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STABILITY OF PARABOLIC HARNACK INEQUALITIES

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ABSTRACT. Let (G, E) be a graph with weights $\{a_{xy}\}$ for which a parabolic Harnack inequality holds with space-time scaling exponent $\beta \geq 2$. Suppose $\{a'_{xy}\}$ is another set of weights that are comparable to $\{a_{xy}\}$. We prove that this parabolic Harnack inequality also holds for (G, E) with the weights $\{a'_{xy}\}$. We also give stable necessary and sufficient conditions for this parabolic Harnack inequality to hold.

1. INTRODUCTION

Consider the elliptic operator in divergence form

$$\mathcal{A}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}\Big)(x) \tag{1.1}$$

acting on functions on $\mathbb{R}^d$, where $a = (a_{ij}(x))$ is bounded, measurable, and uniformly elliptic. A celebrated theorem of Moser [M1] states that an elliptic Harnack inequality (EHI) holds for nonnegative functions h that are harmonic with respect to the operator $\mathcal{A}$. This was extended a few years later in [M2], where Moser proved a parabolic Harnack inequality (PHI) for nonnegative solutions to the heat equation associated with $\mathcal{A}$:

$$\frac{\partial u}{\partial t} = \mathcal{A}u. \tag{1.2}$$

(See also [M3].) These theorems have had a profound influence on linear and nonlinear PDE and differential geometry. To mention just one important result, the EHI lies behind Aronson's proof [A] of Gaussian type bounds for the fundamental solution to (1.2).

We can view Moser's theorems as stability theorems for elliptic and parabolic Harnack inequalities. It is well known that associated to the operator $\mathcal{A}$ is the Dirichlet form

$$\mathcal{E}_{\mathcal{A}}(f,f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x)\,dx.$$

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Of course, the Dirichlet form associated with the Laplacian is

$$\mathcal{E}_\Delta(f,f) = \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx,$$

and the uniform ellipticity of $\mathcal{A}$ implies that the Dirichlet forms $\mathcal{E}_\Delta$ and $\mathcal{E}_\mathcal{A}$ are comparable in the sense that there exists $c_1 > 1$ such that

$$c_1^{-1}\mathcal{E}_\Delta(f,f) \le \mathcal{E}_\mathcal{A}(f,f) \le c_1 \mathcal{E}_\Delta(f,f), \qquad f \in \mathbb{C}_0^1(\mathbb{R}^d). \tag{1.3}$$

Then we can rephrase Moser's theorems as:

> Suppose the state space is $\mathbb{R}^d$. Whenever one has a Dirichlet form that is comparable to that of the Laplacian, then the EHI and PHI hold for the corresponding elliptic operator.

Moser's methods, and an alternate approach due to Fabes-Stroock [FS], building on work of Nash [N] and Davies [Da], have been used successfully in a variety of contexts: certain manifolds, domains in $\mathbb{R}^d$, graphs, and metric spaces. In particular, [Gr] and [SC] gave stable necessary and sufficient conditions for PHI to hold, by proving that the PHI with space-time scaling exponent 2, denoted PHI(2), holds on a manifold M if and only if M satisfies a volume doubling condition and a Poincaré inequality with scaling exponent 2. See [D1] and [St] for extensions of this to certain graphs and metric spaces.

No equivalent characterization for the EHI is known, and it was for some time an open problem whether the EHI and PHI(2) were equivalent. The (negative) answer was actually implicit in a number of papers on diffusions on fractals, before being made explicit in [BB1]. To explain the difference between these two Harnack inequalities further, consider the Markov process associated with the operator $\mathcal{A}$. PHI(2) implies that X satisfies the standard space-time scaling relation:

$$\mathbb{E}^x|X_t - x|^2 \approx t.$$

In the early 1980s mathematical physicists discovered "fractal" lattices for which *anomalous diffusion* occurs:

$$\mathbb{E}^x d(X_n, x)^2 \approx n^{2/\beta}, \quad n \ge 1,$$

for some $\beta > 2$. This property does not destroy the EHI, since the space-time scaling does not arise in the Laplace equation $\mathcal{A}h = 0$, but it does affect the PHI, where the space-time scaling occurs explicitly. Thus these spaces do not satisfy PHI(2), but in many cases they do satisfy the EHI; see, for example, the graphs studied in [Jo] and [BB2]. The proofs of the EHI in these contexts generally employ some special features of the state space such as finite ramification or very strong symmetry conditions and do not give stability of the EHI.

