Pointwise resistance estimates for the Sierpinski carpet

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This note gives a sketch proof that for the two-dimensional Sierpinski carpet one has

$$R_{\text{eff}}(x,y) \asymp |x-y|^{d_w-d_f}. \tag{0.1}$$

We work on the infinite pre-SC graph F. (This is the graph denoted G_n on p. 346 of [1].) We denote $F_n = [0,3^n]^2 \cap F$. We write $\partial_L F_n$ and $\partial_R F_n$ for the left and right sides of F_n, and $\partial_T F_n$, $\partial_B F_n$ for the top and bottom. We write $\mathcal{E}_n$ for the Dirichlet form in the graph F_n.

Let R_n be the resistance across F_n, i.e. the resistance between $\partial_L F_n$ and $\partial_R F_n$ when these two sets are wired or shorted. It is proved in [1] that

$$c_1^{-1} R_n R_m \le R_{n+m} \le c_1 R_n R_m,$$

from which it follows that there exists ρ such that

$$c_1^{-1}\rho^n \le R_n \le c_1 \rho^n. \tag{0.2}$$

Easy estimates for the standard SC give $\rho > 1$. We have

$$d_w - d_f = \frac{\log \rho}{\log 3}. \tag{0.3}$$

Lower bound. This is straightforward. As in [1] one can construct two functions f_n and g_n which satisfy $\mathcal{E}_n(h,h) \le c R_n^{-1}$ for $h = f_n, g_n$. f_n has Dirichlet boundary conditions 1 on $\partial_L F_n$, 0 on $\partial_R F_n$, and Neumann b.c. on the remainder of the boundary. The function g_n satisfies $g_n(0) = 1$ and $g_n(z) = 0$ for the other 3 corners of F_n, is symmetric about the line $x_1 = x_2$ in $\mathbb{R}^2$ and satisfies $g_n = f_n$ on $\partial_B F_n$.

Let $x, y \in F$ with $|x-y| = r$. Choose k so that $3^{r-4} \le r \le 3^r$. Let Q_x and Q_y be squares side 3^k containing x and y. Using the functions f_k and g_k one can build a function φ with $\varphi = 1$ on Q_x, $\varphi = 0$ on every square side 3^k which does not touch Q_x, and

$$\mathcal{E}(\varphi,\varphi) \le 8(\mathcal{E}_n(f_n,f_n) \vee \mathcal{E}_n(g_n,g_n)) \le c\rho^{-n}.$$

It follows that

$$R_{\text{eff}}(x,y) \ge c\rho^{-n} \ge c|x-y|^{d_w-d_f}. \tag{0.4}$$

Upper bound. Let $x, y \in F$ with $3^n \asymp |x-y|$. We need to build a unit flow J on F from x to y with energy

$$E(J) = \sum_{e \in E(F)} |J_e|^2 \le c\rho^n;$$

Let μ_L and μ_R be input/output distributions on $\partial_L F_n$ and $\partial_R F_b n$, with total flux 1, i.e. $\sum_z \mu_L(x) = \sum_x \mu_R(x) = 1$. (See figure).

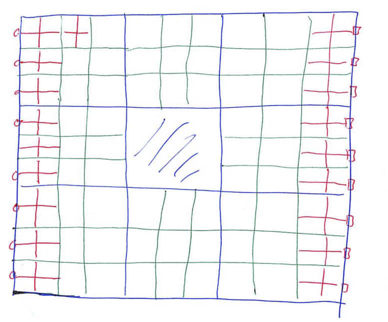

Figure 1: The network F_2. The 'wires' in the network are marked in red, the input points are marked with small red circles, and the output points with red squares.

Let $I_n(\mu_1, \mu_2)$ be the minimal energy flow in F_n with input μ_L and output μ_R. By [1] there exists a distribution ν_n such that

$$R_n = E_n(I(\nu_n, \nu_n)).$$

Set

$$Q_n = \max_{x \in \partial_L F_n, y \in \partial_R F_n} I_n(\delta_x, \delta_y).$$

Note that $Q_0 = R_0 = 1$. (The idea of looking at a max of this kind may have come from [2].) An easy calculation using Cauchy-Schwarz gives for any μ_L, μ_R that

$$E_n(I(\mu_L, \mu_R)) \le Q_n.$$

Now let $m, n \ge 0$, and let $x \in \partial_L F_{n+m}$, $y \in \partial_R F_{n+m}$. We regard F_{n+m} as being made up of 'micro' squares side 3^m, all copies of F_m, arranged according to the pattern F_n.

