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ON THE LIOUVILLE PROPERTY FOR DIVERGENCE FORM OPERATORS

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ABSTRACT. In this paper we construct a bounded strictly positive function σ such that the Liouville property fails for the divergence form operator $L = \nabla(\sigma^2\nabla)$. Since in addition $\Delta\sigma/\sigma$ is bounded, this example also gives a negative answer to a problem of Berestycki, Caffarelli and Nirenberg concerning linear Schrödinger operators.

1. **Introduction.** In a paper on the qualitative properties of solutions of non-linear PDE of the form $\Delta u+F(u) = 0$, Berestycki, Caffarelli and Nirenberg posed the following problem. (See [BCN, Theorem 1.7]).

PROBLEM 1. Let V be a smooth bounded function on $\mathbb{R}^d$, and let $K = K[V]$ be the (Schrödinger) operator

$$K = -\Delta - V.$$

Suppose that a bounded and sign-changing solution u exists to $Ku = 0$. Set

$$\lambda_1(K) = \inf\Big\{\int_{\mathbb{R}^d} |\nabla\psi|^2 - V|\psi|^2 : \psi \in C_0^\infty, \|\psi\|_2 = 1\Big\}.$$

Then is $\lambda_1(K) < 0$?

[BCN, Theorem 1.7] proved that if $d = 1$ or 2 then the answer to Problem 1 is "yes". In [GG] Ghoussoub and Gui proved that the answer is "no" if $d \geq 7$, and made explicit the connection (implicit in the proof of [BCN, Theorem 1.7]) between Problem 1 and the following question on the Liouville property for divergence form operators.

PROBLEM 2. Let σ be a strictly positive C^2 function on $\mathbb{R}^d$, and let $L = L[\sigma]$ be the divergence form operator $L = \nabla(\sigma^2\nabla)$. Let ψ be a solution to $L\psi = 0$. If $\sigma\psi$ is bounded, then is ψ constant? (If this is the case we will say that L has the Liouville property).

It is well-known that if σ is uniformly bounded away from 0 (so that $\sigma > \epsilon > 0$) then $L[\sigma]$ has the Liouville property. The proof of [BCN, Theorem 1.7] implies that the answer to Problem 2 is "yes" if $d = 1, 2$, while [GG] give an example which proves that the answer to Problem 2 is "no" if $d \geq 7$. In those spaces to which the answer to Problem 1 is "yes" this result provides a powerful technique for the study of non-linear PDE—see [GG].

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To see the connection between the two problems note first that if $\sigma > 0$ is C^2 then

$$L[\sigma]\varphi = -\sigma K[-\sigma^{-1}\Delta\sigma](\sigma\varphi). \tag{1.1}$$

THEOREM 1 ([GG, PROPOSITION 2.3, LEMMA 2.1]). *Let V be smooth and bounded.*
(a) If a bounded non-zero C^2 solution u to $K[V]u = 0$ exists, then $\lambda_1(K[V]) \leq 0$.
(b) $\lambda_1(K[V]) < 0$ if and only if $K[V]u = 0$ has no positive solutions.

THEOREM 2. *(See [GG, Proposition 2.8], [BCN, Theorem 1.7]).*
(a) Let V be bounded and smooth, and suppose a bounded sign-changing solution u to $K[V]u = 0$ exists. If $\lambda_1(K[V]) = 0$ then the equation $K[V]\sigma = 0$ has positive solutions, and for any positive solution σ the Liouville property fails for $L[\sigma]$.
(b) Let $\sigma > 0$ be smooth, and such that $V = -\sigma^{-1}\Delta\sigma$ is bounded. Suppose there exists a sign-changing function φ such that $\sigma\varphi$ is bounded, and $L[\sigma]\varphi = 0$. Then there exists a sign-changing solution u to $K[V]u = 0$, but $\lambda_1(K[V]) = 0$.

PROOF. (a) If K, u, σ are as above, set $\varphi = u/\sigma$. By (1.1) $L[\sigma]\varphi = 0$, while φ is sign-changing, and therefore non-constant.

(b) Set $u = \sigma\varphi$: by (1.1) u is a bounded sign-changing solution to $K[V]u = 0$. So, by Theorem 1(a) $K[V] \leq 0$. On the other hand since $\sigma > 0$ also satisfies $K[V]\sigma = 0$, by Theorem 1(b) $\lambda_1(K[V]) = 0$. ∎

REMARKS. 1. The proof above is given in [GG], but is included here for completeness.