The techniques of [Gr] and [SC] do not apply to these spaces. The chief obstacle is that the Moser argument needs the existence of sufficiently many cut-off functions φ, with approximately minimal energy, such that the L^2 and L^∞ norms of $\nabla\varphi$, suitably normalized, are comparable. Functions of this kind exist only if $\beta = 2$.

Thus there exists a large family of spaces (see for example [B1] and [BCG]), corresponding to $\beta \in (2,\infty)$, for which EHI holds but PHI(2) fails. A parabolic Harnack inequality does hold for these spaces, but the space-time scaling exponent will be β rather than 2; we denote this by PHI(β). In this paper we prove stability results for PHI(β) (Theorem 1.1) and give stable necessary and sufficient conditions for PHI(β) to hold (Theorem 1.5). These conditions include the obvious extensions of the $\beta = 2$ case, that is, volume doubling and a rescaled Poincaré inequality, PI(β), but also a new inequality CS(β), which implies the existence of enough 'low energy' cut-off functions on the space.

The general question of the stability of EHI remains open. An example in [D2] shows that EHI can hold without a volume doubling property and also that, essentially, there exist spaces satisfying EHI with different β in different regions. For other recent work on the relation between the EHI and PHI see [HSC].

We now introduce some notation and terminology to describe our results. It is generally the case that techniques for estimating heat kernels are quite robust and can be adapted to a variety of different spaces: graphs, manifolds, subsets of $\mathbb{R}^d$, fractals, and general metric spaces. We have chosen to work on graphs, since this provides the simplest context to employ our methods. However, we expect that our methods will transfer with only fairly minor changes to these other spaces.

We use the letter c with subscripts to denote finite positive constants which depend only on the graph and whose exact value is unimportant. We use the notation $A \approx B$ to mean $c_1 A \le B \le c_2 A$, where c_1 and c_2 are as above.

Let (G, E) be an infinite connected graph; G is the set of vertices and E is the set of edges. We suppose throughout that each vertex belongs to at most finitely many edges. We write $x \sim y$ to mean that $\{x, y\}$ is an edge. We call $a = (a_{xy})$, $x, y \in G$, a *conductance matrix* if $a_{xy} \ge 0$, $a_{xy} = a_{yx}$ for all $x, y \in G$ and $a_{xy} = 0$ if $\{x, y\}$ is not an edge in E. We call (G, E, a) a *weighted graph*.

We set $\mu_0(\{x\}) = \sum_y a_{xy}$ and we extend μ_0 to a measure on G by setting $\mu_0(A) = \sum_{x\in A} \mu_0(\{x\})$. Let $d(x, y)$ be the usual graph distance on G, and for $x \in G$, $r \in [0, \infty)$, set

$$B(x,r) = \{y : d(x,y) < r\}, \qquad V_0(x,r) = \mu_0(B(x,r)).$$

Given a ball $B = B(x, r)$, we write $B^* = B(x, 2r)$.

We will introduce a number of conditions that a graph may or may not satisfy. For the convenience of the reader we give a table which summarizes what the abbreviations mean and where the definitions may be found.

(VD)	Volume doubling	(1.4)
(p_0)	Lower bound on 1-step transition probabilities	(1.5)
(E_β)	Walk dimension β	(1.6)
(EHI)	Elliptic Harnack inequality	(1.8)
(PHI(β))	Parabolic Harnack inequality with exponent β	(1.9)
(R_β)	Resistance exponent β	(1.11)
$(PI(\beta))$	Poincaré inequality with exponent β	(1.14)
(CS(β))	Cut-off Sobolev inequality	(1.15)
(FVG)	Fast volume growth	(2.2)

The weighted graph (G, E, a) satisfies the *volume doubling* condition (VD) if there exists $c_1 > 1$ such that

$$V_0(x, 2R) \le c_1 V_0(x, R) \qquad \text{for all } x \in G,\ R \ge 1. \tag{1.4}$$

The simple random walk $X = (X_n, n \ge 0, \mathbb{P}^x, x \in G)$ on (G, E, a) is the μ_0-symmetric G-valued Markov chain with transition probabilities given by

$$p_{xy} = \mathbb{P}^{\cdot}(X_{n+1} = y | X_n = x) = \frac{a_{xy}}{\mu_0(\{x\})}, \qquad x, y \in G,\ n \ge 0.$$

We need a regularity condition which connects the p_{xy} with the graph structure. We say that (G,E,a) satisfies the (p_0) *condition* if there exists $p_0>0$ such that

$$p_{xy}=\frac{a_{xy}}{\mu_0(\{x\})}\geq p_0 \quad \text{whenever } \{x,y\} \text{ is an edge in } E. \tag{1.5}$$

Note that this implies that vertices have degree at most p_0^{-1}.

