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# Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets

by

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ABSTRACT. – We construct Brownian motion on a class of fractals which are spatially homogeneous but which do not have any exact self-similarity. We obtain transition density estimates for this process which are up to constants best possible.

RÉSUMÉ. – Nous construisons un mouvement Brownien sur une classe de fractals homogènes en espace mais n'ayant pas d'exacte propriété d'auto-similarité. Nous obtenons des estimées du noyau de transition de ce processus qui sont, à une constante près, les meilleures possibles.

## 1. INTRODUCTION

There is now a fairly extensive literature on the heat equation on fractal spaces, and on the spectral properties of such spaces. Most of these papers

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treat sets $F$ which have exact self- similarity, so that there exist 1-1 contractions $\psi_i : F \to F$ such that $\psi_i(F) \cap \psi_j(F)$ is (in some sense) small when $i \neq j$, and

$$F = \cup_i \psi_i(F). \tag{1.1}$$

In the simplest cases, such as the nested fractals of Lindstrøm [18], $F \subset \mathbb{R}^d$, the $\psi_i$ are linear, and $\psi_i(F) \cap \psi_j(F)$ is finite when $i \neq j$. For very regular fractals such as nested fractals, or Sierpinski carpets, it is possible to construct a diffusion $X_t$ with a semigroup $P_t$ which is symmetric with respect to $\mu$, the Hausdorff measure on $F$, and to obtain estimates on the density $p_t(x, y)$ of $P_t$ with respect to $\mu$. In these cases (see [3, 15]) there exist constants $d_w$, $d_s$ (called, following the physics literature, the *walk* and *spectral dimensions* of $F$) such that

$$p_t(x,y) \leq c_1 t^{-d_s/2} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right), \quad t \in (0,1), x, y, \in F, \tag{1.2}$$

with a lower bound of the same form but different constants. Here $|x - y|$ is the Euclidean metric in $\mathbb{R}^2$.

In the mathematical physics literature, the main interest is not in regular fractals, (except as models), but in irregular objects such as percolation clusters, which are believed to exhibit "fractal" properties. It is therefore of interest to investigate the extent to which bounds such as (1.2) hold for less regular sets with some "fractal" structure.

In this paper we will study a family of sets $F$, based on the Sierpinski gasket, which are locally spatially homogeneous, but which do not satisfy any exact scaling relation of the form (1.1). To give the essential flavour of our results we consider a fractal first discussed in [10]. Consider two regular fractals, the standard Sierpinski gasket SG(2) and a variant SG(3) - see Figure 1. Each of these sets may be defined by

$$F = \cap_{n=0}^{\infty} F_n$$

where (for $a = 2$ or 3) $F_n$ is obtained from $F_{n-1}$ by subdividing each triangle in $F_{n-1}$ into $a^2$ smaller triangles, and deleting the 'downward facing' ones. Thus we can write $F_n = \Phi^{(a)}(F_{n-1})$ for $a = 2, 3$. (A more precise definition of the maps $\Phi^{(a)}$ is given in Section 2.)

Let $\Xi = \{2, 3\}^{\mathbb{N}}$, and let $\xi = (\xi_1, \ldots) \in \Xi$; we call $\xi$ an *environment sequence*. Given $\xi$ we can construct a set $F^{(\xi)} = \cap_n F_n^{(\xi)}$ where we use $\xi_n$ to determine which construction to use at level $n$: we have $F_n^{(\xi)} = \Phi^{(\xi_n)} F_{n-1}^{(\xi)}$.

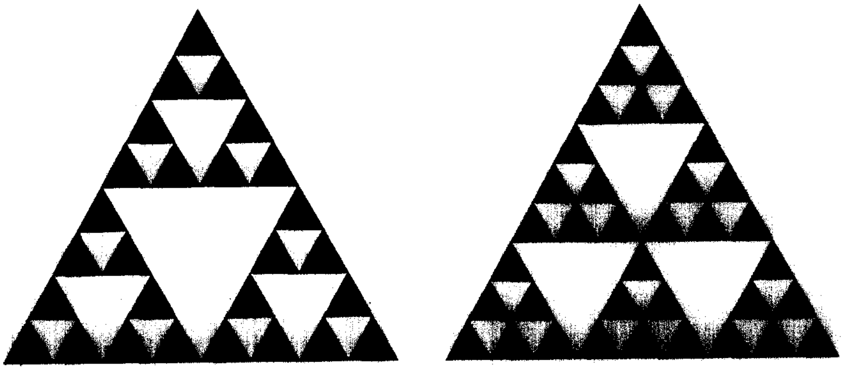

FIG. 1. – The first stages in the construction of SG(2) and SG(3).

Unless the sequence $\xi$ is periodic $F^{(\xi)}$ does not have any exact scaling property, but it is spatially homogeneous in the sense that all triangles of a given size in $F^{(\xi)}$ are identical. Figure 2 shows the first 3 levels in the construction of the set $F$ associated with the sequence $\xi = (2, 3, 2, ...)$.

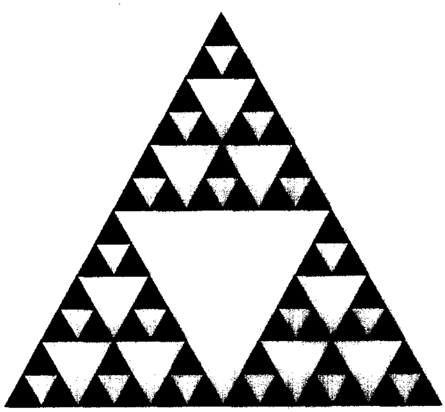

FIG. 2. – The first three levels of a scale irregular Sierpinski gasket.

A previous paper by one of us [10] considered the case when the environment sequence $\xi$ was a sequence of i.i.d. random variables; the sets obtained were called 'homogeneous random Sierpinski gaskets'. We use a different term here, as the sets studied in this paper are not necessarily random. An example of such a scale irregular Sierpinski gasket was discussed in Section 9 of [10]. We also remark that if, at each level, one chooses a different (random) procedure for subdividing each small triangle, then one obtains an example of the random recursive fractals studied in [19], and that diffusions on some sets of this type are studied in [11].

For the case described above our main results take the following form. For $a = 2, 3$ write $(l_a, m_a, t_a)$ for the length, mass and time scaling factors (see [18]) associated with SG($a$). Here (see [10]) we have $(l_2, m_2, t_2) = (2, 3, 5)$ and $(l_3, m_3, t_3) = (3, 6, 90/7)$. Let $L_0 = M_0 = T_0 = 1$, and set for $n \geq 1$,

$$L_n = \prod_{i=1}^{n} l_{\xi_i}, \quad M_n = \prod_{i=1}^{n} m_{\xi_i}, \quad T_n = \prod_{i=1}^{n} t_{\xi_i}. \tag{1.3}$$

There is a natural 'flat' measure $\mu$ on $F = F^{(\xi)}$ which is characterised by the property that it assigns mass $M_n^{-1}$ to each triangle in $F$ of side $L_n^{-1}$. In section 3 we will construct a $\mu$-symmetric diffusion $X_t$, with semigroup $P_t$, on $F$. We do this analytically, by constructing a regular local Dirichlet form $\mathcal{E}$ on $L^2(F, \mu)$. Here we follow the ideas of [16], [14], [8]; though the arguments of these papers do not directly cover the case treated here, they can be adapted without difficulty to our situation.

Once we have constructed $P_t$, we can prove the existence of a density $p_t(x, y)$ with respect to $\mu$, and obtain bounds on $p_t$, by using similar techniques to those developed for regular fractals in [3], [7].

To maintain consistency with notation for more general SGs introduced later, set $B_n = L_n$ and let

$$d_w(n) = \log T_n / \log B_n, \quad d_s(n) = 2 \log M_n / \log T_n, \tag{1.4}$$

and for $n, m \geq 0$ set

$$k = k(m, n) = \inf\{j \geq 0 : T_{m+j}/B_{m+j} \geq T_n/B_m\}. \tag{1.5}$$

Note that $k(m, n) = 0$ if $m \geq n$, and that if $m < n$ then $n < m + k(m, n) < \infty$.

THEOREM 1.1. – *(a) $P_t$ has a continuous density $p_t(x, y)$ with respect to $\mu$.* *(b) There exist constants $c_1, c_2, c_3, c_4$ (not depending on $\xi$) such that if $L_m^{-1} \leq |x - y| < L_{m-1}^{-1}$, $T_n^{-1} \leq t < T_{n-1}^{-1}$, then*

$$p_t(x, y) \leq c_1 t^{-d_s(n)/2} \exp\left( -c_2 \left( \frac{|x - y|^{d_w(m+k)}}{t} \right)^{1/(d_w(m+k)-1)} \right), \tag{1.6}$$

*and*

$$p_t(x, y) \geq c_3 t^{-d_s(n)/2} \exp\left( -c_4 \left( \frac{|x - y|^{d_w(m+k)}}{t} \right)^{1/(d_w(m+k)-1)} \right). \tag{1.7}$$

To understand these estimates intuitively first note that if $\xi_n \equiv a$ (where $a = 2$ or 3) then $d_w(n) \equiv \log t_a / \log l_a$, $d_s(n) \equiv 2 \log m_a / \log t_a$, and we recover the estimates for the heat kernels on the fractals SG(2) and SG(3) obtained in [4, 15]. For non-constant $\xi$, $d_w(n)$ and $d_s(n)$ are the 'effective walk and spectral dimensions at level $n$'. For given $t$, $x$, $y$, let $m, n$ be as in the Theorem, so that $T_n^{-1} \approx t$ and $L_m^{-1} \approx |x - y|$.