In this paper we give an example which shows that the answer to Problems 1 and 2 is “no” for $d \geq 3$. In view of Theorem 2 we can concentrate on the Liouville property, and seek a bounded function $\sigma > 0$ such that $\Delta\sigma/\sigma$ is bounded, but $L[\sigma]$ has non-trivial bounded harmonic functions. Our intuition and proofs are probabilistic. Associated with $\frac{1}{2}L[\sigma]$ is a diffusion process $\tilde{X} = (\tilde{X}_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$, such that $\frac{1}{2}L[\sigma]\varphi = 0$ if and only if $\varphi(\tilde{X}_t)$ is a $\mathbb{P}^x$-martingale for all $x \in \mathbb{R}^d$. (For accounts of the connection between elliptic operators and diffusions see for example the books [Bas], [RW]). Suppose that there exist open disjoint regions D_1, D_2 in $\mathbb{R}^d$ such that if $G_i = \{\tilde{X}_t \in D_i \text{ for all sufficiently large } t\}$ then

$$0 < \psi_i(x) = \mathbb{P}^x(\tilde{X}_t \in D_i \text{ for all sufficiently large } t) < 1, \quad i = 1, 2, \tag{1.2}$$

for some (and so all) $x \in \mathbb{R}^d$. Then since ψ_i are bounded and harmonic (with respect to L), by the martingale convergence theorem

$$\psi_i(\tilde{X}_t) \longrightarrow I_{G_i} \quad \text{as } t \longrightarrow \infty,\ \mathbb{P}^x - \text{a.s.}$$

Thus ψ_i are non-constant, and it is easy to construct from them a bounded sign-changing L-harmonic function: $\psi = \psi_1 - \psi_2$, for example.

For the regions D_i we will take $D_1 = \{x \in \mathbb{R}^d : x_1 > 0\}$, $D_2 = \{x : x_1 < 0\}$. If we take σ small in a neighbourhood of $\{x_1 = 0\}$ this creates a (partial) barrier to the process $\tilde{X}$ crossing between the regions D_1 and D_2: note that $\tilde{X}$ satisfies the SDE

$$d\tilde{X}_t = \sigma(\tilde{X}_t)^2 d\tilde{B}_t + \sigma(\tilde{X}_t)\nabla\sigma(\tilde{X}_t)\,dt, \tag{1.3}$$

where $\tilde{B}$ is a d-dimensional Brownian motion. If $\sigma(x) \longrightarrow 0$ sufficiently fast as $|x| \longrightarrow \infty$ on the set $\{x : x_1 = 0\}$, then this barrier is strong enough so that $\tilde{X}$ only crosses between the regions D_i a finite number of times, a.s. (More precisely, with probability 1 there are only finitely many n such that $\tilde{X}_t$, crosses between the regions D_i between times n and $n + 1$). The fact that $\tilde{X}$ is transient is of course crucial here. So $\mathbb{P}(G_1 \cup G_2) = 1$, while $G_1 \cap G_2 = \emptyset$, and this, (with symmetry) proves (1.2) for $x = 0$.

THEOREM 3. *(a) Let $d \geq 3$. There exists a smooth strictly positive bounded function σ on $\mathbb{R}^d$ such that $V = -\sigma^{-1}\Delta\sigma$ is bounded, and the equation $\nabla(\sigma^2\nabla\varphi) = 0$ has a bounded sign-changing solution φ.*

(b) If $K = -\Delta - V$, then $Ku = 0$ has a bounded sign changing solution u, and $\lambda_1(K) = 0$.

In Section 2 we collect together some (mainly standard) properties of Bessel processes and related diffusions, and in Section 3 we give the construction of the function σ.

We use c_i to denote fixed positive real constants, whose value only depends on the dimension d, and c, c' etc. to denote positive constants (depending only on d) whose value may change from line to line. We write $x \in \mathbb{R}^d$ as $x = (x_1, x^{(1)})$, where $x^{(1)} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. All the functions on $\mathbb{R}^d$ in this paper will depend on x only through $u = x_1$, $y = |x^{(1)}|$. λ_d denotes d-dimensional Lebesgue measure, and $a \wedge b = \min(a, b)$.

2. Some preliminary estimates.

We begin by collecting some estimates on Bessel processes and related potentials.

