x,s)).$$

Proof. By Lemma 5.21, we may assume without loss of generality that $A > 16K$, where $K \ge 2$ is the constant in RBC(K) condition. By Remark 5.11, Corollary 5.14, Lemma 5.10 and domain monotonicity, we have

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \asymp g_{B(x,Ar)}(x,r)^{-1} \lesssim \left(\frac{r}{s}\right)^{\theta} g_{B(x,Ar)}(x,s)^{-1} \lesssim \operatorname{Cap}_{B(x,As)}(B(x,s))$$

for all $x \in \mathcal{X}, 0 < s \le r \lesssim \operatorname{diam}(\mathcal{X}, d)$, where $\theta > 0$ is as given in Corollary 5.14.

For the reverse inequality, we use Corollary 5.18 repeatedly and domain monotonicity to obtain

$$\operatorname{Cap}_{B(x,As)}(B(x,s)) \lesssim \left(\frac{r}{s}\right)^{\theta_1} \operatorname{Cap}_{B(x,Ar)}(B(x,s)) \le \left(\frac{r}{s}\right)^{\theta_1} \operatorname{Cap}_{B(x,Ar)}(B(x,r)),$$

for all $x \in \mathcal{X}, 0 < s \le r \lesssim \operatorname{diam}(\mathcal{X}, d)$, where $\theta_1 = \log_2 C_1 > 0$, where C_1 is as given in Corollary 5.18. Setting $\gamma = \max(\theta, \theta_1)$, we obtain the desired conclusion. □

6 Good doubling measures

As in [BM1, Section 4] we now use the argument of Volberg and Konyagin [VK] to construct a new measure μ such that $(\mathcal{X}, d, \mu)$ satisfies *volume doubling*; that is, there is a constant $c > 1$ so that $\mu(B(x, 2r)) \le c\,\mu(B(x,r))$ for all $x \in \mathcal{X}$ and $r > 0$. We need further that μ relates well with capacities – see Definition 6.1 below. One key difference from [BM1] is that we do not assume bounded geometry condition on the original MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$). Another difference from [BM1] is that we do not have any cutoff at small length scales. This means that μ need not be absolutely continuous with respect to m, and it is not a priori clear that μ is a smooth measure having full quasi support on $\mathcal{X}$. This property is established in Proposition 6.17 of this section. The key inputs from the previous section are inequalities controlling capacities Corollary 5.18, Lemma 5.19 and Proposition 5.20.

6.1 Construction of a doubling measure

The following definition is a simplification of [BM1, Definition 4.1]: we do not require absolute continuity with respect to the reference measure m. We do not require the volume doubling property for the measure ν either – this will follow from Lemma 6.3.

Definition 6.1. Let D be either a finite ball $B(x_0, R) \subset \mathcal{X}$ or the whole space $\mathcal{X}$. If $D = \mathcal{X}$, fix $x_0 \in \mathcal{X}$. Let $C_0 \in (1, \infty)$ and $0 < \beta_1 \le \beta_2$. Let A denote the constant in Corollary 5.18. If $D = B(x_0, R)$ with $R < \infty$, let $I = (0, A^{-2}R^2)$; and if $D = \mathcal{X}$ with $\operatorname{diam}(\mathcal{X}, d) = \infty$, then let $I = (0, \infty)$. We say a Borel measure ν on D is $(C_0, A, \beta_1, \beta_2)$-*capacity good* if for all $x \in B(x_0, R)$ and $s_1, s_2 \in I$ with $s_1 < s_2$,

$$C_0^{-1} \left(\frac{s_2}{s_1}\right)^{\beta_1} \le \frac{\nu(B(x,s_2)) \operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1)) \operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \le C_0 \left(\frac{s_2}{s_1}\right)^{\beta_2}. \tag{6.1}$$

Since ν is locally finite, any capacity good measure ν is a Radon measure.

In this section, we often make the following assumption.

Assumption 6.2. We assume that $(\mathcal{X}, d)$ is a complete locally compact separable metric space, that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the (scale invariant) EHI with constants C_H, δ_H. Furthermore, we assume that the metric space (X, d) satisfies one (and hence all) of the equivalent conditions in Theorem 5.4.

Under Assumption 6.2, we observe by Corollary 5.18 that the second inequality in (6.1) of Definition 6.1 implies the volume doubling property for ν.

Lemma 6.3. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. Let ν be a $(C_0, A, \beta_1, \beta_2)$-capacity good measure on $\mathcal{X}$. Then it satisfies the volume doubling property.*

Proof. If $\mathrm{diam}(\mathcal{X}, d) = \infty$, then the volume doubling property follows from Corollary 5.18 and domain monotonicity of capacity, since $\mathrm{Cap}_{B(x,As)}(B(x,s)$ and $\mathrm{Cap}_{B(x,2As)}(B(x,2s)$ are comparable.

In the case $\mathrm{diam}(\mathcal{X}, d) < \infty$, we view $\mathcal{X}$ as the closed ball $\overline{B(x_0, \mathrm{diam}(\mathcal{X}, d))}$ and use Corollary 5.18 to obtain the volume doubling property for balls $B(x,s)$ with $s \lesssim \mathrm{diam}(\mathcal{X}, d)$. The volume doubling property for larger balls follows from a covering argument and the metric doubling property. $\square$

The following is the main result of this section.

Theorem 6.4 (Construction of a doubling measure)**.** *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. Then there exist constants $C_0 > 1$, $0 < \beta_1 \le \beta_2$ and a measure μ on $\mathcal{X}$ which is $(C_0, A, \beta_1, \beta_2)$-capacity good.*

The proof of Theorem 6.4 requires a preparation of a few results. We begin by adapting the argument in [VK] to construct a measure with the desired properties on a family of compact sets. We then follow [LuS] and obtain μ as a weak* limit of measures defined on an increasing family of compact sets.

The proof uses a family of generalized dyadic cubes, which provide a family of nested partitions of a space. Such a decomposition of space was introduced by Christ [Chr, Theorem 11]. The following is a slight modification of the construction in [KRS, Theorem 2.1]. Since the requirement (f) and (g) are new, we provide some details.

Lemma 6.5. *Let $(\mathcal{X}, d)$ be a complete, metric space satisfying RBC(K) property. Let $x_0 \in \mathcal{X}$ and $A \ge 8$. Then there exists a collection $\{Q_{k,i} : k \in \mathbb{Z}, i \in I_k \subset \mathbb{Z}_+\}$ of Borel sets satisfying the following properties:*

(a) *$\mathcal{X} = \cup_{i \in I_k} Q_{k,i}$ for all $k \in \mathbb{Z}_+$.*

(b) *If $m \le n$ and $i \in I_n$, $j \in I_m$, then either $Q_{n,i} \cap Q_{m,j} = \emptyset$ or $Q_{n,i} \subset Q_{m,j}$.*

(c) *For every $k \in \mathbb{Z}$ and $i \in I_k$, there exists $x_{k,i} \in Q_{k,i}$ such that*

$$B(x_{k,i}, c_A A^{-k}) \subset Q_{k,i} \subset \overline{B(x_{k,i}, C_A A^{-k})},$$

where $c_A = \frac{1}{2} - \frac{1}{A-1}$, and $C_A = \frac{A}{A-1}$.

(d) *The sets $N_k = \{x_{k,i} : i \in I_k\}$, where $x_{k,i}$ are as defined in* (c)*, are increasing in k and $x_0 \in N_k$ for all $k \in \mathbb{Z}$; that is $N_k \subset N_{k+1}$ for all $k \in \mathbb{Z}$ and $x_0 \in \cap_{k \in \mathbb{Z}} N_k$.*

(e) *Property* (b) *defines a partial order* $\prec$ *on* $\mathcal{I} = \{(k,i) : k \in \mathbb{Z}_+, i \in I_k\}$ *by inclusion, where* $(k,i) \prec (m,j)$ *whenever* $Q_{k,i} \subset Q_{m,j}$.

(f) *There exists* $C_M = C_M(A) > 0$ *such that, for all* $k \in \mathbb{Z}_+$ *and for all* $x_{k,i} \in N_k$*, the 'successors'*

$$S_k(x_{k,i}) = \{x_{k+1,j} : (k+1,j) \prec (k,i)\}$$

satisfy

$$|S_k(x_{k,i})| \leq C_M \quad \text{for all } k \in \mathbb{Z}, i \in I_k. \tag{6.2}$$

Furthermore, we have $d(x_{k,i}, y) \leq A^{-k} r$ *for all* $y \in S_k(x_{k,i})$.

(g) *Let*

$$k_0 = \inf \{k \in \mathbb{Z} : |I_k| > 1\}, \tag{6.3}$$

where $|I_k|$ *denotes the cardinality of* I_k*. Then* $k_0 \in \mathbb{Z} \cup \{-\infty\}$ *satisfies*

$$c_A A^{-k_0} \leq \operatorname{diam}(\mathcal{X}, d) \leq 2C_A A^{1-k_0}. \tag{6.4}$$

For all $k \geq k_0$*,* $k \in \mathbb{Z}$ *and* $i \in I_k$*, we have* $|S_k(x_{k,i})| \geq 2$.

(h) *For all* $k \in \mathbb{Z}$*,* $Q_{k,0}$ *is compact and* $x_{k,0} = x_0$.

Proof. The sets $Q_{k,j}, k \in \mathbb{Z}, j \in I_k$ are refered to as 'generalized dyadic cubes'. We follow the construction in [KRS] with a minor modification so as to ensure the property (h).

We choose $N_0 \subset \mathcal{X}$ such that $x_0 \in N_0$ and $N_0 = \{x_{0,i} : i \in I_0\}$ is a maximal subset of $\mathcal{X}$ such that $d(x_{0,i}, x_{0,j}) \geq 1$ for all $i \neq j$ with $i, j \in I_0$. For $k > 0$, we define $N_k = \{x_{k,i} : i \in I_k\}$ is a maximal subset of $\mathcal{X}$ such that $N_{k-1} \subset N_k$ $d(x_{k,i}, x_{k,j}) \geq A^{-k}$ for all distinct $x_{k,i}, x_{k,j} \in N_k$. For $k < 0$, we define $N_k = \{x_{k,i} : i \in I_k\}$ as a maximal set such that $x_0 \in N_k \subset N_{k+1}$, $d(x_{k,i}, x_{k,j}) \geq A^{-k}$ for all distinct $x_{k,i}, x_{k,j} \in N_k$.

We label the indices I_k such that $x_{k,0} = x_0$ for all $k \in \mathbb{Z}$. For each $(k,i) \in \mathbb{Z} \times I_k$, we pick an element $(k-1, j) \in \mathbb{Z} \times I_{k-1}$ such that

$$d(x_{k,i}, x_{k-1,j}) = \min_{l \in I_{k-1}} d(x_{k,i}, x_{k-1,l}).$$

We define $\prec$ as the smallest partial order that contains the relations $(k,i) \prec (k-1,j)$ for all $(k,i) \in \mathbb{Z} \times I_k$, where $(k-1,j) \in \mathbb{Z} \times I_{k-1}$ is chosen as above.