If $m \geq n$ then $k(m, n) = 0$, and the term in the exponential is of order 1, so that

$$p_t(x, y) \approx t^{-d_s(n)/2} \approx M_n.$$

Since $\mu\{y : |x - y| \leq L_n^{-1}\} \approx M_n^{-1}$, it follows that in time $T_n^{-1}$ the diffusion $X$ moves a distance $O(L_n^{-1})$.

If $m < n$, so that $|x - y|$ is large relative to $t$, then $n < m + k$, and the estimates (1.6), (1.7) involve the two 'dimensions' at different levels of the set. For the time factor we have $d_s(n)$ as before, but the exponent $d_w(m + k)$ involves the structure of $F$ at a level finer than either the 'space level' $m$ or the 'time level' $n$. In both cases we see that the heat kernel at time $t$ is not greatly affected by structures in the set $F$ which appear at a length scale finer than $L_{m+k}^{-1}$; that is by $\xi_i$ for $i \geq m + k$.

In Section 6 we consider the case when $d_s(n)$ and $d_w(n)$ converge to limits $d_s$ and $d_w$ respectively, and in Theorem 6.1 we show that the bounds given in Theorem 1.1 can be written in terms of the limiting dimensions with correction terms. It is worth noting that we only obtain bounds of the form (1.2) if the convergence of $d_s(n)$ and $d_w(n)$ is essentially as fast as possible. (See Theorem 6.2 and the remark following).

If the environment sequence $\xi_i$ are i.i.d. random variables, then it is clear that $d_s(n)$ and $d_w(n)$ converge a.s. In this case the results we obtain improve and extend those obtained in [10]; see Corollary 6.3 for the exact correction functions hidden by the $\varepsilon$ used in that paper.

In Section 2 we define the fractal $F$, and set up our notation. The construction of the process is outlined in Section 3, where we also establish the key inequalities involving the Dirichlet form $\mathcal{E}$. Sections 4 and 5 deal with the transition density estimates, which lead to our main results Theorems 4.5 and 5.4, of which Theorem 1.1 is a special case. In Section 6 we look at some examples, and in Section 7 we use (1.6), (1.7) to estimate the eigenvalue counting function $N(\lambda)$.

## 2. SCALE IRREGULAR SIERPINSKI GASKETS

As the building blocks for our scale irregular Sierpinski gaskets will all be nested fractals, we begin by recalling from Lindstrøm [18] the definition of a nested fractal. See [18] for a fuller account of the motivation and definitions.

For $\alpha > 1$, an *$\alpha$-similitude* is a map $\psi : \mathbb{R}^D \to \mathbb{R}^D$ such that

$$\psi(x) = \alpha^{-1}U(x) + x_0, \tag{2.1}$$

where $U$ is a unitary, linear map and $x_0 \in \mathbb{R}^D$. Let $\Psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of $\alpha$-similitudes. For $B \subset \mathbb{R}^D$, define

$$\Phi(B) = \cup_{i=1}^m \psi_i(B),$$

and let

$$\Phi_n(B) = \Phi \circ \ldots \circ \Phi(B).$$

By Hutchinson [12], the map $\Phi$ on the set of compact subsets of $\mathbb{R}^D$ has a unique fixed point $F$, which is a self-similar set satisfying $F = \Phi(F)$.

As each $\psi_i$ is a contraction, it has a unique fixed point. Let $F'$ be the set of fixed points of the mappings $\psi_i$, $1 \leq i \leq m$. A point $x \in F'$ is called an *essential fixed point* if there exist $i, j \in \{1, \ldots, m\}$, $i \neq j$ and $y \in F'$ such that $\psi_i(x) = \psi_j(y)$. We write $F_0$ for the set of essential fixed points. Now define

$$\psi_{i_1,\ldots,i_n}(B) = \psi_{i_1} \circ \ldots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

We will call the set $\psi_{i_1,\ldots,i_n}(F_0)$ an *$n$-cell* and $\psi_{i_1,\ldots,i_n}(F)$ an *$n$-complex*. The *lattice of fixed points* $F_n$ is defined by

$$F_n = \Phi_n(F_0), \tag{2.2}$$

and the set $F$ can be recovered from the essential fixed points by setting

$$F = cl(\cup_{n=0}^\infty F_n).$$

We can now define a nested fractal as follows.

DEFINITION 2.1. – The set $F$ is a nested fractal if $\{\psi_1, \ldots, \psi_m\}$ satisfy:
(A1) *(Connectivity)* For any 1-cells $C$ and $C'$, there is a sequence $\{C_i : i = 0, \ldots, n\}$ of 1-cells such that $C_0 = C, C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset$, $i = 1, \ldots, n$.
(A2) *(Symmetry)* If $x, y \in F_0$ then reflection in the hyperplane $H_{xy} = \{z : |z - x| = |z - y|\}$ maps $F_n$ to itself.

(A3) *(Nesting)* If $\{i_1, \dots, i_n\}, \{j_1, \dots, j_n\}$ are distinct sequences then

$$\psi_{i_1,\dots,i_n}(F) \bigcap \psi_{j_1,\dots,j_n}(F) = \psi_{i_1,\dots,i_n}(F_0) \bigcap \psi_{j_1,\dots,j_n}(F_0).$$

(A4) *(Open set condition)* There is a non-empty, bounded, open set $V$ such that the $\psi_i(V)$ are disjoint and $\cup_{i=1}^m \psi_i(V) \subset V$.

We now define the family of scale irregular Sierpinski gaskets. Let $F_0 = \{z_0, z_1, z_2\}$ be the vertices of a unit equilateral triangle in $\mathbb{R}^2$. Let $A$ be a finite set, for $a \in A$ let $l_a \in (1, \infty)$, $m_a \in \mathbb{N}$, and for each $a \in A$ let

$$\Psi^{(a)} = \{\psi_1^{(a)}, \dots, \psi_{m_a}^{(a)}\}, \quad a \in A,$$

be a family of $l_a$-similitudes on $\mathbb{R}^2$, with set of essential fixed points $F_0$, which satisfies the axioms for nested fractals. Write $F^{(a)}$ for the nested fractal associated with $\Psi^{(a)}$, and let $t_a$ be the time scaling factor (see [18])) of $F^{(a)}$. (Note that the definition of $t_a$ just involves the sets $F_0$ and $F_1^{(a)}$).

Let $\Xi = A^{\mathbb{N}}$; we call $\xi \in \Xi$ an *environment*. We will occasionally need a left shift $\theta$ on $\Xi$: if $\xi = (\xi_1, \xi_2, ...)$ then $\theta\xi = (\xi_2, \xi_3, ...)$. For $B \subset \mathbb{R}^2$ set

$$\Phi^{(a)}(B) = \bigcup_{j=1}^{m_a} \psi_j^{(a)}(B),$$

$$\Phi_n^{(\xi)}(B) = \Phi^{(\xi_1)} \circ \dots \circ \Phi^{(\xi_n)}(B).$$

Then the fractal $F^{(\xi)}$ associated with the environment sequence $\xi$ is defined by

$$F^{(\xi)} = cl(\cup_n \Phi_n^{(\xi)}(F_0)). \tag{2.3}$$

This set is not in general self-similar, but the family $\{F^{(\xi)}, \xi \in \Xi\}$ does satisfy the equation $F^{(\xi)} = \Phi^{(\xi_1)}(F^{(\theta\xi)})$. Let $H$ be the closed convex hull of $F_0$. For many examples the families of maps $\Psi^{(a)}$ will have the additional property that $\Phi^{(a)}(H) \subset H$ for each $a \in A$, and in this case we have a slightly simpler description of $F^{(\xi)}$:

$$F^{(\xi)} = \bigcap_{n=0}^{\infty} \Phi_n^{(\xi)}(H).$$

At this point we fix an environment sequence $\xi$, and, except where clarity requires it, will drop $\xi$ from our notation.

We will use $c$, $c'$ to denote unimportant positive constants, which may change in value from line to line, and $c_i$ to denote positive constants which will be fixed in each section. Outside Section $i$ we will refer to the $j$-th constant of Section $i$ as $c_{i.j}$. These constants will in general depend on the family of nested fractals specified by $\Psi^{(a)}$, $a \in A$, but will be independent of the particular environment sequence $\xi$.