LEMMA 2.1. *Let $d \geq 3$ and X be a* Bes(d) *process. Then*

$$\mathbb{P}^x(X_s \leq y \text{ for some } s \geq t) \leq t^{-1/2}y. \tag{2.1}$$

PROOF. Using a comparison theorem for SDEs (see [IW, p. 353]) we can assume that $x = 0$ and $d = 3$. By Pitman's decomposition [P] we can write $X_t = 2M_t - B_t$, where B_t is a one-dimensional Brownian motion with $B_0 = 0$, and $M_t = \sup_{s \leq t} B_s$. Then $\inf_{s \geq t} X_s = M_t$. By the reflection principle $\mathbb{P}(B_t^+ > y) = 2\mathbb{P}(B_t > y) = \mathbb{P}(|B_t| > y)$, so

$$\mathbb{P}^x(X_s \leq y \text{ for some } s \geq t) = \mathbb{P}(|B_t| \leq y) \leq 2yt^{-1/2}(2\pi)^{-1/2} < t^{-1/2}y. \qquad \blacksquare$$

LEMMA 2.2. *Let U_t be a 1-dimensional diffusion with generator $Lf(u) = \frac{1}{2}\big(\sigma^2(u)f'(u)\big)'$, where $\sigma(u) > \varepsilon > 0$. If $0 < x < y$ then*

$$\mathbb{P}^x(U \text{ hits } 0 \text{ before } y) = \frac{\Phi(x)}{\Phi(0)}, \tag{2.2}$$

where $\Phi(x) = \int_x^y \sigma^{-2}(u)\,du$.

PROOF. Writing $\varphi(x) = \mathbb{P}^x(U \text{ hits } 0 \text{ before } y)$, we have that $L\varphi = 0$, so that $\varphi'(x) = -c\sigma^{-2}(x)$. Since $\varphi(0) = 1$, $\varphi(y) = 0$, (2.2) follows. $\blacksquare$

Let G be the usual Green operator on $\mathbb{R}^d$, given by

$$G\mu(x) = \int |x - x'|^{2-d} \mu(dx'),$$

where μ is a measure on $\mathbb{R}^d$. Set

$$J(a, r) = \{x = (x_1, x^{(1)}) : |x_1| \le a, r - a \le |x^{(1)}| \le r + a\}.$$

LEMMA 2.3. *Let ν be Lebesgue measure restricted to $J(a, r)$. Then $G\nu$ is symmetric in x_1, and $x_1(\partial G\nu/\partial x_1) \le 0$. Also $G\nu$ depends on $x^{(1)}$ only through $y = |x^{(1)}|$. If $r \ge \max(4a, a^2)$ then there exist constants c_1, c_2 such that*

$$c_1 a^2 \le G\nu(x) \le c_2 a^2 \quad \text{if } |x| < \frac{1}{2} r, \tag{2.3}$$

$$c_1 a^2 \log r \le G\nu(x) \le c_2 a^2 \log r \quad \text{if } x \in J(a, r), \tag{2.4}$$

$$G\nu(x) \le c_2 a^2 (|x|/r)^{2-d} \quad \text{if } |x| > 2r. \tag{2.5}$$

PROOF. The first two properties of $G\nu$ are clear from the definition and the symmetry of J.

We have $ca^2 r^{d-2} \le \nu\big(J(a, r)\big) \le c' a^2 r^{d-2}$, and $\frac{3}{4} r \le |x| \le \frac{3}{2} r$ for $x \in J(a, r)$. So if $|x'| \le \frac{1}{2} r$, $ca^2 \le G\nu(x') \le c' a^2$, proving (2.3).

Let $x \in J(a, r)$. Then

$$G\nu(x) = \int_J |x - x'|^{2-d} \, dx' \ge \int_{J \cap B(x,2a)^c \cap B(x,r)} |x - x'|^{2-d} \, dx'.$$

If $a < s < r$ then $\lambda_{d-1}\big(\partial B(x, s) \cap J\big) \ge ca^2 s^{d-3}$, so that

$$G\nu(x) \ge \int_a^r ca^2 s^{-1} \, ds = ca^2 \log(r/a).$$

Also, if $r \ge a^2$ then $\log(r/a) \ge \log r^{1/2} = \frac{1}{2} \log r$. A similar calculation proves the other bound in (2.4).