The *heat kernel* on (G,E,a) is the density of X_n with respect to the measure μ_0:

$$p_n(x,y)=\frac{\mathbb{P}^x(X_n=y)}{\mu_0(\{y\})},$$

and is easily seen to be symmetric: $p_n(x,y)=p_n(y,x)$. Here we use the Markov theory notation $\mathbb{P}^x(\cdot)=\mathbb{P}(\cdot\mid X_0=x)$ and denote the corresponding expectation operator by $\mathbb{E}^x$. We say that (G,E,a) has *walk dimension* β and that (G,E,a) satisfies (E_β) if for some constant $c_1\geq 1$,

$$c_1^{-1}r^\beta\leq \mathbb{E}^x\tau^X_{B(x,r)}\leq c_1 r^\beta, \qquad r\in[1,\infty),\ x\in G, \tag{1.6}$$

where $\tau^X_{B(x,r)}=\min\{n: X_n\notin B(x,r)\}$. The term walk dimension comes from the mathematical physics literature; cf. [BB1], Proposition 8.3.

We next define (EHI), the *elliptic Harnack inequality.* The graph Laplacian $\mathcal{L}_G$ is defined by

$$\mathcal{L}_G f(x)=\frac{1}{\mu_0(\{x\})}\sum_y a_{xy}(f(y)-f(x)), \qquad x\in G. \tag{1.7}$$

Note that $\mathcal{L}_G$ depends on the conductance matrix a. A function $h:\overline{A}\to\mathbb{R}$ is *harmonic on* $A\subset G$ if

$$\mathcal{L}_G h(x)=0, \qquad x\in A,$$

where $\overline{A}=\{y\in G: d(x,y)\leq 1 \text{ for some } x\in A\}$. (G,E,a) satisfies an *elliptic Harnack inequality* (EHI) if there exists $c_1>0$ such that: whenever $x\in G$, $R\geq 1$, and $h:G\to\mathbb{R}$ is nonnegative and harmonic in $B(x,2R)$,

$$\sup_{B(x,R)} h\leq c_1 \inf_{B(x,R)} h. \tag{1.8}$$

We have taken balls $B(x,R)\subset B(x,2R)$ just for simplicity: if $K>1$ and (1.8) holds whenever $h\geq 0$ is harmonic in $B(x,KR)$, then an easy chaining argument gives (EHI) (for a different constant c_1). We remark that under the (p_0) condition any graph satisfies a local Harnack inequality; see Lemma 2.8. The point of the condition (EHI) is that the constant c_1 is independent of x and R.

Let $\beta\geq 2$. (G,E,a) satisfies PHI(β), a *parabolic Harnack inequality of order* β, if whenever $u(n,x)\geq 0$ is defined on $[0,4N]\times\overline{B}(y,2R)$ and satisfies

$$u(n+1,x)-u(n,x)=\mathcal{L}_G u(n,x), \qquad (n,x)\in[0,4N)\times B(y,2R),$$

then

$$\max_{\substack{N\leq n\leq 2N\\ x\in B(y,R)}} u(n,x)\leq c_1 \min_{\substack{3N\leq n\leq 4N\\ x\in B(y,R)}} (u(n,x)+u(n+1,x)) \tag{1.9}$$

where $N\geq 2R$ and $N\approx R^\beta$. The usual parabolic Harnack inequality is the case $\beta=2$; this extension was introduced in [BB1].

Suppose that a and a' are conductance matrices on (G,E). We say a and a' are *equivalent* if there exists a constant c_1 such that

$$c_1^{-1}a_{xy}\leq a'_{xy}\leq c_1 a_{xy}, \qquad \{x,y\}\in E.$$

We call a property *stable* if whenever it holds for (G, E, a) and a and a' are equivalent, then it holds for (G, E, a'). Our first main result is that (PHI(β)) is stable.