We relabel the indices of I_0 of N_0 such that for all $k < 0$,

$$l_1 < l_2 \quad \text{for all } k < 0, l_1 \in \{i \in I_0 : (0,i) \prec (k,0)\} \text{ and } l_2 \in I_0 \setminus \{i \in I_0 : (0,i) \prec (k,0)\}. \tag{6.5}$$

This relabeling exists since $\{i \in I_0 : (0,i) \prec (k,i)\})$ is finite for all $k < 0$ (by the doubling property).

Define the sets $Q_{0,i}$ as

$$Q_{0,i} = \overline{\{x_{l,k} : (l,k) \prec (0,i)\}} \setminus \bigcup_{j < i, j \in I_0} Q_{0,j}$$

For $k < 0$, we define the sets $Q_{k,i}$ inductively as

$$Q_{k,i} = \bigcup_{(k+1,j) \prec (k,i)} Q_{k+1,j},$$

whereas for $k > 0$, we define

$$Q_{k,i} = Q_{k-1,j} \cap \overline{\{x_{l,j} : (l,j) \prec (k,i)\}} \setminus \bigcup_{j<i, j\in I_k} Q_{k,j}, \quad \text{where } (k,i) \prec (k-1,j).$$

Properties (a)-(f) are contained in [KRS, Theorem 2.1].
(g) The estimate $|S_k(x_{k,i})| \geq 2$ relies on the following consequence of RBC(K): $r \leq \operatorname{diam}(B(x,r)) \leq 2r$ for all $B(x,r) \neq \mathcal{X}$. Since $2C_A/c_A = 4A/(A-3) < A$ for all $A \geq 8$, we have

$$\operatorname{diam}(Q_{k,i}) \geq c_A A^{-k} > 2C_A A^{-k-1} \geq \operatorname{diam}(Q_{k+1,j}) \quad \text{for all } k > k_0, k \in \mathbb{Z}.$$

Hence $Q_{k,i} \neq Q_{k+1,j}$ for all $k > k_0, i \in I_k, j \in I_{k+1}$, and therefore $|S_k(x_{k,i})| > 2$ for all $k > k_0$.

Clearly by (c), $\operatorname{diam}(\mathcal{X}, d) = \infty$ if and only if $k_0 = -\infty$. If $k_0 \in \mathbb{Z}$, the estimate (6.4) follows from $B(x_0, c_A A^{-k_0}) \subset Q_{k_0,0} \subsetneq \mathcal{X} = Q_{k_0-1,0} \subset B(x_0, C_A A^{-k_0+1})$.
(h) By (6.4), $Q_{k,0}$ is closed for all $k \in \mathbb{Z}$, since $Q_{k,0} = \overline{\{(l,j) : (l,j) \prec (k,0)\}}$. By (c) and (MD), $Q_{k,0}$ is compact for all $k \geq 0$. $\square$

We fix a family

$$\{Q_{k,i} : k \in \mathbb{Z}, i \in I_k \subset \mathbb{Z}_+\},$$

of generalized dyadic cubes as given by Lemma 6.5, and define the nets N_k and successors $S_k(x)$ as in the lemma.

Definition 6.6. We define the *predecessor* $P_k(x)$ of $x \in N_k$ to be the unique element of N_{k-1} such that $x \in S_{k-1}(P_k(x))$. Note that for $x \in N_k$, $S_k(x) \subset N_{k+1}$ whereas $P_k(x) \in N_{k-1}$. For $x \in B_0$, we denote by $Q_k(x)$ the unique $Q_{k,i}$ such that $x \in Q_{k,i}$.

Let $k_0 \in \mathbb{Z} \cup \{-\infty\}$ be as defined in (6.3). For $k \in \mathbb{Z}$ with $k \geq k_0 + 2$, denote by $c_k(x)$ the relative capacity

$$c_k(x) = \operatorname{Cap}_{B(x, A^{-k+1})}(Q_k(x)). \tag{6.6}$$

The following lemma provides useful estimates on c_k. Note that if $k \geq k_0 + 2$, then

$$A^{-k+1} \leq A^{-k_0-1} \leq c_A^{-1} A^{-1} \operatorname{diam}(\mathcal{X}, d) = \frac{2(A-1)}{(A-3)A} \operatorname{diam}(\mathcal{X}, d).$$

Lemma 6.7 (Relative capacity estimates for generalized dyadic cubes)**.** *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. There exists $A_0 \geq 8$ such that the following hold.*

(a) *For all $A \geq A_0$, there exists $C_1 = C_1(A)$ such that for all $k \geq k_0 + 2, x, y \in N_k$ with $d(x,y) \leq 4A^{-k}$, we have*
$$C_1^{-1} c_k(y) \leq c_k(x) \leq C_1 c_k(y). \tag{6.7}$$

(b) *For all $A \geq A_0$, there exists $C_1 = C_1(A)$ such that for all $k \geq k_0 + 2, x \in N_k$ and $y \in S_k(x)$, we have*
$$C_1^{-1} c_k(x) \leq c_{k+1}(y) \leq C_1 c_k(x). \tag{6.8}$$

(c) *For all $A \geq A_0$, there exists $C_1 = C_1(A)$ such that for all $x \in \mathcal{X}$ and $s < \operatorname{diam}(\mathcal{X}, d)/A^4$,*
$$C_1^{-1} c_k(x) \leq \operatorname{Cap}(B(x,s), B(x, As)) \leq C_1 c_k(x) \tag{6.9}$$
where $k \in \mathbb{Z}$ is the unique integer such that $A^{-k} \leq s < A^{-k+1}$.

Proof. We use domain monotonicity of capacity along with Corollary 5.18 and Lemma 5.19 to obtain the above the estimates. For (c), note that $A^{-k} \le s < \mathrm{diam}(\mathcal{X}, d)/A^4 \le 2C_A A^{-k_0-3} < A^{-k_0-2}$ implies $k \ge k_0 + 2$. $\square$

We record one more estimate regarding the subadditivity of c_k, which will play an essential role in ensuring (6.1).

Lemma 6.8. *([BM1, Lemma 4.6]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. There exists $A_0 \ge 4, \delta = \delta(A) \in (0, 1)$ such that for all $k \in \mathbb{Z}$, $k \ge k_0 + 2$, $A \ge A_0$, for all $x \in N_k$, we have*

$$c_k(x) \le (1 - \delta) \sum_{y \in S_k(x)} c_{k+1}(y).$$

Henceforth, we fix an $A \ge 8$ large enough such that the conclusions of Lemmas 6.7 and 6.8 hold.

We need the following modification of [VK, Lemma, p. 631], which was stated in [BM1, Lemma 4.7] without a proof. For reader's convenience, we provide its full proof below.

Lemma 6.9. *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. Let $c_k(\cdot)$, $k \ge k_0 + 2$, $k \in \mathbb{Z}$ denote the capacities of the corresponding generalized dyadic cubes as defined in* (6.6)*. There exists $C > 1$ satisfying the following. Let μ_k be a probability measure on N_k such that*

$$\frac{\mu_k(e')}{c_k(e')} \le C^2 \frac{\mu_k(e'')}{c_k(e'')} \quad \text{for all } e', e'' \in N_k \text{ with } d(e', e'') \le 4A^{-k}. \tag{6.10}$$

Then there exists a probability measure μ_{k+1} on N_{k+1} such that

(1) *For all $g', g'' \in N_{k+1}$ with $d(g', g'') \le 4A^{-k-1}$ we have*

$$\frac{\mu_{k+1}(g')}{c_{k+1}(g')} \le C^2 \frac{\mu_{k+1}(g'')}{c_{k+1}(g'')}. \tag{6.11}$$

(2) *Let $\delta \in (0, 1)$ be the constant in Lemma 6.8. For all points $e \in N_k$ and $g \in S_k(e)$,*

$$C^{-1} \frac{\mu_k(e)}{c_k(e)} \le \frac{\mu_{k+1}(g)}{c_{k+1}(g)} \le (1 - \delta) \frac{\mu_k(e)}{c_k(e)}. \tag{6.12}$$

(3) *The construction of the measure μ_{k+1} from the measure μ_k can be regarded as the transfer of masses from the points N_k to those of N_{k+1}, with no mass transferred over a distance greater than $(1 + 4/A)A^{-k}$.*

Proof. By the metric doubling property

$$\sup_{k \in \mathbb{Z}} \sup_{x \in N_k} |S_k(x)| = S < \infty, \tag{6.13}$$

where $S_k(x)$ is as define in Lemma 6.5(f). We choose

$$C = C_1 S,$$

where C_1 is chosen such that (6.7), (6.8) and (6.9) hold. For any probability measure μ_k on N_k supported on N_k such that

$$\frac{\mu_k(e')}{c_k(e')} \le C^2 \frac{\mu_k(e'')}{c_k(e'')}$$

for any points $e', e'' \in N_k$ with $d(e', e'') \le 4A^{-k}$.

The transfer of mass is accomplished in two steps. In the first step we distribute the mass $\mu_k(e)$ to all its successors $S_k(e)$ such that mass of $g \in S_k(e)$ is proportional to $c_{k+1}(g)$, that is

$$f_1(g) = \frac{c_{k+1}(g)}{\sum_{g' \in S_k(e)} c_{k+1}(g')} \mu_k(e),$$

for all $e \in N_k$ and $g \in S_k(e)$.

By (6.13), Lemma 6.7 and Lemma 6.8, we have

$$C^{-1} \frac{\mu_k(e)}{c_k(e)} \le \frac{f_0(g)}{c_{k+1}(g)} \le (1-\delta) \frac{\mu_k(e)}{c_k(e)}, \tag{6.14}$$

for all points $e \in N_k$ and $g \in S_k(e)$. If the measure f_1 on N_{k+1} satisfies condition (1) of the Lemma, we set $\mu_{k+1} = f_1$. This is the desired measure. Condition (2) is satisfied by (6.14), and (3) and (4) are obviously satisfied by Lemma 6.5(c). The second step is not necessary in this case.

But if f_0 does not satisfy condition (1) of the Lemma, then we proceed as follows at the second step. Let $p_1, \ldots, p_T$ be the indexed pairs of points $\{g', g''\}$ with $g', g'' \in N_{k+1}$ with $0 < d(g', g'') \le 4A^{-k-1}r$. Take the pair $p_1 = \{g_1', g_1''\}$. If $\frac{f_0(g_1')}{c_{k+1}(g_1')} \le C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}$ and $\frac{f_0(g_1'')}{c_{k+1}(g_1'')} \le C^2 \frac{f_0(g_1')}{c_{k+1}(g_1')}$, then we set $f_1 = f_0$. Assume one of the inequalities is violated, say $\frac{f_0(g_1')}{c_{k+1}(g_1')} > C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}$. Then we construct a measure f_1 from f_0 such that

$$\begin{aligned} f_1(g_1') &= f_0(g_1') - \alpha_1, \\ f_1(g_1'') &= f_0(g_1'') + \alpha_1, \\ f_1(g) &= f_0(g), \quad g \ne g_1', g_1'', \end{aligned}$$

where $\alpha_1 > 0$ is chosen such that

$$\alpha_1 \left(\frac{C^2}{c_{k+1}(g_1'')} + \frac{1}{c_{k+1}(g_1')} \right) = \frac{f_0(g_1')}{c_{k+1}(g_1')} - C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}.$$

It is clear that $\frac{f_1(g_1')}{c_{k+1}(g_1')} = C^2 \frac{f_1(g_1'')}{c_{k+1}(g_1'')}$.