For (2.5), since $|x - x'| \ge \frac{1}{2}|x|$ for $x' \in J$, and $|x| > 2r$, we have

$$G\nu(x) \ge ca^2 r^{d-2} \left(\frac{1}{2}|x|\right)^{2-d} \ge c' a^2 (|x|/r)^{2-d}. \qquad \blacksquare$$

Now set $n_k = e^{2^k}$, $a_k = 2^{k+1}$, and let $J_k = J(a_k, n_k)$ for $k \ge 0$. Set $A = \bigcup_{k=3}^\infty J_k$.

PROPOSITION 2.4. *There exists $\varphi > 0$ on $\mathbb{R}^d$ with the following properties.*

(a) φ is superharmonic, and $\Delta\varphi = 0$ on A^c.

(b) $\varphi \ge 1$ on A.

(c) $x_1 \partial\varphi/\partial x_1 > 0$.

(d) φ depends on x only through $u = x_1$, $y = |x^{(1)}|$.

(e) If $\gamma(t)$ is any path in $\mathbb{R}^d$ such that $\limsup_{t\to\infty}|\gamma(t)| = \infty$ *then* $\liminf_{t\to\infty}\varphi\big(\gamma(t)\big) = 0$.

PROOF. Let ν_k be Lebesgue measure restricted to J_k, and

$$\varphi_k = c_1^{-1}a_k^{-2}(\log n_k)^{-1}G\nu_k.$$

By Lemma 2.3 we have $\varphi_k \geq 1$ on J_k, and $\varphi_k(x) \leq c(\log n_k)^{-1} = c2^{-k}$, provided $|x| \leq \frac{1}{2}n_k$. Set

$$\varphi(x) = \sum_{k=3}^{\infty}\varphi_k(x).$$

Clearly $0 < \varphi(x) < \infty$ for all x. Since each φ_k is superharmonic, and harmonic on J_k^c, φ clearly satisfies (a) and (b). (c) and (d) follow from the corresponding property for $G\nu_k$.

To prove (e), let $x_k \in \mathbb{R}^d$ be such that $|x_k| = \frac{1}{2}n_{k+1}$. Then by Lemma 2.3, if $i \leq k$,

$$\varphi_i(x_k) \leq c(\log n_i)^{-1}(|x_k|/n_i)^{2-d} \leq c(2n_k/n_{k+1}) = c'e^{-2^k},$$

while $\varphi_i(x_k) \leq c2^{-k}$ if $i \geq k+1$. So,

$$\varphi(x_k) \leq cke^{-2^k} + c'2^{-k}.$$

Since $|\gamma(t)| = \frac{1}{2}n_{k+1}$ for infinitely many t, it follows that

$$\liminf_{t\to\infty}\varphi\big(\gamma(t)\big) \leq \liminf_{k\to\infty}\varphi(x_k) = 0. \qquad \blacksquare$$

Let X_t, $t \geq 0$ be a process in $\mathbb{R}^d$. We define the event

$$\{X \text{ ultimately avoids } A\} = \bigcup_{n=0}^{\infty}\{X_t \notin A \text{ for all } t \geq n\}.$$

COROLLARY 2.5. *Let B be a Brownian motion in $\mathbb{R}^d$. Then $\mathbb{P}^x$(B ultimately avoids A) $= 1$.*

PROOF. $\varphi(B_t)$ is a positive supermartingale, and so converges a.s. Using Proposition 2.4(e) we see that $\lim_{t\to\infty}\varphi(B_t) = 0$ a.s. Since $\varphi(x) \geq 1$ on A, it follows that B ultimately avoids A, a.s. $\blacksquare$

3. **The counterexample.** Let $\sigma > 0$, f be functions on $\mathbb{R}^d$ which depend on x only through u and y. Then if $L[\sigma] = \nabla(\sigma^2\nabla)$,

$$\frac{1}{2}L[\sigma]f = \frac{1}{2}\sigma^2(f_{uu} + f_{yy}) + \sigma\sigma_u f_u + \left(\sigma\sigma_y + \sigma^2\frac{d-2}{2y}\right)f_y. \tag{3.1}$$

We will restrict our attention to operators on $\mathbb{R}^d$ of this form. Recall the definitions of n_k, J_k, A from Section 2. For $k \geq 1$ let

$$\bar{\sigma}_k(u) = 1 \wedge n_k^{-1}e^{|u|}.$$

Let $\bar{\sigma}(u, y)$ be given by

(3.2) $$\bar{\sigma}(u,y) = \bar{\sigma}_k(u),\ n_{k-1} + 2^{k-1} \le y \le n_k, \quad k \ge 4$$