We define $L_n$, $T_n$ and $M_n$ by (1.3). We define the *word space* $W$ associated with $F$ by

$$W = \bigotimes_{i=1}^{\infty} \{1, \ldots, m_{\xi_i}\} = \{(w_1, w_2, \ldots) : 1 \leq w_i \leq m_{\xi_i}\}. \tag{2.4}$$

For $w \in W$ write $w|n = (w_1, \ldots, w_n)$, and

$$\psi_{w|n} = \psi_{w_1}^{(\xi_1)} \circ \ldots \circ \psi_{w_n}^{(\xi_n)}. \tag{2.5}$$

We write $W_n = \{(w_1, \ldots, w_n) : 1 \leq w_i \leq m_{\xi_i}, 1 \leq i \leq n\}$ for the set of words of length $n$. Let $\mu$ be the unique measure on $F$ such that $\mu(\psi_{w|n}(F^{(\theta^n \xi)})) = M_n^{-1}$ for all $w \in W$, $n \geq 0$. As for nested fractals we define $F_n = \cup_{w \in W_n} \psi_w(F_0)$, and call sets of the form $\psi_{w|n}(F_0)$ *n-cells*, and the sets $\psi_{w|n}(F^{(\theta^n \xi)})$ *n-complexes*. We define a natural graph structure on $F_n$ by letting $\{x, y\}$ be an edge if and only if $x, y$ both belong to the same $n$-cell. This graph is connected by (A1); write $\rho_n(x, y)$ for the graph distance in $F_n$. (So $\rho_n(x, y)$ is the length of the shortest chain of edges in the graph $F_n$ connecting $x$ and $y$.)

DEFINITION 2.2. – *Let* $b_a = \rho_1(z_0, z_1)$ *on the graph* $F_1^{(a)}$, *and set*

$$B_n = \prod_{i=1}^{n} b_{\xi_i}. \tag{2.6}$$

The scaling factors $(l_a, m_a, t_a, b_a)$ play a fundamental role in what follows. We note the following elementary facts:

$$l_a > 1, \quad b_a \geq 2, \quad b_a \geq l_a, \quad m_a \geq 3, \quad a \in A. \tag{2.7}$$

Write $m^* = \max_a m_a$, $t^* = \max_a t_a$, $b^* = \max_a b_a$.

For many simple nested fractals, such as the SG(2) and SG(3) discussed in the introduction, we have $l_a = b_a$. In this case it is easy to see that there exists $c$ such that if $x, y \in F$ then $x, y$ are joined by a piecewise linear arc (with in general infinitely many segments) of length less than $c|x - y|$. In general however we can have $b_a > l_a$, and then we will have to define an intrinsic metric on $F$. For general nested fractals this takes some work – see [15], [7], but here the simple nature of the Sierpinski gaskets makes it straightforward.

Let

$$b'_a = \max\{\rho_1(x,y) : x, y \in F_1^{(a)}\},$$

and write $b^+ = \max_a b'_a$. Since $A$ is finite, $b'_a/b_a \le c$ for some $c < \infty$. It is then easy to verify that if $x, y \in F_n$ and $m \ge n$ then $\rho_m(x,y) = (B_m/B_n)\rho_n(x,y)$, and that

$$\rho_n(x,y) \le c_1 B_n/B_k \quad \text{if } x, y \in F_n \text{ belong to the same } k\text{-complex}. \tag{2.8}$$

Now define

$$d(x,y) = B_n^{-1}\rho_n(x,y) \quad \text{for } x, y \in F_n, \quad n \ge 0. \tag{2.9}$$

Then $d$ is well-defined, and from (2.8) we deduce that $d$ extends from $\cup_n F_n$ to a metric $d$ on $F$. It follows from (2.8) that

$$d(x,y) \le c_1 B_k^{-1} \quad \text{if } x, y \text{ belong to the same } k\text{-complex}. \tag{2.10}$$

Note also that if $d(x,y) \le B_k^{-1}$ then $x, y$ are either in the same $k$-complex or in adjacent $k$-complexes. If $B(x,r) = \{y \in F : d(x,y) < r\}$, then as the $\mu$-measure of each $k$-complex is $M_k^{-1}$, we have $c_2 M_k^{-1} \le \mu(B(x, B_k^{-1})) \le c_3 M_k^{-1}$. Set

$$d_f(n) = \frac{\log M_n}{\log B_n}; \tag{2.11}$$

it follows that if $B_n^{-1} \le r \le B_{n-1}^{-1}$,

$$c_4 r^{d_f(n)} \le \mu(B(x,r)) \le c_5 r^{d_f(n)}, \quad x \in F. \tag{2.12}$$

Write $\dim_{H,d}(.)$ and $\dim_{P,d}(.)$ for Hausdorff and packing dimension with respect to the metric $d$. The following result follows easily from (2.12) and the density theorems for Hausdorff and packing measure – see [6].

LEMMA 2.3. – *(a)* $\dim_{H,d}(F) = \liminf_{n\to\infty} d_f(n)$,
*(b)* $\dim_{P,d}(F) = \limsup_{n\to\infty} d_f(n)$.

For some simple fractals the distance $d$ is equivalent to Euclidean distance. We just prove this for the examples given in the introduction.

LEMMA 2.4. – *Suppose that* $A = \{2, 3\}$, *and* $F^{(a)}$ *is the SG(a) defined in the introduction. Then*

$$|x - y| \le d(x,y) \le c_6 |x - y| \quad x, y \in F.$$

*Proof.* – Note that as $l_a = b_a$ for each $a \in A$, $L_n = B_n$ for all $n$. If $x$, $y \in F_n$ then there exists a path in $F_n$ connecting $x$ and $y$ of length

$L_n^{-1}\rho_n(x,y)$. So $d(x,y) \geq |x-y|$ for $x$, $y \in F_n$, and this inequality extends to $F$.

The other inequality requires a little more work. For $x \in F$ let $\kappa_n(x)$ denote the corner of the $n$-complex containing $x$ which is closest (in Euclidean distance) to $x$, where we adopt some procedure for breaking ties. (If $x \in F_n$ then $\kappa_n(x) = x$). We have $\rho_{n+1}(\kappa_{n+1}(x), \kappa_n(x)) \leq 3$, so that $d(\kappa_{n+1}(x), \kappa_n(x)) \leq 3L_{n+1}^{-1}$. So $d(x, \kappa_n(x)) \leq 3L_n^{-1}$. For $x \in F$ let $D_n(x)$ denote the union of the $n$-complexes containing $\kappa_n(x)$. Write $c_7 = \sqrt{3}/4$, and note that $B(x, c_7 L_n^{-1}) \cap F \subset D_n(x)$.

Now let $x, y \in F$, and choose $m$ such that $y \in D_m(x) - D_{m+1}(x)$. Then $|x-y| \geq c_7 L_{m+1}^{-1}$, while $y$ and $\kappa_m(x)$ are in the same $m$-complex. Since $d(\kappa_m(x), \kappa_m(y)) \leq L_m^{-1}$, we have

$$d(x,y) \leq 7L_m^{-1} \leq c|x-y|.$$

□

## 3. DIRICHLET FORM AND BROWNIAN MOTION

We now construct a Dirichlet form $\mathcal{E}$ on $L^2(F,\mu)$, following the ideas of [8, 14, 10]. It will be useful to keep in mind the interpretation of Dirichlet forms in terms of electrical networks – see [5, 14]. Note that as $F_n$ is a discrete set, the space $C(F_n)$ of continuous functions on $F_n$ is just the space of all functions on $F_n$. For $f \in C(F_0)$ define

$$\mathcal{E}_0(f,g) = \frac{1}{2} \sum_{x,y \in F_0} (f(x) - f(y))(g(x) - g(y)). \tag{3.1}$$

Set $r_a = t_a/m_a$: we call $r_a$ the *resistance scaling factor* of the nested fractal $F^{(a)}$. Set

$$R_n = \prod_{i=1}^{n} r_{\xi_i}, \tag{3.2}$$

$$\mathcal{E}_n(f,g) = R_n \sum_{w \in W_n} \mathcal{E}_0(f \circ \psi_w, g \circ \psi_w). \tag{3.3}$$

Then we can write

$$\mathcal{E}_n(f,g) = \frac{1}{2} R_n \sum_{x,y \in F_n} e_n(x,y)(f(x) - f(y))(g(x) - g(y)), \tag{3.4}$$

where $e_n(x,y) = 1$ if there exists $w \in W_n$ such that $x, y \in \psi_w(F_0)$, and $e_n(x,y) = 0$ otherwise.

The choice of $R_n$ above ensures that the Dirichlet forms $\mathcal{E}_n$ have the decimation property

$$\mathcal{E}_{n-1}(g,g) = \inf\{\mathcal{E}_n(f,f) : f|_{F_{n-1}} = g\} \quad \text{for } g \in C(F_{n-1}), \tag{3.5}$$

– see [8] for details. We need some further inequalities relating $t_a, m_a$ and $l_a$.