The next step is the construction of measure f_2 from f_1 in exactly the same way that f_1 was constructed from f_0. Here we consider the pair p_2. A measure f_3 is next constructed from f_2 and so on. We claim that $\mu_{k+1} = f_T$ is the desired measure in the lemma.

We first verify that for all $e \in N_k$, for all $g \in S_k(e)$ and for all $s = 0, 1, \ldots, T$, we have

$$C^{-1} \frac{\mu_k(e)}{c_k(e)} \le \frac{f_s(g)}{c_{k+1}(g)} \le (1-\delta) \frac{\mu_k(e)}{c_k(e)}. \tag{6.15}$$

By (6.14), it is clear that (6.15) holds for $s = 0$. We now show (6.15) by induction. Suppose (6.15) holds for $s = j$, we will verify it for $s = j+1$. Let $p_{j+1} = (g', g'')$, $e' = P_{k+1}(g'), e'' = P_{k+1}(g'')$. If $f_j = f_{j+1}$, there is nothing to prove. But if $f_{j+1} \ne f_j$, then assume, say, that

$$\frac{f_j(g')}{c_{k+1}(g')} > C^2 \frac{f_j(g'')}{c_{k+1}(g'')}. \tag{6.16}$$

By (6.16) and the construction, we have

$$f_{j+1}(g') < f_j(g'), \quad f_{j+1}(g'') > f_j(g''). \tag{6.17}$$

Therefore by the induction hypothesis (6.15) for $s = j$ and (6.17), we have

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} \le (1-\delta)\frac{\mu_k(e')}{c_k(e')}, \quad \frac{f_{j+1}(g'')}{c_{k+1}(g'')} \ge C^{-1}\frac{\mu_k(e')}{c_k(e')}.$$

Therefore it suffices to verify that

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} \ge C^{-1}\frac{\mu_k(e')}{c_k(e')}, \quad \frac{f_{j+1}(g'')}{c_{k+1}(g'')} \le (1-\delta)\frac{\mu_k(e'')}{c_k(e'')}. \tag{6.18}$$

Suppose the first inequality in (6.18) fails to be true, then by construction, (6.17) and the induction hypothesis (6.15) for $s = j$, we have

$$C^{-1}\frac{\mu_k(e')}{c_k(e')} > \frac{f_{j+1}(g')}{c_{k+1}(g')} = C^2\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > C^2\frac{f_j(g'')}{c_{k+1}(g'')} \ge C\frac{\mu_k(e'')}{c_k(e'')}, \tag{6.19}$$

which implies $\frac{\mu_k(e')}{c_k(e')} > C^2\frac{\mu_k(e'')}{c_k(e'')}$. However $\frac{\mu_k(e')}{c_k(e')} \le C^2\frac{\mu_k(e'')}{c_k(e'')}$, by the assumption on μ_k, since

$$d(e', e'') \le d(e', g') + d(g', g'') + d(e'', g'') \le 2A^{-k}r + 4A^{-k-1}r \le 4A^{-k}r.$$

This proves the first inequality in (6.18). The proof of the second inequality in (6.18) is similar. Indeed, assume to the contrary that $\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > (1-\delta)\frac{\mu_k(e'')}{c_k(e'')}$; then we have

$$(1-\delta)\frac{\mu_k(e')}{c_k(e')} \ge \frac{f_j(g')}{c_{k+1}(g')} > \frac{f_{j+1}(g')}{c_{k+1}(g')} = C^2\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > C^2(1-\delta)\frac{\mu_k(e'')}{c_k(e'')}, \tag{6.20}$$

which again implies $\frac{\mu_k(e')}{c_k(e')} > C^2\frac{\mu_k(e'')}{c_k(e'')}$. Therefore (6.15) follows by induction. In particular, $\mu_{k+1} = f_T$ satisfies condition (2) of the lemma.

We now verify condition (1) for $\mu_{k+1} = f_T$. For this, it suffices to prove the following assertion:if

$$C^{-2}\frac{f_j(g'')}{c_{k+1}(g'')} \le \frac{f_j(g')}{c_{k+1}(g')} \le C^2\frac{f_j(g'')}{c_{k+1}(g'')} \tag{6.21}$$

holds for a pair of points $g', g'' \in N_{k+1}$ such that $d(g', g'') \le 4A^{-k-1}r$, then the same inequalities hold when f_j is replaced by f_{j+1}.

We now prove this. If $p_{j+1} = \{g', g''\}$, then $f_{j+1} = f_j$ and there is nothing to prove. If $p \cap p_{j+1} = \emptyset$, then again there is nothing to prove. Let $p_{j+1} = \{g_1, g_2\}$. Without loss of generality, we assume $p_{j+1} \cap \{g', g''\} = \{g_1\}$ where $g_1 = g''$ and $f_j(g'')/c_{k+1}(g'') > C^2 f_j(g_2)/c_{k+1}(g_2)$. Then

$$\frac{f_{j+1}(g'')}{c_{k+1}(g'')} = C^2\frac{f_{j+1}(g_2)}{c_{k+1}(g_2)}, \quad f_{j+1}(g'') < f_j(g''), \quad f_{j+1}(g') = f_j(g'). \tag{6.22}$$

Therefore, only the second inequality in (6.21) can fail for f_{j+1}. Suppose that this happens, that is

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} > C^2\frac{f_{j+1}(g'')}{c_{k+1}(g'')}. \tag{6.23}$$

Let $e' = P_{k+1}(g')$ and $e_2 = P_{k+1}(g_2)$. Then by (6.23), (6.22) and (6.15)

$$(1-\delta)\frac{\mu_k(e')}{c_k(e')} \geq \frac{f_{j+1}(g')}{c_{k+1}(g')} > C^2 \frac{f_{j+1}(g'')}{c_{k+1}(g'')} = C^4 \frac{f_{j+1}(g_2)}{c_{k+1}(g_2)} \geq C^3 \frac{\mu_k(e_2)}{c_k(e_2)}, \tag{6.24}$$

which implies that $\frac{\mu_k(e')}{c_k(e')} > C^2 \frac{\mu_k(e_2)}{c_k(e_2)}$. However since $d(e', e_2) \leq d(e', g') + d(g', g'') + d(g_1, g_2) + d(g_2, e_2) \leq 2(A^{-k} + 4A^{-k-1}) \leq 4A^{-k}$, we have a contradiction and hence (6.23) is false. This shows (6.21) for the case $f_j(g'')/c_{k+1}(g'') > C^2 f_j(g_2)/c_{k+1}(g_2)$. The case $f_j(g'')/c_{k+1}(g'') < C^{-2} f_j(g_2)/c_{k+1}(g_2)$ is analyzed similarly and therefore the assertion given by (6.21) is proved. It remains to observe that this assertion proves condition (2) of the lemma for the measure $\mu_{k+1} = f_T$. Along the path from f_0 to f_T, we "correct" the measure at all pairs of points where condition (1) is violated, and the assertion given by (6.21) shows that once a pair is corrected, it remains corrected when further changes are made.

Condition (3) is easily seen to be true. It remains to verify condition (4). Note that by Lemma 6.5, there was a mass transfer over a distance of at most A^{-k} while passing from μ_k to f_0. Therefore it suffices to verify that while passing from f_0 to $f_T = \mu_{k+1}$ there is a transfer over a distance of at most $4A^{-k-1}$.

We will now verify this. It suffices to verify that there are no pairs $p_l = \{g_1, g_2\}$, $p_m = \{g_2, g_3\}$, $1 \leq l < m \leq T$, such that mass is transferred from g_1 to g_2 (in the transition from f_{l-1} to f_l) and then mass is transferred from g_2 to g_3 (in the transition from f_{m-1} to f_m). Assume the opposite. First note that that the assertion given by (6.21) can be modified as follows. If the second inequality in (6.21) is true for f_j it remains true for f_{j+1}. The same argument as before goes through. Using this modified version of the assertion, as the assumption that there is mass transfer from g_1 to g_2 followed by mass transfer from g_2 to g_3, we have

$$\frac{f_0(g_1)}{c_{k+1}(g_1)} > C^2 \frac{f_0(g_2)}{c_{k+1}(g_2)}, \quad \frac{f_0(g_2)}{c_{k+1}(g_2)} > C^2 \frac{f_0(g_3)}{c_{k+1}(g_3)}. \tag{6.25}$$

If $e_1 = P_{k+1}(g_1)$, $e_3 = P_{k+1}(g_3)$, then

$$d(e_1, e_3) \leq d(e_1, g_1) + d(g_1, g_2) + d(g_2, g_3) + d(g_3, e_3) \leq 2r(A^{-k} + 4A^{-k-1}) \leq 4A^{-k}.$$

Consequently by assumption, $\mu_k(e_1)/c_k(e_1) \leq C^2 \mu_k(e_3)/c_k(e_3)$. However the inequalities (6.25) and (6.15) imply the opposite inequality $\mu_k(e_1)/c_k(e_1) > C^2 \mu_k(e_3)/c_k(e_3)$. We have arrived at the desired contradiction and the proof of the lemma is complete. $\square$

Remark 6.10. If $M_k \subset N_k$ and if μ_k is a probability measure on M_k satisfying the assumptions of Lemma 6.9, then the above construction yields a measure μ_{k+1} on $M_{k+1} \subset N_{k+1}$ satisfying the conclusion on Lemma 6.9, where $M_{k+1} = \bigcup_{y \in M_k} S_{k+1}(y)$.

We now adapt the method in [VK] to construct the doubling measure.

Proposition 6.11 (Measure in a cube). *Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. Let $l \geq k_0 - 1$. There exist constants C_0, $A > 1$ and $0 < \beta_1 \leq \beta_2$ such that for any integer $l \geq k_0 - 1$, there exists a $(C_0, A, \beta_1, \beta_2)$-capacity good measure $\nu = \nu_l$ on $Q_{l,0}$.*

Proof. Choose A large enough such that the conclusion of Lemma 6.9 holds. The cubes $Q_{k,0}$ for $k \leq 0$ and $Q_{k,j} \subset Q_{l,0}$ for $k > l, j \in I_k$ form a generalized dyadic decomposition of the compact space $Q_{k,0}$. Let $N_k, k \in \mathbb{Z}$ be as given in Lemma 6.5(d) for the generalized dyadic decomposition of $Q_{l,0}$ as mentioned above. Let μ_{l+3} be the probability measure on N_{l+3} such that μ_2 is proportional to c_2; that is

$$\mu_{l+3}(x) = \frac{c_{l+3}(x)}{\sum_{y \in N_{l+3}} c_{l+3}(y)}, \quad \text{for all } x \in N_{l+3}.$$

We use Lemma 6.9 and Remark 6.10 to inductively construct probability measures μ_k on N_k for all $k \geq l+3$. We define the measure $\nu = \nu_l$ as a weak (sub-sequential) limit of the measures μ_k as $k \to \infty$ (the existence of such a limit follows from the compactness of $Q_{l,0}$). We claim that

$$\nu \text{ is } (C_0, A, \beta_1, \beta_2)\text{-capacity good for some } C_0, A, \beta_1, \beta_2 > 0. \tag{6.26}$$

For each $x \in Q_{l,0}$ and $k \geq l+3$, we choose a point $e_{x,k} \in N_k$ such that

$$d(x, N_k) = \min_{e \in N_k} d(x, e) = d(x, e_{x,k}) \leq C_A A^{-k}.$$

If $s < A^{-4} \operatorname{diam}(Q_{l,0}) \leq A^{-3} 2 C_A A^{-l} \leq A^{-l-3}$, then $s < A^{-l-3}$. In order to show (6.26), we prove the following two sided estimate on measure of balls: there exists $C_2 \geq 1$ such that

$$C_2^{-1} \mu_n(e_{x,n}) \leq \nu(B(x,s) \cap Q_{l,0}) \leq C_2 \mu_n(e_{x,n}), \quad \text{for all } x \in Q_{l,0}, s < A^{-l-3}, \tag{6.27}$$

where n is the unique integer such that $A^{-n-1} \leq s < A^{-n}$.