(3.3) $$\bar{\sigma}(u,y) = 1 \wedge \exp\left(-2^{k-1} + |u| - (y - n_{k-1})\right),\ n_{k-1} \le y \le n_{k-1} + 2^{k-1}, \quad k \ge 4,$$

(3.4) $$\bar{\sigma}(u,y) = \bar{\sigma}_3(u), \quad 0 \le y \le n_3.$$

Let ψ be a symmetric C^∞ function supported on $(-\frac{1}{2}, \frac{1}{2})$, and set

$$\sigma_k(u) = \int \psi(u-u')\bar{\sigma}(u')\,du', \quad \sigma(u,y) = \iint \psi(u-u')\psi(y-y')\sigma(u',y')\,du'\,dy'.$$

It is straightforward to verify

LEMMA 3.1. *σ_k and σ are bounded smooth strictly positive functions on $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}_+$ which satisfy:*

(3.5) $$\bar{\sigma}_k(u) = \bar{\sigma}(-u), \quad \sigma(u,y) = \sigma(-u,y),$$

(3.6 $$|\Delta\sigma| \le c_3\sigma,$$

(3.7) $$u\sigma_u \ge 0, \quad \sigma_y = 0 \text{ on } A^c,$$

(3.8) $$\sigma(u,y) = \bar{\sigma}_k(u) \quad \text{if } n_{k-1} + 2^k \le y \le n_k - 2^{k+1}$$

(3.9) $$\int_{2^k-1}^{2^k} \sigma_k^{-2}(u)\,du \le c_4, \quad \int_0^1 \sigma_k^{-2}(u)\,du \ge c_5 n_k^2.$$

Now let L_1 be the operator given by

(3. 10) $$L_1 f = \frac{1}{2}\sigma^2(f_{uu} + f_{yy}) + \sigma\sigma_u f_u + \left(\sigma\sigma_y + \sigma^2\frac{d-2}{2y}\right)f_y,$$

and set $L_2 = \sigma^{-2}L_1$. Let $Z_t = \left((U_t, Y_t), t \ge 0, \mathbb{P}^z, z \in \mathbb{R} \times \mathbb{R}_+\right)$ be the diffusion associated with L_2. Then Z is (the unique) solution to the SDE

$$dU_t = dB_t + \left(\frac{\sigma_u(Z_t)}{\sigma(Z_t)}\right)dt,$$

(3.11) $$dY_t = dB'_t + \left(\frac{\sigma_y(Z_t)}{\sigma(Z_t)} + \frac{d-2}{2Y_t}\right)dt,$$

where B, B' are independent one-dimensional Brownian motions. Write $g(u, y) = \sigma_u(u, y)/\sigma(u, y)$: by (3.7) $g \ge 0$. Set $V_t = U_t^2$: then by Itô's formula

(3. 12) $$\begin{aligned} dV_t &= 2V_t^{1/2}\operatorname{sgn}(U_t)\,dB_t + \left(1 + 2V_t^{1/2}g(V_t^{1/2})\right)dt \\ &= 2V_t^{1/2}\,d\bar{B}_t + \left(1 + 2V_t^{1/2}g(V_t^{1/2})\right)dt. \end{aligned}$$

Here

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0, \end{cases}$$

and $\bar{B}_t$, given by

$$\bar{B}_t = \int_0^t \operatorname{sgn}(U_t)\, dB_t,$$

is another one-dimensional Brownian motion—see [RW, p. 63]. Let $\bar{V}$ be the solution to

$$d\bar{V}_t = 2\bar{V}_t^{1/2} d\bar{B}_t + dt, \quad \bar{V}_0 = V_0. \tag{3.13}$$

By a comparison theorem for SDEs (see [IW, p. 353]) it follows that $\bar{V}_t \le V_t = U_t^2$ for all $t \ge 0$. However, (3.13) implies that $\bar{V}^{1/2}$ is a Bes(1) process, and so equal in law to the absolute value of a Brownian motion. (See [RW, p. 69]).

Set

$$T_A = \inf\{t \ge 0 : (U_t, Y_t) \in A\},$$
$$\bar{T}_A = \inf\{t \ge 0 : (\bar{V}_t^{1/2}, Y_t) \in A\}.$$

We have $\bar{T}_A \le T_A$. From (3.7) and (3.11) we deduce that if $\bar{Y}$ is the solution to

$$d\bar{Y}_t = dB'_t + \frac{d-2}{2\bar{Y}_t}\, dt, \quad \bar{Y}_0 = Y_0, \tag{3.14}$$

then $\bar{Y}$ is a Bes$(d-1)$ process, and $\bar{Y}_t = Y_t$ for $0 \le t \le T_A$. Let also $\bar{Z}_t = (U_t, \bar{Y}_t)$, and $\bar{R}_t = \left(\bar{V}_t + \bar{Y}_t^2\right)^{1/2}$: then $\bar{R}$ is a Bes(d) process, and $|\bar{Z}_t| \ge \bar{R}_t$.