LEMMA 3.1. – *For each $a \in A$,*

$$r_a \geq \tfrac{3}{2}, \tag{3.6}$$

$$t_a \geq b_a^2 \geq 2b_a. \tag{3.7}$$

*Proof.* – Let $g(z_0) = 0$, $g(z_1) = g(z_2) = 1$, so that $\mathcal{E}_0(g,g) = 2$. We let $\xi_1 = a$ and apply (3.5) in the case $n = 1$. For (3.6) let $f(x) = \lambda$ for $x \in F_1 - F_0$. Then

$$\mathcal{E}_1(f,f) = r_a(2\lambda^2 + 4(1-\lambda)^2),$$

so that, taking $\lambda = 2/3$, we obtain $r_a \geq 3/2$.

To prove (3.7) let $f(x) = \min(1, \rho_1(z_0,x)/b_a)$, for $x \in F_1$. Let $i \in \{1,\ldots,m_a\}$, and consider the 1-cell $\psi_i^{(a)}(F_0) = \{y_1,y_2,y_3\}$ say. Since the distance (in the graph $F_1^{(a)}$) between each pair $y_j$, $y_k$ is 1, we have $|f(y_j) - f(y_k)| \leq b_a^{-1}$, for each $j,k$, and at least two of the $f(y_j)$ must be equal. Therefore $\mathcal{E}_0(f \circ \psi_i^{(a)}, f \circ \psi_i^{(a)}) \leq 2b_a^{-2}$, so that $2 = \mathcal{E}_0(g,g) \leq r_a m_a (2b_a^{-2})$. The second inequality in (3.7) is immediate from (2.7). □

LEMMA 3.2. – *For all $n \geq 0$, $f \in C(F_n)$, $0 \leq m \leq n$ we have*

$$|f(x)-f(y)|^2 \leq c_1 R_m^{-1}\mathcal{E}_n(f,f) \text{ if } x,y \text{ are in the same } m\text{-complex.} \tag{3.8}$$

*Proof.* – We can view $F_n$ as an electrical network with associated Dirichlet form $\mathcal{E}_n$ – see [5]. Note that the resistance of an edge in $F_n$ is $R_n^{-1}$. Write $r(x,y)$ for the effective resistance between the points $x$ and $y$ in the network $F_n$. Then (see [14]) $r$ is a metric and for $f \in C(F_n)$

$$|f(x) - f(y)|^2 \leq r(x,y)\mathcal{E}_n(f,f). \tag{3.9}$$

Note first that if $k \leq n$, $x, y \in F_k$ and $\rho_k(x,y) = 1$ then $r(x,y) \leq 1/R_k$. So if $x, y \in F_k$ are in the same $(k-1)$-complex then $r(x,y) \leq b^+/R_k$.

Now let $x, y \in F_n$, and suppose that $x, y$ are in the same $m$-complex. Choose $z_m \in F_m$ in the same $m$-complex as $x, y$. Then there exists a chain $z_m = x_m, x_{m+1}, \dots, x_n = x$ such that $x_k \in F_k$ and $x_{k-1}, x_k$ are in the same $(k-1)$-complex. Hence

$$r(z_m, x) \le \sum_{k=m+1}^{n} r(x_{k-1}, x_k) \le b^+ \sum_{k=m+1}^{n} 1/R_k$$
$$\le b^+ R_m^{-1} \sum_{j=1}^{n-m} (2/3)^j < 2b^+ R_m^{-1},$$

where we used (3.7) in the last line. Combining this with (3.9) proves the lemma with $c_1 = 4b^+$. □

The decimation property (3.5) implies that if $f : F \to \mathbb{R}$ then $\mathcal{E}_n(f|_{F_n}, f|_{F_n})$ is non-decreasing in $n$. This enables us to define a limiting bilinear form $(\mathcal{E}, \mathcal{F})$ by

$$\mathcal{F} = \{f \in C(F) : \lim_{n\to\infty} \mathcal{E}_n(f, f) < \infty\},$$

and

$$\mathcal{E}(f, f) = \mathcal{E}^{(\xi)}(f, f) = \lim_{n\to\infty} \mathcal{E}_n(f, f), \qquad f \in \mathcal{F}.$$

The following result is proved from Lemma 3.2 in the same way as Theorem 4.14 of [16].

THEOREM 3.3. – *(a) The bilinear form $(\mathcal{E}, \mathcal{F})$ is a regular local Dirichlet form on $L^2(F, \mu)$.*
*(b) $|f(x) - f(y)|^2 \le c_1 \mathcal{E}(f, f)$ for all $f \in \mathcal{F}$.*

Note also that from (3.8) we deduce for $f \in \mathcal{F}$

$$|f(x) - f(y)|^2 \le c_1 R_m^{-1} \mathcal{E}(f, f) \text{ if } x, y \text{ are in the same } m\text{-complex.} \quad (3.10)$$

We need some further properties of the Dirichlet form $\mathcal{E}$, and begin by proving the following Poincaré inequality. For $u \in C(F)$ we write $\bar{u} = \int_F u d\mu$.

LEMMA 3.4. – *For $f \in \mathcal{F}$*

$$\mathcal{E}(f, f) \ge c_2 \|f - \bar{f}\|_2^2. \quad (3.11)$$

*Proof.* – Let $g = f - \bar{f}$. Then from Lemma 3.2, for $x, y \in F$, $(g(x) - g(y))^2 = (f(x) - f(y))^2 \le c_1 \mathcal{E}(f, f)$. So,

$$c_1 \mathcal{E}(f, f) = c_1 \int \int \mathcal{E}(f, f) \mu(dx) \mu(dy) \ge \int \int (g(x) - g(y)^2 \mu(dx) \mu(dy)$$
$$= 2 \int g(x)^2 \mu(dx). \qquad □$$

The following decomposition of Dirichlet forms is along the same lines as that given in [15], but the non-constant environment gives it a more cumbersome form. We use notation such as $R_n(\xi)$ to denote the quantity $R_n$ associated with the environment sequence $\xi$.

LEMMA 3.5. – *For* $f \in \mathcal{F}$, $n \geq 0$,

$$\mathcal{E}^{(\xi)}(f,f) = \sum_{w \in W_n(\xi)} R_n(\xi) \mathcal{E}^{(\theta^n \xi)}(f \circ \psi_w, f \circ \psi_w). \tag{3.12}$$

*Proof.* – If $m \geq n$ then

$$\begin{aligned}
&\mathcal{E}_m^{(\xi)}(f,f) \\
&= \sum_{w \in W_m(\xi)} R_m(\xi) \mathcal{E}_0(f \circ \psi_w, f \circ \psi_w) \\
&= \sum_{w \in W_n(\xi)} \sum_{v \in W_{m-n}(\theta^n \xi)} R_n(\xi) R_{m-n}(\theta^n \xi) \mathcal{E}_0(f \circ \psi_w \circ \psi_v, f \circ \psi_w \circ \psi_v) \\
&= \sum_{w \in W_n(\xi)} R_n(\xi) \mathcal{E}_{m-n}^{(\theta^n \xi)}(f \circ \psi_w, f \circ \psi_w).
\end{aligned}$$

Letting $m \to \infty$ the result follows. □

## 4. TRANSITION DENSITY ESTIMATES: UPPER BOUNDS

Let $P_t$ be the semigroup of positive operators associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$, and let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be the infinitesmal generator of $(P_t)$ – see [9]. As $(\mathcal{E}, \mathcal{F})$ is regular and local, there exists a Feller diffusion $(X_t, t \geq 0, P^x, x \in F)$ with semigroup $P_t$, which we will call *Brownian motion on* $F$. As in [8] we deduce from Theorem 3.3 that $G_\lambda = \int e^{-\lambda t} P_t dt$ has a bounded symmetric density $g_\lambda(x,y)$ with respect to $\mu$. As $g_\lambda(x,y) \in \mathcal{F} \subset C(F)$, $g_\lambda(x,.)$ is continuous for each $x$. As in Lemma 2.9 of [7], it follows that $P_t$ has a bounded symmetric density $p_t(x,y)$ with respect to $\mu$, and that $p_t(x,y)$ satisfies the Chapman-Kolmogorov equations. We now obtain upper bounds on $p_t(x,y)$, beginning with the on-diagonal upper bound, where we follow closely the argument of [17].

LEMMA 4.1. – *There is a constant* $c_1$ *such that if* $T_n^{-1} \leq t \leq T_{n-1}^{-1}$ *then*

$$||P_t||_{1 \to \infty} \leq c_1 M_n. \tag{4.1}$$

*Proof.* – For $w \in W_n$ write $f_w = f \circ \psi_w$ and

$$\bar{f}_w = \int_{F^{(\theta^n \xi)}} f_w(x) \mu^{(\theta^n \xi)}(dx).$$

Note that for $v \in C(F_n)$, $\bar{v} = \int v d\mu = \sum_{w \in W_n} M_n^{-1} \bar{v}_w$.