Note that, by Lemma 6.9 the mass from $e \in N_k$ travels a distance of at most

$$(1 + 4A^{-1}) \sum_{l=k}^{\infty} A^{-l} = C_3 A^{-k}, \quad \text{where } C_3 := (1 + 4A^{-1})(1 - A^{-1})^{-1} \leq \tfrac{12}{7}. \tag{6.28}$$

Therefore, none of the mass outside $N_n \cap B(x, (1 + C_3)A^{-n})$ falls in $B(x,s)$, and therefore

$$\nu(B(x,s)) \leq \mu_n\left(N_n \cap B(x, (1 + C_3)A^{-n})\right) \quad \text{for all } x \in Q_{l,0}, s \in (0, A^{-l-3}). \tag{6.29}$$

By the triangle inequality, if $e \in N_n \cap B(x, (1 + C_3)A^{-n})$, then

$$d(e, e_{x,n}) \leq d(e, x) + d(x, e_{x,n}) \leq (1 + C_3)A^{-n} + C_A A^{-n} \leq 4A^{-n}.$$

Therefore by (6.29), (6.10), (6.7), and the metric doubling property, we obtain the upper bound $\nu(B(x,s)) \lesssim \mu_n(e_{x,n})$ in (6.27).

For the lower bound in (6.27), using (6.28), we have that for all $x \in Q_{l,0}$ and for all $s < A^{-l-3}$ with $A^{-n-1} \leq s < A^{-n}, n \in \mathbb{Z}_+$, the mass from $e_{x,n+2}$ travels a distance of at most $C_3 A^{-n-2} \leq \frac{12}{7} A^{-n-2}$ from $e_{x,n+2}$. Since $d(x, e_{n+2,x}) \leq A^{-n-2}$, we have that the mass from $e_{n+2,x}$ stays within

$$B(x, \frac{19}{7} A^{-n-2}) \subset B(x, \frac{19}{7A} s) \subset B(x, s/2).$$

Therefore

$$\nu(B(x,s)) \geq \mu_{n+2}(e_{x,n+2}) \tag{6.30}$$

By (6.12) and (6.8), we obtain that $\mu_{n+2}(e_{x,n+2})$ and $\mu_n(P_{n+1}(P_{n+2}(e_{x,n+2})))$ are comparable, where P_{n+1}, P_{n+2} denote the predecessor as given in Definition 6.6. By triangle inequality, we obtain that $d(e_{x,n}, P_{n+1}(P_{n+2}(e_{x,n+2}))) \leq 4A^{-n}$, and therefore by (6.7), we obtain that $\mu_n(P_{n+1}(P_{n+2}(e_{x,n+2})))$ and $\mu_n(e_{x,n})$ are comparable. Combining the above with (6.30), we obtain the lower bound $\nu(B(x,s)) \gtrsim \mu_n(e_{x,n})$ in (6.27). This completes the proof of (6.27).

Next, we obtain (6.26) from (6.27). Let $0 < s_1 < s_2 < A^{-l-3}$. Let $n_1, n_2 \in \mathbb{Z}$ such that $A^{-n_i-1} \le s_i < A^{-n_i}$ for $i = 1, 2$. For $x \in Q_{l,0}$, let $x_{n_i} \in N_{n_i}$ be unique point in N_{n_i} such that $x_{n_i} \in Q_{n_i}(x)$. By (6.27) and Lemma 6.7(c), we have

$$\frac{\nu(B(x,s_2)) \operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1)) \operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \asymp \frac{\mu_{n_2}(x_{n_2})c_{n_1}(x)}{\mu_{n_1}(x_{n_1})c_{n_2}(x)}$$

Let $\tilde{x}_{n_2}$ be the unique point in N_{n_2} such that $x_{n_1} \in Q_{n_1}(\tilde{x}_{n_2})$. By the triangle inequality and Lemma 6.9(1), we obtain

$$\frac{\nu(B(x,s_2)) \operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1)) \operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \asymp \frac{\mu_{n_2}(\tilde{x}_{n_2})c_{n_1}(x_{n_1})}{\mu_{n_1}(x_{n_1})c_{n_2}(\tilde{x}_{n_2})}$$

Next, by using Lemma 6.9(2), we obtain

$$(1-\delta)^{n_1-n_2} \lesssim \frac{\nu(B(x,s_2)) \operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1)) \operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \asymp \frac{\mu_{n_2}(\tilde{x}_{n_2})c_{n_1}(x_{n_1})}{\mu_{n_1}(x_{n_1})c_{n_2}(\tilde{x}_{n_2})} \lesssim C_2^{n_2-n_1}.$$

The desired estimate (6.26) follows by setting $\beta_1 = -\log(1-\delta)/\log A$ and $\beta_2 = \log C_2/\log A$. □

We are now in the position to give the

Proof of Theorem 6.4. The compact case follows by choosing $l = k_0 - 1$ in Proposition 6.11.

It suffices to consider the non-compact case. For $l \le -1, l \in \mathbb{Z}$ let ν_l be the measure given by Proposition 6.11 on $Q_{l,0}$, and choose $a_n > 0$

$$a_l \nu_l(B(x_0,1)) = 1, \quad \text{for all } l \in \mathbb{Z}, l < 0$$

By a compactness argument similar to that in [LuS] yields the existence of a measure ν which is a sub-sequential weak* limit of the sequence of measures $a_l\nu_l$ as $l \to -\infty$, bounded on compacts, such that it is $(C_0, A, \beta_1, \beta_2)$-capacity good. □

6.2 A criterion for smoothness of measure

In this section, we will provide a useful sufficient condition for a doubling measure to be smooth. The definition of a smooth measure is given in Definition 2.4.

Lemma 6.12. *(See [GH, Lemma 7.1 and 7.4] and [GNY, Lemma 2.5]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.2. There exists $C, A \ge 0$ such that for any ball $B(x,r), n \in \mathbb{N}$ with $A^n r < \operatorname{diam}(\mathcal{X}, d)/A$, denoting $B_k = B(x, A^k r)$, we have*

$$\sum_{k=0}^{n-1} \operatorname{Cap}_{B_{i+1}}(B_i)^{-1} \le \operatorname{Cap}_{B_n}(B_0)^{-1} \le C \sum_{k=0}^{n-1} \operatorname{Cap}_{B_{i+1}}(B_i)^{-1}.$$

Proof. The upper bound is contained in [GH, Lemma 7.1 and 7.4]. The upper bound in [GH] is under the additional volume doubling property assumption but the proof only uses the weaker metric doubling assumption. The lower bound is a general fact that does not require the EHI – see [GNY, Lemma 2.5]. □

The following lemma follows immediately from [CF, Theorem 3.3.8] or [FOT, Theorem 4.4.3] and the countable subadditivity for capacities.

Lemma 6.13. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a MMD space. Let $\{B_i : i \in I\}$ be a countable family of open balls such that $\cup_{i\in I} B_i = X$. Let $U \subset X$ be a Borel set. Then A has zero capacity for $(\mathcal{E}, \mathcal{F})$ if and only if $U_i := U \cap B_i$ has zero capacity for the part Dirichlet form $(\mathcal{E}^{2\cdot B_i}, \mathcal{F}^{2\cdot B_i})$ for all $i \in I$.*

Proposition 6.14. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies the EHI. Let μ be a capacity good measure. Then μ is a smooth Radon measure.*

Proof. Let A denote the constant in Lemma 6.12. Let $B = B(x_0, r)$ denote any ball such that $r \le \operatorname{diam}(\mathcal{X}, d)/A^2$. For $x \in \mathcal{X}, s < \operatorname{diam}(\mathcal{X}, d)/A^2$, we set $\Psi(x, s) = \frac{\mu(B(x,s))}{\operatorname{Cap}_{B(x,As)}(B(x,s))}$.

We will show that $x \mapsto \int_B g_B(x, y)\, \mu(dy)$ is bounded uniformly in B.

Fix $x \in B$ and set $B_i = B(x, A^{1-i} d(x, y))$, $A_i = B_i \setminus B_{i+1}$ for $i \in \mathbb{N}_{\ge 0}$.

$$
\begin{aligned}
\int_B g_B(x,y)\,\mu(dy) &\le \int_B g_{B(x,4r)}(x,y)\,\mu(dy) \text{ (by domain monotonicity)}\\
&\le \sum_{i=1}^{\infty} \int_{B\cap A_i} g_{B(x,Ar)}(x,y)\,\mu(dy) \quad (\text{since } \mu(\{x\}) = 0)\\
&\lesssim \sum_{i=0}^{\infty} \int_{B\cap A_i} \operatorname{Cap}_{B_0}(B_{i+1})^{-1}\,\mu(dy) \quad \text{(by Lemma 5.10)}\\
&\lesssim \sum_{i=0}^{\infty} \int_{B\cap A_i} \sum_{j=0}^{i} \operatorname{Cap}_{B_j}(B_{j+1})^{-1}\,\mu(dy) \quad \text{(by Lemma 6.12)}\\
&\lesssim \sum_{j=0}^{\infty} \operatorname{Cap}_{B_j}(B_{j+1})^{-1} \sum_{i=j}^{\infty} \int_{B\cap A_i} d\mu \lesssim \sum_{j=0}^{\infty} \operatorname{Cap}_{B_j}(B_{j+1})^{-1} \mu(B_j)\\
&\lesssim \sum_{j=0}^{\infty} \Psi(x, 2^{2-j} r) \lesssim \Psi(x_0, r). \quad \text{(by (6.1))}.
\end{aligned} \tag{6.31}
$$

Since the above bound is uniform in $x \in \mathcal{X}$, we obtain that

$$\int_B \int_B g_B(x, y)\, \mu(dx)\, \mu(dy) \lesssim \Psi(x_0, r)\mu(B) < \infty$$

for any ball $B = B(x_0, r)$ with $r < \operatorname{diam}(\mathcal{X}, d)/A^2$. This implies that $\mu\big|_B$ is of finite energy for the part Dirichlet form $(\mathcal{E}^B, \mathcal{F}^B)$ of $(\mathcal{E}, \mathcal{F})$ on B, and hence smooth for $(\mathcal{E}^B, \mathcal{F}^B)$. Since balls of the form $B = B(x_0, r)$ with $r < \operatorname{diam}(\mathcal{X}, d)/A^2$ can be used to form a countable cover of X, by [FOT, Lemma 2.2.3] and Lemma 6.13, the Radon measure μ assigns zero measure to every set of capacity zero. Hence μ is smooth for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. □