Now set

$$H_k(t) = \{(u, y) : n_{k-1} + 2^k \le y \le n_k - 2^{k+1}, |u| = t\}, \quad k \ge 4,$$
$$I_k = [-2^k, 2^k] \times [n_{k-1} + 2^k, n_k - 2^{k+1}], \quad k \ge 4,$$
$$H_3(t) = \{(u, y) : 0 \le y \le n_3, |u| = t\}.$$

Fix $k \ge 4$ and define stopping times S_i, T_i by

$$T_0 = 0,$$
$$S_n = \inf\{t \ge T_{n-1} : Z_t \in H_k(2^k - 1)\},$$
$$T_n = \inf\{t \ge S_n : Z_t \in H_k(0) \cup H_k(2^k) \cup A\}.$$

Note that $Z_t \in I_k$ for $S_n \le t \le T_n$, and that if Z hits $H_k(0)$ and $T_A = \infty$ then $Z_{T_n} \in H_k(0)$ for some n.

LEMMA 3.2. *On* $\{S_n < \infty\}$,

$$\mathbb{P}^z(Z_{T_n} \in H_k(0), T_n < T_A \mid \mathcal{F}_{S_n}) \le c n_k^{-2}. \tag{3.15}$$

PROOF. Using the Markov property of Z, we can assume $n = 1$ and $S_1 = 0$, $Z_0 = (u_0, y_0) \in H_k(2^k - 1)$. On $0 \le t \le T_1$ we therefore have that U satisfies the SDE

$$U_t = u_0 + B_t + \int_0^t g_k(U_s)\, ds, \tag{3.16}$$

where $g_k = \sigma_k^{-1}\partial\sigma_k/\partial u$. If U' is the solution to (3.16) for $0 \le t < \infty$, then $U = U'$ on $[0, T_1]$. Set $T' = \inf\{t : U'_t \in \{0, 2^k\}\}$. So

$$\begin{aligned}\mathbb{P}(U_{T_1} = 0, T_1 < T_A) &= \mathbb{P}(U'_{T_1} = 0, T_1 < T_A)\\ &\le \mathbb{P}(U_{T'} = 0)\\ &\le \int_{2^k-1}^{2^k} \sigma_k^{-2}(u)\,du \Big/ \int_0^{2^k} \sigma_k^{-2}(u)\,du \le c_6 n_k^{-2}.\end{aligned}$$

Here we used Lemma 2.2 and the estimate (3.9) in the last line. ∎

Now set

$$t_k = 4^k n_k^2, \quad m_k = k t_k^{1/2} = k2^k n_k.$$

LEMMA 3.3. *On* $\{T_{n-1} < \infty\} \cap \{T_{n-1} < T_A\} \cap \{|U_{T_{n-1}}| \ge 2^k\}$

$$\mathbb{P}^z(S_n - T_{n-1} > t_k \mid F_{T_{n-1}}) \ge c_7 t_k^{-1/2}.$$

PROOF. As in the previous proof, it is enough to obtain the estimate for $S_1 - T_0$ in the case when $Z_0 = (u_0, y_0) \in H_k(2^k)$. Using the comparison between U_t and $\overline{V}_t^{1/2}$ we have

$$\mathbb{P}(S_1 - T_0 > t_k) \ge \mathbb{P}\big(T_{-1}(\beta) > t_k\big),$$

where β is a one-dimensional Brownian motion started at 0, and $T_{-1}(\beta) = \inf\{s : \beta_s = -1\}$. However using the reflection principle as in Lemma 2.1,

$$\mathbb{P}\big(T_{-1}(\beta) > t\big) = \mathbb{P}(|B_t| < 1) \sim ct^{-1/2}, \quad \text{as } t \to \infty.$$ ∎

Set

$$\begin{aligned}N_k &= \max\{n : S_n < \infty\},\\ G &= \{U_{T_n} = 0 \text{ for some } n \le m_k \wedge N_k\},\\ \eta &= \max_{1\le n\le N_k\wedge m_k}(S_n - T_{n-1}),\end{aligned}$$