Let $u_0 \in \mathcal{D}(\mathcal{A})$ with $u_0 \geq 0$ and $\|u_0\|_1 = 1$. Set $u_t(x) = (P_t u_0)(x)$ and $g(t) = \|u_t\|_2^2$. We remark that $g$ is continuous and decreasing. As the semigroup is Markov, $\|u_t\|_1 = 1$, and using Lemmas 3.5 and 3.4,

$$\begin{aligned} \frac{d}{dt} g(t) &= -2\mathcal{E}(u_t, u_t) \\ &= -2 \sum_{w \in W_n} R_n \mathcal{E}^{(\theta^n \xi)}(u_t \circ \psi_w, u_t \circ \psi_w) \quad \text{(by 3.12)} \\ &\leq -2c_{3.1} R_n \sum_w \int (u_{t,w} - \bar{u}_{t,w})^2 d\mu^{(\theta^n \xi)} \\ &= -2c_{3.1} R_n M_n \int u_t^2 d\mu + 2c_{3.1} R_n \sum_w \bar{u}_{t,w}^2 \\ &\leq -2c_{3.1} R_n M_n \|u_t\|_2^2 + 2c_{3.1} R_n M_n^2. \end{aligned} \tag{4.2}$$

Since $M_n R_n = T_n$, we have $g'(t) \leq -cT_n(g(t) - M_n)$, for all $n \geq 0$. Therefore

$$-\frac{d}{dt} \log (g(t) - M_n) \geq cT_n, \text{ if } g(t) > M_n. \tag{4.3}$$

Let $s_n = \inf\{t \geq 0 : g(t) \leq M_n\}$ for $n \in \mathbb{N}$. Thus (4.3) holds for $0 < t < s_n$. Integrating (4.3) from $s_{n+2}$ to $s_{n+1}$ we obtain

$$\begin{aligned} cT_n(s_{n+1} - s_{n+2}) &\leq -\log (g(s_{n+1}) - M_n) + \log (g(s_{n+2}) - M_n) \\ &= \log (M_{n+2} - M_n)/(M_{n+1} - M_n) \leq \log (m^* + 1). \end{aligned}$$

Thus $s_{n+1} - s_{n+2} \leq c(T_n)^{-1}$, and iterating this we have

$$s_n \leq c \sum_{k=n-1}^{\infty} (T_k)^{-1} \leq c_2 (T_n)^{-1}.$$

This implies that $g(c_2/T_n) \leq g(s_n) = M_n$. It follows that there exists $c_1 < \infty$ such that if $T_n^{-1} \leq t < T_{n-1}^{-1}$ then

$$g(t) \leq c_1 M_n.$$

Finally

$$\|P_t\|_{1\to\infty} = \|P_t\|_{1\to 2}^2 = \sup_{u_0} g(t),$$

and as $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{F}$, this proves the Lemma. □

As in [7], Lemma 4.6 we can now use the symmetry of $p_t(x,y)$, and the fact that it satisfies the Chapman-Kolmogorov equations, to deduce that $p_t(x,y)$ is jointly continuous in $x, y$ for each $t$. We therefore obtain from Lemma 4.1 the pointwise bound

$$p_t(x,y) \le c_1 M_n, \quad x, y \in F. \tag{4.4}$$

For any process $Z$ on $F$ define the stopping times $S_i^k(Z)$ by $S_0^k(Z) = \inf\{t \ge 0 : Z_t \in F_k\}$, and

$$S_i^k(Z) = \inf\{t > S_{i-1}^k(Z) : Z_t \in F_k \backslash \{Z_{S_{i-1}^k(Z)}\}\};$$

these are the times of the successive visits to $F_k$ by $Z$. We define the crossing times on level $k$ by $W_i^k(Z) = S_i^k(Z) - S_{i-1}^k(Z)$, and write $S_i^k = S_i^k(X)$, $W_i^k = W_i^k(X)$. We now recall some properties of $X$ and the crossing times – see [4, 18] for details. Let $Y_i^n = X_{S_i^n}$; then $Y^n$ is a simple random walk on $F_n$. The 'Einstein relation' $t_a = m_a r_a$ implies that $E\, W_i^n(Y^m) = T_m/T_n$ for $i \ge 1$, $n \le m$. If $X_t^n = Y_{[T_n t]}^n$ then, as in [4], we have that the processes $X^n$ converge a.s. to $X$. We also have $W_i^n(X^m) \to W_i^n(X)$ a.s. and in $L^2$ as $m \to \infty$, from which we deduce that $EW_i^n(X) = T_n^{-1}$ for $n \ge 0$, $i \ge 1$.

Now fix $z \in F_n$, and $B$ be the union of the $n$-complexes $\psi_w(F)$, $w \in W_n$ which contain $z$. Write $S_B = \inf\{t \ge 0 : X_t \notin B\}$, and note that $E^z S_B = T_n^{-1}$. For $x \in B$ we have $S_B \le S_1^n$ $P^x$ -a.s., and since $S_1^m$, $m \ge n$ is a decreasing sequence with limit 0 (as $X$ is non-constant), we deduce

$$S_B \le \sum_{i=n}^{\infty} (S_1^i - S_1^{i+1}). \tag{4.5}$$

As $X_{S_1^{i+1}} \in F_{i+1}$, we have $E(S_1^i - S_1^{i+1}) \le \gamma(\xi_{i+1}) T_{i+1}^{-1}$, where $\gamma(a)$ is such that if $\xi_1 = a$ and $S_0 = \inf\{r \ge 0 : Y_r^1 \in F_0\}$, then

$$\max_{y \in F_1^{(a)}} E^y S_0 = \gamma(a).$$

(Note that as $Y^1$ is for each $a$ a random walk on the irreducible set $F_1^{(a)}$, $\gamma(a)$ is finite.) Let $c_3 = \max_a \gamma(a)$. From (4.5) we have, for $x \in B$,

$$E^x S_B \le c_3 \sum_{i=n}^{\infty} T_{i+1}^{-1} \le c_4 T_n^{-1}. \tag{4.6}$$

Since $S_B \leq t + 1_{(S_B>t)}(S_B - t)$ we have, from (4.6),

$$\begin{aligned} E^z S_B &\leq t + E^z\big(1_{(S_B>t)} E^{X_t}(S_B)\big) \\ &\leq t + P^z(S_B > t) c_4 T_n^{-1}. \end{aligned}$$

So $P^z(S_B \leq t) \leq c_4^{-1} T_n t + (1 - c_4^{-1})$, and as $S_B = W_1^n$ $P^z$-a.s., we deduce there exist $c_5 > 0$, $c_6 \in (0,1)$ such that

$$P^z(W_1^n \leq t) \leq c_5 T_n t + c_6, \quad t \geq 0. \tag{4.7}$$

This bound is quite crude, but we can now, as in [2], use it to derive a much better estimate on $P^z(W_1^n \leq t)$.

We first define

$$k = k(m,n) = \inf\left\{ j \geq 0 : \frac{T_{m+j}}{B_{m+j}} \geq \frac{T_n}{B_m} \right\}. \tag{4.8}$$

As the function $k(m,n)$ plays a crucial role in our bounds, we need to spend a little time exploring its properties. First, we recall the inequalities $2 \leq b_a \leq b^*$, $4 \leq t_a \leq t^*$, $2 \leq b_a \leq t_a/b_a \leq t^*/2$, from (2.7) and Lemma 3.1.

If $m \geq n$ then $T_m/B_m \geq T_n/B_m$, so $k(m,n) = 0$. If $m < n$ then as $T_n/B_n < T_n/B_m$ we deduce that $k(m,n) > n - m$. On the other hand, writing $k = k(m,n)$, we have

$$2^{k-1} \leq \frac{T_{m+k-1}/T_m}{B_{m+k-1}/B_m} < T_n/T_m \leq (t^*)^{n-m},$$

so that

$$n - m < k(m,n) \leq c_7(n-m) \quad \text{when} \quad m < n. \tag{4.9}$$

Note also from (4.9) and the remarks preceding that if $m < n$ then $n < m + k \leq m + c_7(n-m) < (1+c_7)n$. Therefore,

$$n \leq m + k(m,n) \leq (1+c_7)n \quad \text{if } m < n. \tag{4.10}$$

Using the bounds on $t_a/b_a$ above we have, for $i \geq 0$,

$$2^{i+1}\frac{T_{m+l}}{B_{m+l}} \leq \frac{T_{m+1+l+i}}{B_{m+1+l+i}} \leq (t^*/2)^{i+1}\frac{T_{m+l}}{B_{m+l}},$$

from which it follows that

$$|k(m+1,n) - k(m,n)| \leq c_8, \text{ for all } m,n. \tag{4.11}$$

So, we have,

$$\left|\log\left(\frac{B_{m'+k(m',n)}}{B_{m'}}\right)-\log\left(\frac{B_{m+k(m,n)}}{B_m}\right)\right|\leq (1+c_8)|m'-m|\log b^*. \quad (4.12)$$

We now define the approximate walk and spectral dimensions,

$$d_w(m)=\frac{\log T_m}{\log B_m},\qquad d_s(m)=\frac{2\log M_m}{\log T_m}. \quad (4.13)$$

LEMMA 4.2. – *Let* $0<t<1$, $0<r<1$, *and let* $n,m$ *satisfy*

$$T_n^{-1}\leq t<T_{n-1}^{-1},\quad B_m^{-1}\leq r<B_{m-1}^{-1}.$$

*Then writing* $k=k(m,n)$,

$$\begin{aligned}\frac{1}{2}\exp\left(c_9\frac{B_{m+k}}{B_m}\right)&\leq\exp\left(\left(\frac{r^{d_w(m+k)}}{t}\right)^{1/(d_w(m+k)-1)}\right)\\&\leq\exp\left(c_{10}\frac{B_{m+k}}{B_m}\right).\end{aligned} \quad (4.14)$$

*Proof.* – If $m\geq n$ then $k=0$, and so $B_{m+k}/B_m$=1. Since $d_w(m)\leq\log t^*/\log 2\leq c$, and $r\leq cB_m^{-1}$, we have $r^{d_w(m+k)}=r^{\log T_m/\log B_m}\leq cT_m^{-1}$, so that $r^{d_w(m+k)}/t\leq cT_n/T_m\leq c'$. As $r^{d_w(m+k)}/t\geq 0$ the lower bound is clear. It follows that (4.14) holds.