A smooth measure μ on $\mathcal{X}$ uniquely determines a positive continuous additive functional $A^\mu = \{A^\mu_t; t \ge 0\}$ of X. It can be used to define a time-changed process $Y_t := X_{\tau_t}$, where

$$\tau_t := \inf\{r > 0 : A^\mu_r > t\}.$$

Let $S(\mu)$ denote the quasi support of μ (see Definition 2.5) and F be the topological support of μ. Clearly $S(\mu) \subset F$ $\mathcal{E}$-q.e.. and $\mu(F \setminus S(\mu)) = 0$. Suppose μ is a smooth Radon measure. Then the time-changed process Y, after possibly modification on a Borel properly exception set for

X, is an μ-symmetric Hunt process on F and its associated Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ on $L^2(F;\mu)$ is regular. Moreover,

$$\begin{aligned}\mathcal{F}^\mu &= \left\{\phi \in L^2(X,\mu) : \phi = u \quad \mu\text{-a.e. for some } u \in \mathcal{F}_e\right\},\\ \mathcal{E}^\mu(\phi,\phi) &= \mathcal{E}(H_{S(\mu)}u, H_{S(\mu)}u), \quad \text{for } \phi \in \mathcal{F}^\mu, \text{ where } \phi = u \quad \mu\text{-a.e. for some } u \in \mathcal{F}_e, \end{aligned} \tag{6.32}$$

where $\mathcal{F}_e$ is the extended Dirichlet space of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and $H_{S(\mu)}u(x) = \mathbb{E}^x u(X_{\sigma_{S(\mu)}})$ for $x \in \mathcal{X}$. See [CF, Theorem 5.2.13] or [FOT, Theorem 5.1.5 and Theorem 6.2.1]. The Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is called the *trace Dirichlet form* of $(\mathcal{E}, \mathcal{F})$ on $L^2(S(\mu);\mu)$. If μ has full quasi support, then $\mathcal{F}^\mu_e = \mathcal{F}_e$ by [CF, Theorem 5.2.15] and (6.32) can be simplified as

$$\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X},\mu), \quad \mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) \quad \text{for all } u \in \mathcal{F}_e \tag{6.33}$$

Remark 6.15. The above mentioned properties for time-changed processes and Dirichlet forms in fact hold for any smooth measure μ rather than just smooth Radon measures except that the time-changed process is a right process instead of being a Hunt process on F and the trace Dirichlet for $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is quasi-regular on $L^2(S(\mu);\mu)$ instead of being regular on $L^2(F;\mu)$. See [CF, Theorem 5.2.7].

Recall the definition of quasi support of a smooth measure from Definition 2.5. In this work, we are interested in smooth measures with full quasi support as defined below.

Definition 6.16 (Admissible smooth measures)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space. We say that a smooth Radon measure μ on $\mathcal{X}$ is *admissible* if μ has full quasi support. In particular, the time-changed Dirichlet form is given by (6.33).

Proposition 6.17. *Suppose that $(\mathcal{X}, d)$ is relatively K ball connected for some $K > 1$ and $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies the EHI. Let μ be a capacity good (hence smooth) measure. Then μ is admissible.*

Proof. Let $\mathcal{N}_0$ be a Borel properly exceptional set for the Hunt process X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X};m)$. Denote by $S(\mu)$ a quasi support of μ. Following [BK, Proposition 2.6], it suffices to show that

$$\mathbb{P}^x(\sigma_{S(\mu)} = 0) = 1 \qquad \text{for quasi every } x \in \mathcal{X}. \tag{6.34}$$

For the reader's convenience, we recall why (6.34) implies that μ has full quasi support. By [FOT, Theorem 4.6.1(i)] we may assume that $S(\mu)^c$ is nearly Borel and finely open, by adjusting $S(\mu)$ on a set of capacity zero. Then since $S(\mu)^c$ is nearly Borel and finely open for any $x \in S(\mu)^c \setminus \mathcal{N}_0$, we have $\mathbb{P}^x(\sigma_{S(\mu)} > 0) = 1$, which by (6.34) implies that $S(\mu)^c$ has capacity zero.

Note that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible by Theorem 4.6. Let $x \in \mathcal{X} \setminus \mathcal{N}_0$. Let $t > 0$ and $\varepsilon > 0$ be arbitrary. Applying Lemma 3.1 to the subprocess $X^{B(x,R_0)}$ of X killed upon leaving a ball $B(x,R_0)$ whose complement has positive capacity, we can choose $r = r(x,t,\varepsilon)$ so that

$$\mathbb{P}^x(T < t) > 1 - \varepsilon, \quad \text{where } T = \sigma_{B(x,r)^c}. \tag{6.35}$$

By decreasing $r = r(x,t,\varepsilon)$ if necessary, we may assume that $0 < r < \operatorname{diam}(\mathcal{X}, d)/(4A)$, where A is the constant in capacity good condition. Fixing $r = r(x,t,\varepsilon)$ as above, we define

$$\begin{aligned} K_n &= \left(B(x, A^{-n}r) \setminus B(x, A^{-n-1}r)\right) \cap S(\mu),\\ A_n &= \{\sigma_{K_n} < T\}. \end{aligned}$$

We show that there exist constants $c_0 \in (0,1)$ and $N_0 \geq 1$ that depend only on the constants associated with Assumption 6.2) such that

$$\mathbb{P}^x(A_n) \geq c_0 \qquad \text{for all } n \geq N_0. \tag{6.36}$$

Let e_n denote the equilibrium measure for K_n such that $e_n(\overline{K_n}) = \mathrm{Cap}_B(K_n)$, where $B = B(x,r)$. To prove (6.36), by Proposition 5.7, there exists N_0 such that

$$\mathbb{P}^x(A_n) = \int_{\overline{K_n}} g_B(x,y)\, e_n(dy) \asymp g_B(x, A^{-n}r)\, \mathrm{Cap}_B(K_n), \quad \text{for all } n \geq N_0 \text{ and q.e. } x \in \mathcal{X}. \tag{6.37}$$

Using an argument similar to that of (6.31), for all $y \in \mathcal{X}, s > 0$ such that $B(y,2s) \subset B$, we have

$$\int_{B(y,s)} g_B(y,z)\, \mu(dz) \lesssim g_B(y,s)\mu(B(y,s)). \tag{6.38}$$

Using the fact that (X,d) is relatively ball connected, for any $n \geq 1$, there exists $y \in \mathcal{X}$ such that $B(y, A^{-n-1}r(A-1)/3) \subset B(x, A^{-n}r) \setminus B(x, A^{-n-1}r)$. Since μ is a doubling measure and $\mu(S(\mu)^c) = 0$, we obtain

$$\mu(K_n) = \mu\left(B(x, A^{-n}r) \setminus B(x, A^{-n-1}r)\right) \geq \mu\left(B(y, A^{-n-1}r(A-1)/3)\right) \gtrsim \mu\left(B(x, A^{-n}r)\right). \tag{6.39}$$

We recall the following variational characterization of capacity (Kelvin's principle):

$$\mathrm{Cap}_B(K_n)^{-1} = \inf_\nu \int_{\overline{K_n}} \int_{\overline{K_n}} g_B(y,z)\, \nu(dy)\, \nu(dz),$$

where ν varies over all Borel probability measures supported in $\overline{K}_n$. By considering the measure $\nu(\cdot) = \mu(K_n \cap \cdot)/\mu(K_n)$ and using Lemma 5.9, we obtain

$$\begin{aligned}
\mathrm{Cap}_B(K_n)^{-1} &\leq \mu(K_n)^{-2} \int_{K_n} \int_{K_n} g_B(y,z)\, \mu(dy)\, \mu(dz) \\
&\leq \mu(K_n)^{-2} \int_{B(x,A^{-n}r)} \int_{B(x,A^{-n}r)} g_B(y,z)\, \mu(dz)\, \mu(dy) \\
&\lesssim \mu(K_n)^{-2} \int_{B(x,A^{-n}r)} g_B(x, A^{-n}r) \mu(B(x, A^{-n}r))\, \mu(dz) \quad \text{(by (6.38))} \\
&\lesssim g_B(x, A^{-n}r), \quad \text{(by (6.39))}.
\end{aligned} \tag{6.40}$$

Combining (6.37) and (6.40) establishes the claim (6.36). Choosing $\varepsilon = c_0/2$ and $n \geq N_0$, we obtain

$$\begin{aligned}
\mathbb{P}^x(\sigma_{S(\mu)} \leq t) &\geq \mathbb{P}^x(\sigma_{K_n} < T) - \mathbb{P}^x(T \geq t) \quad (\text{since } \{\sigma_{K_n} < T\} \subset \left\{\sigma_{S(\mu)} \leq t\right\} \cup \{T \geq t\}) \\
&> c_0 - \varepsilon > \tfrac{1}{2}c_0. \quad \text{(by (6.35) and (6.36))}
\end{aligned}$$

Since $t > 0$ is arbitrary, the 0-1 law gives $\mathbb{P}^x(\sigma_{S(\mu)} = 0) = 1$. $\square$

7 Quasi-symmetry and stability

Although the assumption that all MMDs are strongly local is in force in this section, we remark that Lemma 7.1, Proposition 7.3, Lemma 7.5(a) in fact hold for general Dirichlet forms as well.

The following is a straightforward consequence of the definition of quasisymmetry.

Lemma 7.1. *([BM1, Lemma 5.3]) Let $(\mathcal{X}, d_i, \mu, \mathcal{E}, \mathcal{F}^\mu), i = 1, 2$ be two MMD spaces such that d_1 and d_2 are quasisymmetric. If $(\mathcal{X}, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI, then so does $(\mathcal{X}, d_1, \mu, \mathcal{E}, \mathcal{F}^\mu)$.*

The next definition is a slight modification of [BM1, Definition 5.4], the change being made so that it applies to both compact and non-compact spaces.

Definition 7.2. We say that a function $\Psi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ on a metric space $(\mathcal{X}, d)$ is a *regular scale function* if $\Psi(x, 0) = 0$ for all x and there exist constants $C_1, \beta_1, \beta_2 > 0$ such that, for all $x, y \in \mathcal{X}$ and finite $0 < s \le r \le \operatorname{diam}(\mathcal{X}, d)$, we have with $R := d(x, y)$

$$C_1^{-1}\Big(\frac{r}{R \vee r}\Big)^{\beta_2}\Big(\frac{R \vee r}{s}\Big)^{\beta_1} \le \frac{\Psi(x, r)}{\Psi(y, s)} \le C_1\Big(\frac{r}{R \vee r}\Big)^{\beta_1}\Big(\frac{R \vee r}{s}\Big)^{\beta_2}. \tag{7.1}$$

Given a regular scale function Ψ on $(\mathcal{X}, d)$, we now define a metric d_Ψ. This is proved as in [BM1] – the proof there still works when $\operatorname{diam}(\mathcal{X}, d) < \infty$.