Then if $z \notin I_k$ and $k \ge 4$,

$$\begin{aligned}&\mathbb{P}^z\big(Z \text{ hits } H_k(0), T_A = \infty\big)\\ &\quad= \mathbb{P}^z\big(Z \text{ hits } H_k(0), G, T_A = \infty\big) + \mathbb{P}^z\big(Z \text{ hits } H_k(0), G^c, T_A = \infty\big).\\ &\quad\le \mathbb{P}^z(G, T_A = \infty) + \mathbb{P}^z(N_k > m_k, G^c, T_A = \infty)\end{aligned}\tag{3.17}$$

By Lemma 3.2 the first term in (3.17) is bounded by $c_2 m_k n_k^{-2}$. If $T_A = \infty$, then $Z = \overline{Z}$, and so $|Z_t| \ge \overline{R}_t$ for all t. We have

$$\begin{aligned}\mathbb{P}(N_k > m_k, G^c, T_A = \infty) &= \mathbb{P}(N_k > m_k, |U_{T_n}| = 2^k \text{ for } 1 \le n \le m_k, \eta < t_k, T_A = \infty)\\ &\quad+ \mathbb{P}(N_k > m_k, G^c, \eta \ge t_k, T_A = \infty).\end{aligned}\tag{3.18}$$

The first term in (3.18) is bounded by

$$\mathbb{P}(N_k > m_k, S_n - T_{n-1} < t_k \text{ for } 1 \le n \le m_k, G^c, T_A = \infty) \le \left(1 - c_7 t_k^{-1/2}\right)^{m_k}, \tag{3.19}$$

by Lemma 3.3. If $N_k > m_k$ and $\eta \ge t_k$ then $Z_{t_0} \in H_k(2^k - 1)$ for some $t_0 > t_k$. Since $|Z_{t_0}|^2 \le (2^k - 1)^2 + n_k^2 \le 4n_k^2$, we deduce from (2.1) that

$$\mathbb{P}^z(N_k > m_k, G^c, \eta \ge t_k, T_A = \infty) \le \mathbb{P}^z(\overline{R_t} < 2n_k \text{ for some } t \ge t_k) \le 2t_k^{-1/2} n_k.$$

Collecting these estimates together, we have

$$\begin{aligned}\mathbb{P}^z(Z \text{ hits } H_k(0), T_A = \infty) &\le c m_k n_k^{-2} + (1 - c_7 t_k^{-1/2})^{m_k} + 2t_k^{-1/2} n_k \\ &\le ck 2^k n_k^{-1} + e^{-c_7 k} + 2^{1-k} = \varepsilon_k,\end{aligned} \tag{3.20}$$

where $\sum_{k=2}^{\infty} \varepsilon_k < \infty$.

LEMMA 3.4. *(a) Z ultimately avoids A, a.s.*

(b) Z is transient.

(c) For any $z \in \mathbb{R} \times \mathbb{R}_+$,

$$\mathbb{P}^z(Z \text{ hits } H_k(0) \text{ for infinitely many } k, T_A = \infty) = 0.$$

PROOF. (a) From the properties of the function φ in Proposition 2.4, we see that if $\overline{\varphi}(u, y)$ is the function such that $\varphi(x) = \overline{\varphi}\big(u(x), y(x)\big)$, then $u\overline{\varphi}_u \ge 0$. Since on A^c $\overline{\varphi}$ satisfies

$$\frac{1}{2}(\overline{\varphi}_{uu} + \overline{\varphi}_{yy}) + \frac{d-2}{2y}\overline{\varphi}_y = 0,$$

we have on A^c

$$L_2 \overline{\varphi} = \sigma^{-1} \sigma_u \varphi_u \le 0.$$

So $1 \wedge \overline{\varphi}(Z_t)$ is a supermartingale, and so converges a.s. to some limit. But since $|Z_t| \ge |U_t| \ge \overline{V}_t^{1/2}$, and $\limsup_{t \to \infty} \overline{V}_t^{1/2} = \infty$, by Proposition 2.4(e) we have that the limit must be 0. Thus, as in Corollary 2.5, Z ultimately avoids A.

(b) This is immediate from (a).