If $m<n$ then writing $\alpha=d_w(m+k)$,

$$r^\alpha/t\leq cT_n/B_m^\alpha\leq cT_{m+k}/(B_{m+k}B_m^{\alpha-1})=c(B_{m+k}/B_m)^{\alpha-1},$$

with a similar lower bound. □

LEMMA 4.3. – *There exist constants* $c_{11},c_{12}$ *such that if* $k=k(m,n)$ *then*

$$P(W_1^m\leq T_n^{-1})\leq c_{11}\exp\left(-c_{12}B_{m+k}/B_m\right). \quad (4.15)$$

*Proof.* – If $j\geq 0$, then for the process $X$ to cross one $m$-complex it must cross at least $N=B_{m+j}/B_m$ $(m+j)$-complexes. So

$$W_1^m\geq\sum_{i=1}^{B_{j+m}/B_m}V_i,$$

where $V_i$ are i.i.d. and have distribution $W_1^{m+j}$. Lemma 1.1 of [2] states that if $P(V_i < s) \leq p_0 + \alpha s$, where $p_0 \in (0,1)$ and $\alpha > 0$, then

$$\log P\left(\sum_1^N V_i \leq t\right) \leq 2(\alpha N t/p_0)^{1/2} - N\log(1/p_0). \tag{4.16}$$

Thus, using (4.7) and (4.16), we have

$$\begin{aligned}\log P(W_1^m \leq T_n^{-1}) \\ \leq c_{13}(B_{m+j}/B_m)^{1/2}[(T_{m+j}/T_n)^{1/2} - c_{14}(B_{m+j}/B_m)^{1/2}].\end{aligned} \tag{4.17}$$

Given $k = k(m,n)$ as above, there exists $c_{15}$ and $k_0$ such that $k - c_{15} \leq k_0 \leq k$, and

$$(T_{m+k_0}/T_n)^{1/2} < \frac{1}{2}c_{14}(B_{m+k_0}/B_m)^{1/2}.$$

Provided $k_0 \geq 1$ we deduce

$$\log P(W_1^m \leq T_n^{-1}) \leq -\frac{1}{2}c_{13}c_{14}B_{m+k_0}/B_m \leq -c_{12}B_{m+k}/B_m.$$

Choosing $c_{11}$ large enough we have $1 < c_{11}\exp(-c_{12}B_{m+k}/B_m)$ whenever $k < c_{15} + 1$, so that (4.15) holds in all cases. □

LEMMA 4.4. – *There exist constants $c_{11}, c_{16}$ such that if $0 < t < 1$, $0 < r < 1$, and $n, m$ satisfy*

$$T_n^{-1} \leq t < T_{n-1}^{-1}, \quad B_m^{-1} \leq r < B_{m-1}^{-1},$$

*and $k = k(m,n)$ then for $x \in F$*

$$\begin{aligned}P^x(\sup_{0\leq s\leq t} d(X_s,x) \geq r) \\ \leq c_{11}\exp\left(-c_{16}\left(\frac{r^{d_w(m+k)}}{t}\right)^{1/(d_w(m+k)-1)}\right).\end{aligned} \tag{4.18}$$

*Proof.* – Let $m_0$ be such that $2c_{2.1}B_{m_0}^{-1} \leq r \leq 2c_{2.1}B_{m_0-1}^{-1}$. Then $|m - m_0| \leq c$. From (2.10) we have that $d(x,y) \leq c_{2.1}B_l^{-1}$ if $x, y$ are in the same $l$-complex. So, $d(X_s,x) \leq 2c_{2.1}B_{m_0}^{-1} \leq r$ for $0 \leq s \leq S_1^{m_0}$. Therefore, writing $k_0 = k(m_0,n)$,

$$\begin{aligned}P^x(\sup_{0\leq s\leq t} d(X_s,x) \geq r) &\leq P^x(S_1^{m_0} \leq t) \\ &\leq P^x(S_1^{m_0} \leq T_n^{-1}) \\ &\leq c_{11}\exp(-c_{12}B_{m_0+k_0}/B_{m_0}) \\ &\leq c_{11}\exp(-cB_{m+k}/B_m), \qquad \text{(using (4.12))} \\ &\leq c_{11}\exp\left(-c_{16}\left(\frac{r^{d_w(m+k)}}{t}\right)^{1/(d_w(m+k)-1)}\right),\end{aligned}$$

by Lemmas 4.2 and 4.3. □

THEOREM 4.5. – *There exist constants* $c_{17}, c_{18}$ *such that if* $0 < t < 1$, $x, y \in F$, *and* $n, m$ *satisfy*

$$T_n^{-1} \leq t < T_{n-1}^{-1}, \quad B_m^{-1} \leq d(x,y) < B_{m-1}^{-1}, \tag{4.19}$$

*and* $k = k(m,n)$ *then*

$$p_t(x,y) \\ \leq c_{17} t^{-d_s(n)/2} \exp\left(-c_{18}\left(\frac{d(x,y)^{d_w(m+k)}}{t}\right)^{1/(d_w(m+k)-1)}\right). \tag{4.20}$$

*Proof.* – Noting that $M_n \leq ct^{-d_s(n)/2}$, this is proved from (4.4) and Lemma 4.4 by exactly the same argument as in Theorem 6.2 of [3]. □

*Remark.* – Note that the bound (4.20) may also be written in the form

$$p_t(x,y) \leq cM_n \exp(-c' B_{m+k}/B_m), \tag{4.21}$$

where $m, n$ satisfy (4.19), and $k = k(m,n)$.

## 5. LOWER BOUNDS

In this section we use techniques developed in [3], [7] to obtain lower bounds on $p_t(x,y)$ which will be identical, apart from the constants, to the upper bound (4.20).

LEMMA 5.1. – *There exists a constant* $c_1$ *such that if* $T_n^{-1} \leq t$ *then*

$$p_t(x,x) \geq c_1 M_n \quad \textit{for all } x \in F. \tag{5.1}$$

*Proof.* – Note from Lemma 4.4 that if $r = \lambda B_n^{-1}$, with $\lambda > b^*$, then

$$P^x(d(x,X_t) > r) \leq c_{4.11} \exp(-c_{4.16} B_{m+k}/B_m),$$

where $m < n$ satisfies $B_m^{-1} \leq \lambda B_n^{-1} < B_{m-1}^{-1}$, and $k = k(m,n)$. Note that $\lambda \leq (b^*)^{n-m+1}$. Since $m + k > n$ we have $B_{m+k}/B_m > B_n/B_m \geq 2^{n-m}$. Thus

$$B_{m+k}/B_m \geq c\lambda^{\log 2/\log b^*},$$

so that there exists $c_2 > 0$ such that

$$P^x(d(x, X_t) > r) \le c \exp(-c' \lambda^{c_2}). \tag{5.2}$$

Now let $\lambda = \lambda_0$ be large enough so the left hand side of (5.2) equals $\frac{1}{2}$. Then by (2.12) $\mu(B(x, \lambda_0 B_n^{-1})) \le c M_n^{-1}$, and so writing $G = B(x, \lambda_0 B_n^{-1})$ we have $P^x(X_t \in G) \ge \frac{1}{2}$. So, using Cauchy-Schwarz,

$$\begin{aligned} \tfrac{1}{4} \le P^x(X_t \in G))^2 &= \left( \int_G p_t(x,y) \mu(dy) \right)^2, \\ &\le \mu(G) \int_G p_t(x,y)^2 \mu(dy) \le c M_n^{-1} p_{2t}(x,x). \end{aligned}$$

If $t \ge T_n^{-1}$ then $t/2 \ge T_{n+1}^{-1}$, so we deduce that $p_t(x,x) \ge c M_{n+1} \ge c_1 M_n$.
□

We need to extend this 'on-diagonal lower bound' to a 'near-diagonal lower bound', which we do via an estimate on the Hölder continuity of the heat kernel.