Proposition 7.3. *([BM1, Proposition 5.7]) Let Ψ be a regular scale function on a metric space $(\mathcal{X}, d)$. There exists a metric $d_\Psi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying the following properties:*

(a) *There exist $C, \beta > 0$ such that for all $x, y \in \mathcal{X}$,*

$$C^{-1}\Psi(x, d(x, y)) \le d_\Psi(x, y)^\beta \le C\Psi(x, d(x, y)). \tag{7.2}$$

(b) *d and d_Ψ are quasisymmetric.*

(c) *Assume in addition that $(\mathcal{X}, d)$ (or equivalently $(\mathcal{X}, d_\Psi)$) is uniformly perfect. Fix $A > 1$. Let B_Ψ and B denote metric balls in $(\mathcal{X}, d_\Psi)$ and $(\mathcal{X}, d)$ respectively. If either $B_\Psi(x, s) \subset B(x, r) \subset B_\Psi(x, As) \subsetneq \mathcal{X}$ or $B(x, r) \subset B_\Psi(x, s) \subset B(x, Ar) \subsetneq \mathcal{X}$ holds for some $x \in \mathcal{X}$, $r > 0$ and $s > 0$, then there is a constant $C_1 > 1$ (which does not depend on $x \in \mathcal{X}, r > 0, s > 0$) such that*

$$C_1^{-1}s^\beta \le \Psi(x, r) \le C_1 s^\beta, \tag{7.3}$$

where $\beta > 0$ is as given by (7.2).

We now introduce Poincaré, cutoff energy inequalities, and capacity bounds with respect to a regular scale function Ψ on $(\mathcal{X}, d)$. This is again a slight modification of [BM1, Definition 5.8 and 5.13], so as to include both bounded and unbounded spaces. Recall that a *cutoff function* φ for $B_1 \subset B_2$ is any function $\varphi \in \mathcal{F}^\mu$ such that $0 \le \varphi \le 1$ in $\mathcal{X}$, $\varphi \equiv 1$ in an open neighbourhood of $\overline{B_1}$, and $\operatorname{supp}\varphi \Subset B_2$. Recall also that $\mu_{\langle f\rangle}$ is the energy measure of $f \in \mathcal{F}$; see Section 2

Definition 7.4. Let Ψ be a regular scale function on $(\mathcal{X}, d)$, and $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ a MMD space.

(i) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *Poincaré inequality* PI(Ψ), if there exists constants $C, A_1, A_2 \ge 1$ such that for all $x \in \mathcal{X}$, $R \in (0, \operatorname{diam}(\mathcal{X}, d)/A_2)$ and $f \in \mathcal{F}^\mu$

$$\int_{B(x,R)} (f - \overline{f})^2\, d\mu \le C\Psi(x, R)\, \mu_{\langle f\rangle}(B(x, A_1R)), \tag{PI(Ψ)}$$

where $\overline{f} = \frac{1}{\mu(B(x,r))}\int_{B(x,R)} f\, d\mu$.

(ii) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *cutoff energy inequality* $\mathrm{CS}(\Psi)$, if there exist $C_1, C_2 > 0, A_1, A_2 > 1$ such that the following holds. For all $R \in (0, \operatorname{diam}(\mathcal{X}, d)/A_2)$, $x \in \mathcal{X}$ with $B_1 = B(x, R)$ and $B_2 = B(x, A_1 R)$, there exists a cutoff function φ for $B_1 \subset B_2$ such that for any $u \in \mathcal{F}^\mu \cap L^\infty$,

$$\int_{B_2 \setminus B_1} u^2 d\mu_{\langle \varphi \rangle} \le C_1 \mu_{\langle u \rangle}(B_2 \setminus B_1) + \frac{C_2}{\Psi(x,R)} \int_{B_2 \setminus B_1} u^2 \, d\mu. \qquad \mathrm{CS}(\Psi)$$

(iii) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *capacity estimate* $\mathrm{cap}(\Psi)$ if there exist positive constants $C_1, A_1, A_2 > 1$ such that for all $R \in (0, \operatorname{diam}(\mathcal{X}, d)/A_2)$ and $x \in \mathcal{X}$

$$C_1^{-1} \frac{\mu(B(x,R)}{\Psi(x,R)} \le \operatorname{Cap}\left(B(x,R), B(x, A_1 R)^c\right) \le C_1 \frac{\mu(B(x,R)}{\Psi(x,R)}. \qquad \mathrm{cap}(\Psi)$$

If $\Psi(r) = r^\beta$, we denote $\mathrm{PI}(\Psi), \mathrm{CS}(\Psi), \mathrm{cap}(\Psi)$ by $\mathrm{PI}(\beta), \mathrm{CS}(\beta), \mathrm{cap}(\beta)$ respectively.

The following lemma shows that the Poincaré and cutoff energy inequalities take a much simpler form with respect to the metric d_Ψ.

Lemma 7.5. *([BM1, Lemma 5.9]) Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a uniformly perfect MMD space and let Ψ be a regular scale function. Let d_Ψ be the metric constructed in Proposition 7.3 with $\beta > 0$ as given in* (7.2)*. Then*

(a) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{PI}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{PI}(\beta)$.

(b) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{CS}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{CS}(\beta)$.

The following comparison of annuli follows readily from the definition.

Lemma 7.6. *([MT, Lemma 1.2.18]) Let the identity map $\mathrm{Id} : (\mathcal{X}, d_1) \to (\mathcal{X}, d_2)$ be an η-quasisymmetry for some distortion function η. Then for all $A > 1, x \in \mathcal{X}, r > 0$, there exists $s > 0$ such that, writing B_i for balls in $(\mathcal{X}, d_i)$*

$$B_2(x,s) \subset B_1(x,r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s). \tag{7.4}$$

In (7.4)*, s can be defined as*

$$s = \sup\left\{0 \le s_2 < 2 \operatorname{diam}(\mathcal{X}, d_1) : B_2(x, s_2) \subset B_1(x,r)\right\}$$

Moreover, for all $A > 1$, $x \in \mathcal{X}$ and $r > 0$, there exists $t > 0$ such that

$$B_1(x,r) \subset B_2(x,t) \subset B_2(x, At) \subset B_1(x, A_1 r), \tag{7.5}$$

where $A_1 = 1/\eta^{-1}(A^{-1})$. In (7.5)*, t can be defined as*

$$t = A^{-1} \sup\left\{0 \le r_2 < 2A \operatorname{diam}(\mathcal{X}, d_2) : B_2(x, Ar_2) \subset B_1(x, A_1 r)\right\}.$$

The following is an analogue of Lemma 7.5 for the capacity estimate $\mathrm{cap}(\Psi)$.

Lemma 7.7. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a quasi-arc connected MMD space that satisfies the EHI and let Ψ be a regular scale function. Suppose μ satisfy the volume doubling property on $(\mathcal{X}, d)$. Let d_Ψ be the metric constructed in Proposition 7.3 with $\beta > 0$ as given in* (7.2)*. Then $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\beta)$.*

Proof. Let B and B_Ψ denote balls in the metrics d and d_Ψ respectively. By Lemma 7.1, $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ also satisfies the EHI. Let the identity map $\mathrm{Id} : (\mathcal{X}, d) \to (\mathcal{X}, d_\Psi)$ be an η-quasisymmetry. Note that μ satisfies the volume doubling property with respect to the metric d and d_Ψ.

Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\Psi)$. Set $A_1 = \eta(2)$. By Lemma 5.21, we may assume that

$$\mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)}, \quad \text{for all } x \in \mathcal{X}, 0 < r \lesssim \mathrm{diam}(\mathcal{X}, d). \tag{7.6}$$

By Lemma 7.6 and Proposition 7.3(c), for all $0 < s < \mathrm{diam}(\mathcal{X}, d_\Psi)$, there exists $r > 0$ such that $B(x,r) \subset B_\Psi(x,s) \subset B_\Psi(x,2s) \subset B(x, \eta(2)r)$ and $s^\beta \asymp \Psi(x,r)$. By the volume doubling property $\mu(B(x,r)) \asymp \mu(B_\Psi(x,s))$. By domain monotonicity and (7.6), we have

$$\mathrm{Cap}_{B_\Psi(x,2s)}(B_\Psi(x,s)) \geq \mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)} \asymp \frac{\mu(B_\Psi(x,s))}{s^\beta}, \tag{7.7}$$

for all $x \in \mathcal{X}, 0 < s \lesssim \mathrm{diam}(\mathcal{X}, d_\Psi)$.

Set $A_2 = 1/\eta^{-1}(A_1^{-1})$. By Lemma 7.6 and Proposition 7.3(c), for all $s \in (0, \mathrm{diam}(\mathcal{X}, d_\Psi))$, there exists $r > 0$ such that $B_\Psi(x,s) \subset B(x,r) \subset B(x, A_1 r) \subset B_\Psi(x, A_2 r)$ and $\Psi(x,r) \asymp s^\beta$. By the volume doubling property, $\mu(B(x,r)) \asymp \mu(B_\Psi(x,s))$. By Lemma 5.21, domain monotonicity and (7.6), we have

$$\begin{aligned}\mathrm{Cap}_{B_\Psi(x,2s)}(B_\Psi(x,s)) &\asymp \mathrm{Cap}_{B_\Psi(x,A_2 s)}(B_\Psi(x,s)) \\ &\leq \mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)} \asymp \frac{\mu(B_\Psi(x,s))}{s^\beta}\end{aligned} \tag{7.8}$$

for all $x \in \mathcal{X}, 0 < s \lesssim \mathrm{diam}(\mathcal{X}, d_\Psi)$. By (7.7) and (7.8), $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\beta)$.

The converse follows from a similar argument. $\square$

We will now apply these results in the context of a change of measure on a MMD space. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space which satisfies the EHI and satisfy one (and hence all) of the three equivalent conditions in Theorem 5.4. Let $(\mathcal{E}, \mathcal{F}_e)$ be its corresponding extended Dirichlet space, and μ be the measure constructed in Theorem 6.4. By Propositions 6.14 and 6.17, μ is a positive Radon measure charging no set of capacity zero and possessing full quasi-support. Let $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denote the time-changed Dirichlet space with respect to μ as defined in (6.32). We have $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X}, \mu)$, $\mathcal{E}^\mu(f,f) = \mathcal{E}(f,f)$ for all $f \in \mathcal{F}^\mu$, and $\mathcal{F}_e^\mu = \mathcal{F}_e$ (cf. [CF, Theorems 5.2.2 and 5.2.15]). Moreover, the Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ on $L^2(\mathcal{X}; \mu)$ shares the same quasi notions as the original Dirichlet $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$; see [CF, Theorem 5.2.11].

Theorem 7.8. *Let $(\mathcal{X}, d)$ be complete, locally compact and quasi-arc connected. Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is a MMD space which satisfies the EHI. Let μ be a $(C_0, A, \beta_1, \beta_2)$-capacity good measure. Denote $D = \mathrm{diam}(\mathcal{X}, d)$. Then the function Ψ defined by $\Psi(x,0) = 0$ and*

$$\Psi(x,r) = \begin{cases} \frac{\mu(B(x,r))}{\mathrm{Cap}_{B(x,r/A^4)}(B(x,r/A^5))}, & \text{if } 0 < r < D, \\ \frac{\mu(B(x,D))}{\mathrm{Cap}_{B(x,D/A^4)}(B(x,D/A^5))}, & \text{if } r \geq D \text{ and } D < \infty, \end{cases} \tag{7.9}$$

is a regular scale function on $(\mathcal{X}, d)$. Furthermore, the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality $\mathrm{PI}(\Psi)$*, the cutoff energy inequality* $\mathrm{CS}(\Psi)$ *and the capacity estimate* $\mathrm{cap}(\Psi)$*.*

Proof. By volume doubling and Lemma 5.19(c), there exists $C_2 > 0$ such that for all $r > 0$ and for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$, we have

$$C_2^{-1}\Psi(x,r) \le \Psi(y,r) \le C_2\Psi(x,r). \tag{7.10}$$

If $R \le r$ the inequalities in (7.1) are immediate from Theorem 6.4 and (7.10). If $s < r < R$, then writing

$$\frac{\Psi(x,r)}{\Psi(y,s)} = \frac{\Psi(x,r)}{\Psi(x,R)}.\frac{\Psi(y,R)}{\Psi(y,s)}.\frac{\Psi(x,R)}{\Psi(y,R)},$$

and bounding each of the three terms on the right using Theorem 6.4 and (7.10) gives (7.1). Thus Ψ is a regular scale function.