(c) Since z is in at most one of the sets I_k, this is immediate from the estimate (3.20) and the Borel-Cantelli lemma. ∎

THEOREM 3.5. *Z ultimately avoids $\{u = 0\}$, $\mathbb{P}^z$-a.s.*

PROOF. Since $\mathbb{P}^z(Z \text{ ultimately avoids } A) = 1$, we have

$$0 = \lim_{n \to \infty} \mathbb{P}^z(Z_t \in A, \text{ for some } t \ge n) = \lim_{n \to \infty} \mathbb{E}^z\big(\mathbb{P}^{Z_n}(T_A < \infty)\big). \tag{3.21}$$

Note that $\{u = 0\} \subseteq \Gamma = A \cup \bigcup_{k=3}^{\infty} H_k(0)$. Set $F_n = \{Z_t \in \Gamma \text{ for some } t \ge n\}$, $F = \bigcap_{n=0}^{\infty} F_n$. Then

$$\mathbb{P}^z(F) = \mathbb{P}^z(F \cap \{T_A < \infty\}) + \mathbb{P}^z(F \cap \{T_A = \infty\}).$$

If F occurs then either Z hits infinitely many of the $H_k(0)$, or Z hits one of the components of Γ after time n for infinitely many n. But as Z is transient the second possibility has probability 0. So

$$\mathbb{P}^z(F \cap \{T_A = \infty\}) = \mathbb{P}^z\big(Z \text{ hits } H_k(0) \text{ for infinitely many } k, T_A = \infty\big) = 0$$

by Lemma 3.4(c).

So,

$$\mathbb{P}^z(F) = \mathbb{P}^z(F \cap \{T_A < \infty\}) \quad \text{for } z \in \mathbb{R} \times \mathbb{R}_+.$$

But

$$\mathbb{P}^z(F) = \mathbb{E}^z\big(\mathbb{P}^{Z_n}(F)\big) = \mathbb{E}^z\big(\mathbb{P}^{Z_n}(F \cap \{T_A < \infty\})\big) \le \mathbb{E}^z\mathbb{P}^{Z_n}(T_A < \infty),$$

which converges to 0 as $n \longrightarrow \infty$ by (3.21). So $\mathbb{P}^z(F) = 0$. ∎

By Theorem 3.5 we see that if $D_1 = \{u > 0\}$, $D_2 = \{u < 0\}$ and $G_i = \{Z_t \in D_i$ for all sufficiently large $t\}$, then $G_1 \cap G_2 = \emptyset$, while $\mathbb{P}^z(G_1 \cup G_2) = 1$. By symmetry $\mathbb{P}^0(G_i) = \frac{1}{2}$. Set $\psi_i(z) = \mathbb{P}^z(G_i)$. We have $\psi_1 + \psi_2 = 1$, $0 < \psi_i < 1$ and since $\psi_i(Z_t)$ is a martingale, by the martingale convergence theorem $\psi_i(Z_t) \longrightarrow I_{G_i}$ a.s., which shows that ψ_i are non-constant. So $\psi = \psi_1 - \psi_2$ is a sign-changing function which is harmonic with respect to the operator L_2. Hence $L_1\psi = \sigma^2 L_2 \psi = 0$. We have proved:

COROLLARY 3.5. *The equation $L_1\psi = 0$ has a bounded sign-changing solution.*

PROOF OF THEOREM 3. Recall the notation $x = (x_1, x^{(1)})$, $u = x_1$, $y = |x^{(1)}|$. Let σ, ψ be as above, and define $\tilde\sigma(x) = \sigma\big(u(x), y(x)\big)$, $\tilde\psi(x) = \psi\big(u(x), y(x)\big)$. Then $\tilde\sigma$ and $\tilde\sigma^{-1}\Delta\tilde\sigma$ are bounded, and

$$L[\tilde\sigma]\tilde\psi = 2L_1\psi = 0,$$

so that $\tilde\psi$ is a bounded sign-changing solution of $\nabla\big(\tilde\sigma^2\nabla\tilde\psi\big) = 0$. The final assertion in Theorem 3 is now immediate from Theorem 2. ∎

REFERENCES

[Bas] R. F. Bass, *Diffusions and Elliptic Operators*. Springer, New York, 1997.

[BCN] H. Berestycki, L. Cafarelli and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*. preprint.

[GG] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*. Math. Ann., to appear.

[IW] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion processes*. North-Holland, Kodansha, 1981.

[P] J. Pitman, *One dimensional Brownian motion and the three-dimensional Bessel process.* J. Appl. Probab. **7**(1975), 511–526.

[RW] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales, Volume I*. (2nd edition), Wiley, 1994.

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