LEMMA 5.2. – *Let $m \ge 0$, $n \ge 0$, and $T_n^{-1} < t$, $d(x, x') \le B_{m-1}^{-1}$. Then for each $y \in F$,*

$$|p_t(x,y) - p_t(x',y)| \le c_3 M_n \sqrt{\frac{R_n}{R_m}}. \tag{5.3}$$

*In particular $p_t(.,.)$ is uniformly continuous on $F \times F$ for each $t > 0$.*

*Proof.* – By (3.10) if $x, x'$ are in the same $m$-complex then

$$|p_t(x,y) - p_t(x',y)|^2 \le c R_m^{-1} \mathcal{E}(p_t(.,y), p_t(.,y)). \tag{5.4}$$

As in [7] Lemma 6.4, we have, writing $u(x) = p_{t/2}(x,y)$,

$$\begin{aligned} \mathcal{E}(P_{t/2}u, P_{t/2}u) &\le c(t/2)^{-1} \|u\|_2^2, \\ &\le c t^{-1} p_t(y,y) \le c' t^{-1} M_n \le c'' T_n M_n. \end{aligned}$$

As $T_n = M_n R_n$ we deduce that (5.3) holds if $x, x'$ are in the same $m$-complex. If now we just have $d(x,x') \le B_{m-1}^{-1}$, then there is a chain of at most $b^+$ $m$-complexes linking $x, x'$, and again we have, adjusting the constant $c$, that (5.3) holds. □

LEMMA 5.3. – *There exist $c_4, c_5$ such that if $T_n^{-1} < t$, then*

$$p_t(x,y) \ge c_4 M_n \quad \textit{whenever } d(x,y) \le c_5 B_n^{-1}. \tag{5.5}$$

*Proof.* – We can find $c$ such that there exists $m$ with $n \leq m \leq n+c$ and $R_m/R_n \geq (3/2)^{m-n} \geq 4c_3^2/c_1^2$. As $m-n < c$ we have $B_{m-1}^{-1} \geq c_5 B_n^{-1}$ for some constant $c_5$. So if $d(x,y) \leq c_5 B_n^{-1}$ then by Lemmas 5.1 and 5.2,

$$\begin{aligned} p_t(x,y) &\geq p_t(x,x) - |p_t(x,y) - p_t(x,x)| \\ &\geq M_n\big(c_1 - c_3(R_n/R_m)^{1/2}\big) \geq \frac{1}{2}c_1 M_n. \end{aligned}$$

□

We can now use a standard chaining argument to obtain general lower bounds on $p_t$ from Lemma 5.3.

THEOREM 5.4. – *There exist constants $c_6, c_7$ such that if $x, y$ in $F$, $t \in (0,1)$ and*

$$T_n^{-1} \leq t < T_{n-1}^{-1}, \quad B_m^{-1} \leq d(x,y) < B_{m-1}^{-1},$$

*then*

$$p_t(x,y) \geq c_6 t^{-d_s(n)/2} \exp\left(-c_7\left(\frac{d(x,y)^{d_w(m+k)}}{t}\right)^{1/(d_w(m+k)-1)}\right). \quad (5.6)$$

*Proof.* – Using (5.5) we see that the bound is satisfied if $m \geq n$. Now let $m < n$, write $k = k(m,n)$, and choose $j, l$ with $0 \leq j < l < c$ such that

$$2^{l-j} \geq 3b^*/c_2, \quad (b^*)^l < (2b^*)^j;$$

note that such a choice is possible, with a constant $c$ depending only on $c_2$ and $b^*$. We then have

$$\frac{B_{m+k+l}}{B_{m+k}} \leq \frac{B_{m+k+j}}{B_{m+k}}(b^*)^{l-j} \leq \frac{T_{m+k+j}}{T_{m+k}}2^{-j}(b^*)^{l-j} < \frac{T_{m+k+j}}{T_{m+k}}, \quad (5.7)$$

and

$$\frac{3b^*}{B_{m+k+l}} \leq \frac{3b^* 2^{j-l}}{B_{m+k+j}} \leq \frac{c_2}{B_{m+k+j}}. \quad (5.8)$$

Let $N = B_{m+k+j}/B_m$. Since $d(x,y) \leq b^* B_m^{-1}$ there exists a chain $x = z_0, z_1, \ldots, z_N = y$ with $d(z_{i-1}, z_i) \leq b^* B_{m+k+j}^{-1}$. Let $G_i = B(z_i, b^* B_{m+k+j}^{-1})$; then, if $x_i \in G_i$, we have

$$d(x_{i-1}, x_i) \leq 3b^* B_{m+k+l}^{-1} \leq c_2 B_{m+k+j}^{-1}. \quad (5.9)$$

Let $s = t/N$, then

$$s \geq \frac{B_m}{T_n B_{m+k+l}} \geq \frac{B_{m+k}}{T_{m+k} B_{m+k+l}} > \frac{1}{T_{m+k+j}}. \quad (5.10)$$

From (5.5), (5.9) and (5.10) we have $p_s(x_{i+1}, x_i) \geq cM_{m+k+j} \geq c'M_{m+k}$. Therefore since $\mu(G_i) \geq c_8 M_{m+k}^{-1}$, and $m + k \geq n$,

$$\begin{aligned} p_t(x, y) &\geq \int_{G_1} \cdots \int_{G_{N-1}} p_s(x, x_1) \ldots p_s(x_{N-1}, y)\mu(dx_1) \ldots \mu(dx_{N-1}), \\ &\geq \left( \prod_{i=1}^{N-1} \mu(G_i) \right) (c_8 M_{m+k})^N, \\ &\geq cM_{m+k} \exp(-c_9 N) \geq cM_n \exp(-c_{10} B_{m+k}/B_m). \end{aligned}$$

Using Lemma 4.2 completes the proof. □

*Proof of Theorem 1.1.* – This is an immediate consequence of Lemma 2.4 and Theorems 4.5 and 5.4. □

## 6. EXAMPLES

In this section we apply Theorems 4.5 and 5.4 to see how oscillations in the environment sequence $\xi_i$ relate to oscillations in the transition density.

For the environment sequence $\xi$ set

$$h_a(n) = n^{-1} \sum_{i=1}^{n} 1_{(\xi_i = a)}, \quad a \in A.$$

Let $(p_a)$ be a probability distribution on $A$, and suppose that $\xi$ satisfies, for some regularly varying increasing function $g$,

$$h_a(n) \to p_a \quad \text{as } n \to \infty \quad \text{for each } a \in A, \tag{6.1}$$

$$|h_a(n) - p_a| \leq n^{-1} g(n), \quad n \geq 1, \, a \in A. \tag{6.2}$$

Note that if $0 < p_a < 1$ then $\liminf |nh_a(n) - np_a| > 0$, so that the rate of convergence given by taking $g(n) = O(1)$ is the fastest possible. We take $g(0) = 1$.

We have

$$d_s(n) = \frac{2 \sum_a h_a(n) \log m_a}{\sum_a h_a(n) \log t_a}, \quad d_w(n) = \frac{\sum_a h_a(n) \log t_a}{\sum_a h_a(n) \log b_a}. \tag{6.3}$$

Let

$$d_s = \lim_n d_s(n) = \frac{2 \sum_a p_a \log m_a}{\sum_a p_a \log t_a},$$

and define $d_w$ similarly.

If $(p_a)$, $(q_a)$ are probability distributions on $A$, and for $a \in A$, $u_a$, $v_a$ satisfy $u^* \geq u_a \geq c_1$, $v^* \geq v_a \geq c_1$, then elementary calculations yield

$$\left| \frac{\sum q_a u_a}{\sum q_a v_a} - \frac{\sum p_a u_a}{\sum p_a v_a} \right| \leq c_1^{-2} u^* v^* \max_a |p_a - q_a|. \tag{6.4}$$

Therefore (6.1), (6.2) imply that

$$\frac{1}{2}|d_s(n) - d_s| \leq c_3 n^{-1} g(n), \quad |d_w(n) - d_w| \leq c_3 n^{-1} g(n). \tag{6.5}$$

Let

$$\psi(t) = g(\log(1/t)), \quad t \in (0, 1).$$

THEOREM 6.1. – *Let $\xi$ satisfy (6.1) and (6.2). Then for $0 < t < 1$, $x, y \in F$*

$$p_t(x, y) \leq c_4 t^{-d_s/2} e^{c_5 \psi(t)} \exp\left( -c_6 e^{-c_5 \psi(t)} \left( \frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w - 1)} \right), \tag{6.6}$$

$$p_t(x, y) \geq c_7 t^{-d_s/2} e^{-c_5 \psi(t)} \exp\left( -c_8 e^{c_5 \psi(t)} \left( \frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w - 1)} \right). \tag{6.7}$$