By Lemma 5.22, the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies cap(Ψ).

Let d_Ψ and $\beta > 0$ be as given by Proposition 7.3. By Lemma 7.7, the MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies cap(β). By Lemma 7.1 and Proposition 7.3(b), $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI.

By Lemma 5.2 the space $(\mathcal{X}, d_\Psi)$ is uniformly perfect, and hence the measure μ on $(\mathcal{X}, d_\Psi)$ satisfies (RVD). Therefore by [GH, Theorem 3.14], since $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI and cap(β), it satisfies PI(β) and CS(β). We now conclude using Lemma 7.5. □

The following gives equivalent characterization of the EHI for a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$.

Theorem 7.9. *Let $(\mathcal{X}, d)$ be a complete, locally compact, quasi-arc connected metric space with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. The following are equivalent:*

(a) *$(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI.*

(b) *There exists an admissible smooth doubling Radon measure μ on $(\mathcal{X}, d)$ and a regular scale function Ψ such that the time-changed MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality* PI(Ψ) *and the cutoff energy inequality* CS(Ψ).

(c) *There exists an admissible smooth doubling Radon measure μ on $(\mathcal{X}, d)$, a metric d_Ψ on $\mathcal{X}$ that is quasisymmetric to d, and $\beta > 0$, such that the time-changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies Poincaré inequality* PI(β) *and the cutoff energy inequality* CS(β) *for some $\beta > 0$.*

Proof. (a) ⇒(b) This is immediate from Theorems 6.4 and 7.8.

(b)⇒(c) Let d_Ψ and $\beta > 0$ be as given by Proposition 7.3. Quasisymmetry of d_Ψ follows from Proposition 7.3(b). Then PI(β) and CS(β) for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ follow from Lemma 7.5.

(c)⇒(a) By Lemma 5.2(a,b,d,e), $(\mathcal{X}, d_\Psi)$ is uniformly perfect. Thus μ satisfies (RVD). Since μ is doubling on $(\mathcal{X}, d)$, the space $(\mathcal{X}, d)$ is metric doubling and therefore so is $(\mathcal{X}, d_\Psi)$. So by [BM1, Proposition 5.11 and Remark 5.12], we obtain the condition (CSA) in [GHL]. Then by the implication (CSA) plus PI(β) implies the EHI in [GHL, Theorem 1.2], we obtain the EHI for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$. Since d_Ψ and d are quasisymmetric, the desired EHI follows from Lemma 7.1. □

Remark 7.10. (i) Note that conditions (b) and (c) in the Theorem above do not include the requirement that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the conditions (HC) or (Ha) introduced in Section 3. (It would be undesirable to include (Ha) or (HC), since we do not know if they are stable.) Thus (b) or (c) do not immediately give the existence of Green's functions; however the existence of regular Green functions does follow from the implications (b), (c) $\Rightarrow$ (a) and Theorems 4.6 and 4.7

The proof in [GHL] that (CSA) plus PI(β) implies the EHI does not require the existence of Green's functions.

(ii) The result (a) implies (c) in Theorem 7.9 can be sharpened as follows. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI then for any $\beta > 2$ there exists a metric d_Ψ on $\mathcal{X}$ that is quasisymmetric to d, and an admissible smooth Radon measure μ such that the time-changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies Poincaré inequality PI(β) and the cutoff energy inequality CS(β). The condition $\beta > 2$ is sharp in the sense that any β in property (c) must necessarily satisfy $\beta \geq 2$ and there are examples for which $\beta = 2$ is not possible. These results are contained in [KM].

Proof of Theorem 1.2. The condition that $\mathcal{E}(f,f) \asymp \mathcal{E}'(f,f)$ for all $f \in \mathcal{F}$ implies that the associated energy measures satisfy $\mu_{\langle f \rangle} \asymp \mu'_{\langle f \rangle}$ (by [LJ, Proposition 1.5.5(b)]). Hence the conditions PI(Ψ) and CS(Ψ) hold for $\mathcal{E}'$, and therefore the implication (b) $\Rightarrow$ (a) in Theorem 7.9 implies that the EHI holds for $\mathcal{E}'$. $\square$

The following is an extension of Theorem 1.2, where the symmetrizing measures for the Dirichlet forms may be different.

Theorem 7.11. *Let $(\mathcal{X}, d)$ be a complete, locally compact, relatively ball connected metric space, and let m be a Radon measure on $\mathcal{X}$ with full quasi support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI. Let μ be a smooth Radon measure of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ with full quasi support on $\mathcal{X}$,, and $(\mathcal{E}', \mathcal{F}')$ be another strongly local Dirichlet form on $L^2(\mathcal{X}; \mu)$ such that $\mathcal{F} \cap C_c(\mathcal{X}) = \mathcal{F}' \cap C_c(\mathcal{X})$ and*

$$C^{-1}\mathcal{E}(f,f) \leq \mathcal{E}'(f,f) \leq C\mathcal{E}(f,f) \quad \text{for all } f \in \mathcal{F} \cap C_c(\mathcal{X}). \tag{7.11}$$

Then $(\mathcal{X}, d, \mu, \mathcal{E}', \mathcal{F}')$ satisfies the EHI.

Proof. Let X be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. Since μ is a smooth Radon measure has full quasi-support, its associated positive continuous additive functional A_t is strictly increasing up to the lifetime of X. Thus its time-changed process $Y_t := X_{\tau_t}$, with $\tau_t := \inf\{r > 0 : A_t > t\}$, has the same family of harmonic functions as that of X. By (6.33), the Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ of time-changed process Y is regular on $L^2(\mathcal{X}; \mu)$ and has the property that $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X}; \mu)$, $\mathcal{F}^\mu_e = \mathcal{F}_e$ and $\mathcal{E}^\mu = \mathcal{E}$ on $\mathcal{F}_e$. Since both m and μ are Radon, any $f \in \mathcal{F}$ that has compact support is in $\mathcal{F}^\mu$ and

$$\mathcal{F} \cap C_c(\mathcal{X}) = (\mathcal{F}_e \cap L^2(\mathcal{X}; m)) \cap C_c(\mathcal{X}) = (\mathcal{F}^\mu_e \cap L^2(\mathcal{X}; m)) \cap C_c(\mathcal{X}) = \mathcal{F}^\mu \cap C_c(\mathcal{X}).$$

Hence $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is strongly local and satisfies the EHI. Since $\mathcal{F}' \cap C_c(\mathcal{X}) = \mathcal{F}^\mu \cap C_c(\mathcal{X})$ is dense in $\mathcal{F}'$ and $\mathcal{F}^\mu$ with respect to the norm $\sqrt{\mathcal{E}'_1}$ and $\sqrt{\mathcal{E}^\mu_1}$, respectively, where

$$\mathcal{E}'_1(u,u) := \mathcal{E}'(u,u) + \int_{\mathcal{X}} u(x)^2 \mu(dx) \quad \text{and} \quad \mathcal{E}^\mu_1(u,u) := \mathcal{E}^\mu(u,u) + \int_{\mathcal{X}} u(x)^2 \mu(dx),$$

we have by (7.12) that $\mathcal{F}' = \mathcal{F}^\mu$ and

$$C^{-1}\mathcal{E}^\mu(f,f) \le \mathcal{E}'(f,f) \le C\mathcal{E}^\mu(f,f) \quad \text{for all } f \in \mathcal{F}^\mu. \tag{7.12}$$

The desired conclusion of the theorem now follows from Theorem 1.2 applied to the MMD $(\mathcal{X}, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$. $\square$

Remark 7.12. The stability results of this paper, Theorem 1.2 and Theorem 7.11, hold for the $\mathrm{EHI}_{\le 1}$ as well. We now indicate the needed modifications. All of the results of Section 5 extend easily under the assumption $\mathrm{EHI}_{\le 1}$ except that the conclusions only hold for balls of small enough radii. The main difference is in the construction of the measures ν_l in Proposition 6.11. Instead of the initial condition on N_{l+3} for the inductive construction using Lemma 6.9, we set the initial condition on N_1 to the uniform probability measure on N_1, where N_1 is as given in the generalized dyadic decomposition of the generalized $Q_{l,0}$. Then the weak* subsequential limit as in the proof of Theorem 6.4 will be a capacity good measure (only at small enough scales using the same argument). However, this property is enough so that our construction gives a smooth measure with full quasi support. All the results used in Section 7 (for example, [GHL, Theorem 1.2]) will also admit local versions. Although there is no clear reference in the literature for these results, a careful reading of the proofs in the literature shows that these local versions do hold, with essentially the same proof.

8 Examples

Example 8.1. The following example, based on an example of instability of the Liouville property of Benjamini [Ben2], shows that in general without (MD) the EHI is not stable.

We begin by describing Benjamini's example. Let $(\mathbb{B}, E_\mathbb{B})$ be the (1-sided) binary tree with root $0_\mathbb{B}$. We have

$$\mathbb{B} = \{0_\mathbb{B}\} \cup \bigcup_{n=1}^\infty \{0,1\}^n.$$

We call an edge of the form $\{x,(x,0)\}$ with $x \in \{0,1\}^n$ a 0-edge and an edge of the form $\{x,(x,1)\}$ a 1-edge. Let $E_\mathbb{B}^{(j)}$ be the set of j-edges for $j = 0,1$. Given $f : \mathbb{B} \to \mathbb{R}$ and an edge $e = \{x,y\}$, set $|\nabla f(e)| = |f(x) - f(y)|$.

Let $\alpha_0 = 0$, $\alpha_1 \in (0,1)$ and define the quadratic forms

$$\mathcal{E}^{(j)}(f,f) = \sum_{e \in E_\mathbb{B}^{(0)}} |\nabla f(e)|^2 + (1+\alpha_j) \sum_{e \in E_\mathbb{B}^{(1)}} |\nabla f(e)|^2.$$

Clearly, $\mathcal{E}^{(1)}(f,f)$ and $\mathcal{E}^{(2)}(f,f)$ are comparable and they have the same domain of defnition. Let m be counting measure on $\mathbb{B}$. Let $X^{(j)}$ be the Markov process associated with $\mathcal{E}^{(j)}$. Then (see [Lyo, Ben1]) there exists an infinite subset $A \subset \mathbb{B}$ with the property that started at any point in $\mathbb{B}$, $X^{(0)}$ is a.s. ultimately in A, while $X^{(1)}$ is a.s. ultimately in $\mathbb{B} \setminus A$.