Let G_x be the micro square containing x. We can build a flow J_x on G_x with input δ_x and output ν_m (appropriately translated) on $\partial_R Q_x$, with $E_m(J_x) \le Q_m$. Combining this with the flow $I_m(\nu_m, \nu_m)$, reversed so it goes from right to left, one obtains a flow J'_x on G_x with input δ_x, output ν_m on $\partial_L G_x$, and with zero output on the other 3 sides of G_x. Further

$$E_m(J') \le cQ_m.$$

(We have used here the fact that if we have two flows J_1, J_2 then $E_m(J_1 + J_2) \le 2E(J_1) + 2E(J_2)$.)

Look at the macro cube F_n, and let x', y' be the points in F_n corresponding to the squares G_x and G_y. The flow $I_n(\delta_{x'}, \delta_{y'})$ has energy

$$E_n(I_n(\delta_{x'}, \delta_{y'}) \le Q_n.$$

Using the 'macro' flow $I_n(\delta_{x'}, \delta_{y'})$ and the micro flow $I_m(\nu_m)$ we can as in [1] build a flow J'' on F_{n+m} with input ν_m on G_x, output ν_m on G_y and energy

$$E(J'') \le cQ_nR_m.$$

Combining J'' with the flow J'_x and a similar flow J'_y one obtains a flow I on F_{n+m} with input δ_x and output δ_y. It follows that

$$Q_{n+m} \le c(Q_m + R_mQ_n). \tag{0.5}$$

Set $y_n = \rho^{-n}Q_n$. Note that $y_0 = 1$. As $R_m \le c\rho^n$ we deduce that there exists a constant a such that for $n, m \ge 0$

$$y_{n+m} \le a\rho^{-n}y_m + ay_n. \tag{0.6}$$

Lemma 0.1. *Let $\rho > 1$ and suppose (y_n) satisfies $y_0 = 1$ and (0.6). Then there exists $A = A(a, \rho)$ such that $y_n \le A$ for all n.*

Proof. Choose n so that $a\rho^{-n} \le \frac{1}{2}$. Set $b = a\max_{1\le k\le n} y_k$. (The equation (0.6) enables us to bound n and b in terms of of a and ρ.) Let $H = \{k : y_k \le 2b\}$. Suppose that $m \in H$. Then

$$y_{m+n} \le a\rho^{-n}y_m + ay_n \le \tfrac{1}{2}y_m + b \le b + b = 2b.$$

So $m + n \in H$. It follows that $y_k \le 2b$ for all k. □

Since (y_n) is bounded, we obtain $Q_n \le c\rho^n$ for all n. We have constructed a flow across F_n with energy bounded by $c\rho^n$, and it follows that

$$R_{\text{eff}}(x, y) \le c\rho^n \text{ for all } x \in \partial_L F_n,\ y \in \partial_R F_n.$$

Using the fact that R_{eff} is a metric, the upper bound in (0.1) follows.

Remarks. 1. The sketch above is for the basic S. carpet in $\mathbb{Z}^2$. However, the same argument should work for generalized SC in two dimensions.
2. I do not see any obstacle to using this method for higher dimensional SCs which satisfy $\rho > 1$, i.e. $d_w > d_f$.

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References

[1] M.T. Barlow and R.F. Bass. On the resistance of the Sierpiński carpet. *Proc. Roy. Soc. London Ser. A* **431** (1990), 345-360.

[2] S. Cao, H. Qiu. Dirichlet forms on unconstrained Sierpinski carpets. Preprint 2021.

[3] I. McGillivray. Resistance in higher-dimensional Sierpiński carpets. *Potential Analysis* **16** (2002), 289-303.

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