*Proof.* – Let $T_n^{-1} \leq t < T_{n-1}^{-1}$, $B_m^{-1} \leq r = d(x, y) \leq B_{m-1}^{-1}$. Then, since $4^n \leq T_n \leq (t^*)^n$, and similar bounds hold for $B_m$, we have

$$cn \leq \log(1/t) \leq c'n, \quad cm \leq \log(1/r) \leq c'm. \tag{6.8}$$

So by (6.5)

$$\begin{aligned} t^{-d_s(n)/2} &\leq t^{-d_s/2} t^{-c_3 n^{-1} g(n)} \\ &\leq t^{-d_s/2} \exp(cg(c'n)) \leq t^{-d_s/2} \exp(c_5 \psi(1/t)). \end{aligned} \tag{6.9}$$

For the off-diagonal term we have, writing $u = r^{d_w}/t$,

$$u \leq c \frac{T_n}{B_m^{d_w}} \leq c \frac{T_{m+k}}{B_{m+k} B_m^{d_w - 1}} = c \left( \frac{B_{m+k}}{B_m} \right)^{d_w - 1} B_{m+k}^{d_w(m+k) - d_w},$$

so that if $\gamma = (d_w - d_w(m + k))/(d_w - 1)$ then

$$B_{m+k}/B_m \geq c u^{1/(d_w - 1)} B_{m+k}^{\gamma}. \tag{6.10}$$

If $m < n$ then using (4.10) we have $c'n \leq \log B_{m+k} \leq c''n$, and so

$$\log B_{m+k}^{\gamma} \geq -cn|d_w(m + k) - d_w| \geq -c'g(n), \tag{6.11}$$

while if $m \geq n$ then $B_{m+k}/B_m = 1$. From (4.21) we have

$$p_t(x,y) \leq ct^{-d_s(n)/2} \exp(-cB_{m+k}/B_m),$$

and combining this with (6.9), (6.10) and (6.11) we obtain (6.6).

The lower bound is proved in exactly the same way. □

The on-diagonal bounds here are (up to constants) the best possible. Set

$$q_t(x) = p_t(x,x)t^{d_s/2}.$$

THEOREM 6.2. – *Let $\xi$ satisfy (6.1) and suppose there exists a sequence $n_i \to \infty$ such that*

$$n_i(d_s(n_i) - d_s) > g(n_i), \quad i \geq 1. \tag{6.12}$$

*Then if $s_i = T_{n_i}^{-1}$,*

$$q_{s_i}(x) \geq \exp(c\psi(1/s_i)), \quad i \geq 1. \tag{6.13}$$

*Similarly, if $n_i(d_s(n_i) - d_s) < g(n_i)$, then $q_{s_i}(x) \leq \exp(-c\psi(1/s_i))$ for $i \geq 1$.*

*Proof.* – From Theorem 5.4, and using the calculations in Theorem 6.1 we have

$$q_{s_i}(x) \geq cs_i^{(d_s - d_s(n_i))/2} \geq c\exp(c'g(n_i)) \geq c\exp(c'\psi(1/s_i)),$$

which establishes (6.13). The upper bound is proved in the same way. □

*Remark.* – Theorems 6.1 and 6.2 imply that the bounds on $p_t$ of the kind which hold for regular fractals such as nested fractals or Sierpinski carpets, (see [3, 15]), hold for scale irregular Sierpinski gaskets if and only if the convergence of $d_s(n)$ to $d_s$ is as fast as possible, so that the function $g$ in (6.2) satisfies $g(n) \leq K$ for all $n$.

We can apply Theorem 6.1 to the case when the environment random variables $\xi_i$ (defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$) are i.i.d. with (non-degenerate) distribution $(p_a)$. By the law of the iterated logarithm the random variables $h_a(n)$ satisfy (6.2) with $g(n) = C(\omega)(n \log\log n)^{1/2}$, where $\mathbb{P}(C(\omega) < \infty) = 1$. Applying Theorem 6.1, and writing $\phi(t) = \max\{((\log(1/t)) \log\log\log(1/t))^{1/2}, 1\}$, we have

COROLLARY 6.3. – *There exists a constant* $C = C(\omega) \in (0, \infty)$ *such that for* $0 < t < 1$ *and* $x, y \in F^{(\xi(\omega))}$, $\mathbb{P} - a.s.$,

$$p_t(x,y) \leq c_4 t^{-d_s/2} e^{C\phi(t)} \exp\left(-c_6 e^{-C\phi(t)} \left(\frac{d(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \quad (6.14)$$

*with a similar lower bound.*

*Remark.* – In [10] it was proved that for each $\varepsilon > 0$ there exist $c_7(\varepsilon, \omega)$, $c_8(\varepsilon, \omega)$ such that for $x, y \in F^{(\xi(\omega))}$

$$p_t(x,y) \leq c_7 t^{-d_s/2-\varepsilon} \exp\left(-c_8 \left(\frac{d(x,y)^{d_w+\varepsilon}}{t}\right)^{1/(d_w-1)-\varepsilon}\right). \quad (6.15)$$

Setting $r = d(x,y)$ let $a(r,t)$, $b(r,t)$ denote the right hand sides of (6.14) and (6.15) respectively. Since $\lim_{t \downarrow 0} t^{\varepsilon} e^{c\phi(t)} = 0$, we have that $a(0,t) < b(0,t)$ for all sufficiently small $t$. With a little more labour we can also show that $a(r,t) < b(r,t)$ for all sufficiently small $r, t$, so that, neglecting constants, the bound in (6.14) improves that of (6.15). (Of course, this is to be expected, since Theorem 5.4 shows that the bounds in Theorem 4.5 are, up to constants, the best possible).

Note, however, that for the on diagonal bounds there is less oscillation in the random recursive case [11] than that observed here.

## 7. SPECTRAL RESULTS

Write $\mathcal{L}$ for the infinitesimal generator of the semigroup $(P_t)$: we call $\mathcal{L}$ the *Laplacian* on the fractal $F$. The uniform continuity of $p_t$ (see Lemma 5.2) implies that $P_t$ is a compact operator on $L^2(F, \mu)$, so that $P_t$, and hence $-\mathcal{L}$, has a discrete spectrum. Let $0 \leq \lambda_1 \leq \ldots$ be the eigenvalues of $-\mathcal{L}$, and let $N(\lambda) = \#\{\lambda_i : \lambda_i < \lambda\}$ be the eigenvalue counting function.

Since

$$\int_F p_t(x,x)\mu(dx) = \int_0^\infty e^{-st} N(ds), \quad t > 0,$$

using (4.20) and (5.6) we have

$$c_1 M_n \leq \int_0^\infty e^{-s/T_n} N(ds) \leq c_2 M_n, \quad n \geq 0. \quad (7.1)$$

PROPOSITION 7.1. – *There exist constants* $c_3$, $c_4$, $c_5$ *such that if* $\lambda > c_3$ *and* $n$ *is such that* $T_{n-1} \le \lambda < T_n$ *then*

$$c_4 \lambda^{d_s(n)/2} \le N(\lambda) \le c_5 \lambda^{d_s(n)/2}. \tag{7.2}$$

*Proof.* – It is sufficient to prove that there exists $c_6 > 0$ such that

$$cM_n \le N(T_n) \le c' M_n \quad \text{for } n \ge c_6.$$

The right hand inequality is easy. From (7.1)

$$c_2 M_n \ge \int_0^{T_n} e^{-s/T_n} N(ds) \ge e^{-1} N(T_n).$$

For the left hand inequality, let $r < n$ and note that

$$c_1 M_r \le N(T_n) + \int_{T_n}^{\infty} e^{-s/T_r} N(ds).$$

We have

$$\int_{T_n}^{\infty} e^{-s/T_r} N(ds) = \sum_{k=n}^{\infty} \int_{T_k}^{T_{k+1}} e^{-s/T_r} N(ds) \tag{7.3}$$

$$\le \sum_{k=n}^{\infty} e^{-T_k/T_r} N(T_{k+1}) \tag{7.4}$$

$$\le cM_r \sum_{k=n}^{\infty} m^* (m^*)^{k-r} \exp(-4^{k-r}). \tag{7.5}$$

So there exists $c_6 > 0$ such that if $n > c_6$ then there exists $n - c_6 \le r \le n$ such that

$$\int_{T_n}^{\infty} e^{-s/T_r} N(ds) \le \frac{1}{2} c_1 M_r.$$

We therefore deduce that $N(T_n) \ge \frac{1}{2} c_1 M_r \ge c' M_n$ by the choice of $r$ for $n > c_6$. □

Finally, we consider the case, mentioned in Section 6, when the environment sequence is i.i.d. with non-degenerate distribution $(p_a)$. Let $\phi(\lambda) = ((\log \lambda) \log\log\log \lambda)^{1/2}$. Combining Proposition 7.1 with the calculations made in Section 6 we obtain

COROLLARY 7.2. – *There exists positive constants* $c_7, c_8$ *such that* $\mathbb{P}$*-a.s.*

$$0 < \limsup_{\lambda \to \infty} \frac{N(\lambda) e^{-c_7 \phi(\lambda)}}{\lambda^{d_s/2}} < \liminf_{\lambda \to \infty} \frac{N(\lambda) e^{c_8 \phi(\lambda)}}{\lambda^{d_s/2}} < \infty.$$

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