Let W be the x_1-axis in $\mathbb{Z}^4$. Let $\varphi : A \to W$ be bijective. Let $(\mathbb{V}, E_\mathbb{V})$ be the graph with vertex set $\mathbb{V} = \mathbb{B} \cup \mathbb{Z}^4$ and edges consisting of $E_\mathbb{B}$, the edges of $\mathbb{Z}^4$, and all pairs of the form $\{x, \varphi(x)\}$ for $x \in A$. We extend the forms $\mathcal{E}^{(j)}$ to $\mathbb{V}$ by assigning unit conductance to all edges not in $E_\mathcal{B}$. Write $Y^{(j)}$ for the Markov process associated with $\mathcal{E}^{(j)}$ with counting measure (on $\mathbb{V}$). Then since the SRW on $\mathbb{Z}^4$ hits W only finitely often the process $Y^{(0)}$ will ultimately stay

$\mathbb{Z}^4$, a.s. On the other hand $Y^{(1)}$ has a positive probability of never visiting $\mathbb{Z}^4$. Since $\mathbb{Z}^4$ has the Liouville property, it follows that all bounded $\mathcal{E}^{(0)}$-harmonic functions are constant, while $\mathcal{E}^{(1)}$ has non-constant bounded harmonic functions.

The same property holds for the cable system $(\mathcal{X}, d)$ of the graph $(\mathbb{V}, E_{\mathbb{V}})$. It is easy to verify that (MD) fails for this space. Let m be the measure which assigns a copy of Lebesgue measure on $[0,1]$ to each cable. Let $d'(x,y) = 1 \wedge d(x,y)$. The EHI holds for $(\mathcal{X}, d', m, \mathcal{E}^{(0)}, \mathcal{F})$, but fails for $(\mathcal{X}, d', m, \mathcal{E}^{(1)}, \mathcal{F})$.

Example 8.2. We give an example of a strongly local irreducible MMD space where harmonic functions may be discontinuous and (Ha) fails. Furthermore, (HC) for its part MMD space on a ball. The space consists of three parts: the closure of a domain in $\mathbb{R}^2$, the standard Sierpinski gasket, and a line segment. Let $\mathcal{X}_1$ be the compact Sierpinski gasket, with vertices $A_1 = (0,0)$, $A_2 = (1,0)$ and $A_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\mathcal{X}_2 = [0,1] \times [-1,0]$ a unit closed square, and let $\mathcal{X}_3$ be a smooth curve outside $\mathcal{X}_1 \cup \mathcal{X}_2$ that connects the vertex A_3 of the Sierpinski gasket with the point $A_4 = (1/2, 1)$ at the middle of the right side of the square $\mathcal{X}_2$. We identify $\mathcal{X}_3$ with a closed line segment of length $l > 1$.

Let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$, equipped with Euclidean metric inherited from $\mathbb{R}^2$. Clearly, $(\mathcal{X}, d)$ is a compact separable metric space. Let m_1 be the measure on $\mathcal{X}_1$ which assigns mass 3^{-n} to each triangle of side 2^{-n}, and for $j = 2, 3$, let m_j be Lebesgue measure on $\mathcal{X}_j$. Let m be the measure on $\mathcal{X}$ such that $m|_{\mathcal{X}_i} = m_i$ for each i. Clearly, m is a finite measure on $\mathcal{X}$.

Let $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ be the strongly local Dirichlet form on $L^2(\mathcal{X}_1, m_1)$ associated with the standard diffusion on the Sierpinski gasket – see [Kig, Chapter 3]. It is known that $C^2(\mathcal{X}_1)$, the space of C^2 functions on $\mathbb{R}^2$ restricted to $\mathcal{X}_1$, is $\sqrt{\mathcal{E}_1^{(1)}}$-dense in $\mathcal{F}^{(1)}$, where

$$\mathcal{E}_1^{(1)}(u,u) := \mathcal{E}^{(1)}(u,u) + \int_{\mathcal{X}_1} u(x)^2 m_1(dx).$$

The Dirichlet form $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ on $L^2(\mathcal{X}_1, m_1)$ is a so called resistance form, each point in $\mathcal{X}_1$ is of positive capacity and every $f \in \mathcal{F}^{(1)}$ is Hölder continuous on $\mathcal{X}_1$ – see [Kig].

Denote by $C^2(\mathcal{X})$ be the space of continuous functions f on $\mathcal{X}$ so that $f|_{\mathcal{X}_i} \in C^2(\mathcal{X}_i)$ for $i = 1, 2, 3$, and define for $f, g \in C^{(2)}(\mathcal{X})$,

$$\mathcal{E}(f,g) = \mathcal{E}^{(1)}(f|_{\mathcal{X}_2}, g|_{\mathcal{X}_2}) + \frac{1}{2}\int_{\mathcal{X}_2} \nabla f(x) \cdot \nabla g(x) m_2(dx) + \frac{1}{2}\int_{\mathcal{X}_3} f'(x) g'(x) m_3(dx). \tag{8.1}$$

Clearly, the bilinear form $(\mathcal{E}, C^2(\mathcal{X}))$ is closable in $L^2(\mathcal{X}; m)$ in the sense that if $\{f_n; n \geq 1\} \subset C^2(\mathcal{X})$ is $\mathcal{E}$-Cauchy and f_n converges to 0 in $L^2(\mathcal{X}; m)$, then $\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0$. This is because each term in the right hand side of (8.1) is closable on $L^2(\mathcal{X}_i; m_i)$ for $i = 1, 2, 3$. Let $\mathcal{F}$ be the $\sqrt{\mathcal{E}_1}$-completion of $C^2(\mathcal{X})$, where $\mathcal{E}_1(f,f) := \mathcal{E}(f,f) + \int_{\mathcal{X}} f(x)^2 m(dx)$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathcal{X}; m)$. Clearly it is strongly local, and is irreducible by Theorem 4.5. It is easy to see that $f \in \mathcal{F}$ if and only if $f|_{\mathcal{X}_1} \in \mathcal{F}^{(1)}$, $f|_{\mathcal{X}_i} \in W^{1,2}(\mathcal{X}_i)$ for $i = 2, 3$ and the trace of $f|_{\mathcal{X}_1}$ on $K := \mathcal{X}_1 \cap ([0,1] \times \{0\})$ coincides on K with the trace of $f|_{\mathcal{X}_2}$ on $[0,1] \times \{0\}$.

Let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ be the diffusion process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. The diffusion X is conservative as $1 \in \mathcal{F}$ with $\mathcal{E}(1,1) = 0$. The diffusion X on $\mathcal{X}$ behaves as follows:

(i) when X_t is inside $\mathcal{X}_1$, it behaves like Brownian motion on the Sierpinski gasket $\mathcal{X}_1$ until it reaches the vertex A_3 or the bottom K;

(ii) when X_t is inside $\mathcal{X}_2$, it behaves like two-dimensional Brownian motion in $\mathcal{X}_2$ reflected on $\partial\mathcal{X}_2 \setminus \mathcal{X}_1$;

(iii) when X_t is inside $\mathcal{X}_3$, it behaves like one-dimensional Brownian motion reflected at the end point A_4;

(iv) when X_t is at the vertex A_3, it has positive probability to enter either $\mathcal{X}_1$ and $\mathcal{X}_3$; when X_t is at the Cantor set K, it has positive probability to enter either $\mathcal{X}_1$ and $\mathcal{X}_2$; when the X_t is at A_4, it gets reflected into $\mathcal{X}_3$.

Note that single point in $\mathcal{X}_2$ is polar for reflected Brownian motion in $\mathcal{X}_2$. Thus the process X starting from $\mathcal{X}_2 \setminus \{A_4\}$ can only enter $\mathcal{X}_3$ through the Sierpinksi gasket $\mathcal{X}_1$ via vertex A_3.

For any $r \in (0, 1/2)$, let $h(x) = \mathbb{P}^x(\tau_{B(A_4,r)} \in B(A_4, r) \cap \mathcal{X}_2)$. Clearly h is harmonic in the ball $B(A_4, r)$, $h(x) = 1$ for $x \in B(A_4, r) \cap \mathcal{X}_2 \setminus \{A_4\}$ and $h(x) = 0$ on $B(A_4, r) \cap \mathcal{X}_3$. Thus h does not satisfy the non-scale-variant Harnack inequality. In other words, (Ha) fails for this strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Note that the point A_4 is of positive capacity and $(\mathcal{E}, \mathcal{F})$ is irreducible. On the other hand, the part Dirichlet $(\mathcal{E}, \mathcal{F}^{B(A_4,r)})$ on $L^2(B(A_4, r), m|_{B(A_4,r)})$ is not irreducible for any $r \in (0, 1/2]$; the space $B(A_4, r)$ has two disjoint invariant sets $B(A_4, r) \cap \mathcal{X}_2)$ and $B(B_4, r) \cap \mathcal{X}_3$. (This example also shows that a strongly local regular Dirichlet form does not need to be irreducible even though the underlying metric space is connected.) Thus (HC) fails for $(B(A_4, r), d, m|_{B(A_4,r)}, \mathcal{E}, \mathcal{F}^{B(A_4,r)})$; for instance, $G_{B(A_4,r)}f(x) := \mathbb{E}^x \int_0^{\tau_{B(A_4,r)}} f(X_s)ds$ with $f = 1_{(B(A_4,r)\setminus B(A_4,3r/4))\cap\mathcal{X}_2}$ is bounded away from a positive constant in $B(A_4, r/2) \cap \mathcal{X}_2 \setminus \{A_4\}$ and identically zero on $B(A_4, r/2) \cap \mathcal{X}_3$. However, if we define a new metric ρ on $\mathcal{X}$ so that its restriction on $\mathcal{X}_1 \cup \mathcal{X}_3$ is comparable to d, and on $\mathcal{X}_2 \setminus \{A_4\}$ is locally comparable to d but sending A_3 to infinity from the side of $\mathcal{X}_2 \cap B(A_4, 1/4)$. Then (X, ρ) is a locally compact separable metric space and $(\mathcal{X}, \rho, m, \mathcal{E}, \mathcal{F})$ is a strongly local regular irreducible MMD space. Under this metric, one can in fact show that the EHI holds on balls with radius no larger than 1.

Example 8.3. To give a concrete example of an irreducible strongly local MMD space that fits the setting of Theorem 1.2 but fails to satisfy the local regularity of [BM1] in the compact setting, consider $\mathcal{X}$ to be the join of Vicsek tree (compact) with the unit interval $[0, 1]$, where the symmetrizing measure m is given by the Hausdorff measure on each of the pieces. The space $\mathcal{X}$ satisfies the relatively ball condition. We take $(\mathcal{E}, \mathcal{F})$ to be the strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$ obtained by combining the Dirichlet form associated with Brownian motion on $(0, 1]$ with the Dirichlet form associated with the diffusion on the Vicsek tree, in a similar fashion to the previous example. The argument in [De2] can be adapted to show that this example satisfies the EHI. This example is essentially due to Delmotte [De2].

Acknowledgment. We thank Naotaka Kajino for helpful comments on an earlier draft of this work. The research of Martin Barlow is partially supported by an NSERC grant. The research of Zhen-Qing Chen is partially supported by a Simons Foundation Grant. The research of Mathav Murugan is partially supported by the Canada Research Chairs program and an NSERC grant.

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