Theorem 1.1. *Suppose that (G, E) is an infinite connected graph and that a and a' are equivalent conductance matrices on (G, E) satisfying the (p_0) condition. Suppose that the weighted graph (G, E, a) satisfies $PHI(\beta)$. Then (G, E, a') also satisfies $PHI(\beta)$.*

If A and B are two disjoint subsets of G, define the effective resistance between A and B by

$$[R_{\text{eff}}(A, B)]^{-1} = \inf\Big\{\sum_{x\sim y} |f(x) - f(y)|^2 a_{xy} : f = 1 \text{ on } A, f = 0 \text{ on } B\Big\}. \tag{1.10}$$

We say G satisfies (R_β) and has *resistance exponent* β if

$$R_{\text{eff}}(B(x_0, r), B(x_0, 2r)^c) \approx \frac{r^\beta}{V_0(x_0, r)}, \qquad r \geq 1. \tag{1.11}$$

Note that the property (R_β) is stable.

Many properties of random walks on graphs satisfying PHI(β) are already quite well known; in particular one has good estimates on the transition probabilities $p_n(x, y)$. The following is the main theorem of [GT2].

Theorem 1.2 ([GT2, Theorem 3.1])**.** *Let (G, E) be an infinite connected graph with conductances $a = (a_{xy})$ satisfying the (p_0) condition. The following are equivalent:*

(a) (G, E, a) satisfies $PHI(\beta)$.

(b) (G, E, a) satisfies (VD), (E_β), and (EHI).

(c) For $x, y \in G$, $n \geq d(x, y)$, the transition density of X on (G, E, a) satisfies

$$p_n(x, y) \leq \frac{c_1}{V_0(x, n^{1/\beta})} \exp\left[-\Big(\frac{d(x, y)^\beta}{c_1 n}\Big)^{1/(\beta-1)}\right], \tag{1.12}$$

$$p_n(x, y) + p_{n+1}(x, y) \geq \frac{c_2}{V_0(x, n^{1/\beta})} \exp\left[-\Big(\frac{d(x, y)^\beta}{c_2 n}\Big)^{1/(\beta-1)}\right]. \tag{1.13}$$

(d) (G, E, a) satisfies (VD), (EHI) and (R_β).

Remarks 1.3. 1. We always have $\beta \geq 2$; see, for example, [B1], Lemma 1.1 – the proof extends easily to the volume doubling case.

2. Explicit examples of graphs satisfying these conditions, for various $\beta \geq 2$, are given in [B1], [BB2], and [Jo].

3. Note that $p_n(x, y) = 0$ when $d(x, y) > n$. The purpose of adding p_n and p_{n+1} in (1.13) is to avoid the parity problems which arise if G is bipartite or close to bipartite.

We prove Theorem 1.1 by finding stable necessary and sufficient conditions for PHI(β) to hold for a graph which satisfies (p_0).

(G, E, a) satisfies $(PI(\beta))$, a scaled *Poincaré inequality* with parameter $\beta \geq 2$, if there exists a constant c_1 such that for any ball $B = B(x, R) \subset G$ with $R \geq 1$, and $f : B \to \mathbb{R}$,

$$\sum_{x\in B} (f(x) - \overline{f}_B)^2 \mu_0(x) \leq c_1 R^\beta \sum_{x\in B} \sum_{y\in B} a_{xy}(f(x) - f(y))^2. \tag{1.14}$$

Here $\overline{f}_B = \mu(B)^{-1} \sum_B f(x)\mu_0(x)$. This is a generalization to the anomalous diffusion case of the standard Poincaré inequality.

The following definition is new.

Definition 1.4. CS(β), the *cut-off Sobolev inequality with exponents β and θ.*

Let $\beta \geq 2$ and $\theta \in (0,1]$. We say (G, E, a) satisfies CS(β) if there exist constants c_1 and c_2 such that for every $x_0 \in G$, $R \geq 1$, there exists a cut-off function $\varphi (= \varphi_{x_0,R})$ satisfying properties (a)–(d) below.

(a) $\varphi(x) \geq 1$ for $x \in B(x_0, R/2)$.

(b) $\varphi(x) = 0$ for $x \in B(x_0, R)^c$.

(c) $|\varphi(x) - \varphi(y)| \leq c_1 (d(x,y)/R)^\theta$ for all x,y.

(d) For any ball $B(x_1, s)$ with $1 \leq s \leq R$ and $f : B(x_1, 2s) \to \mathbb{R}$

(1.15)

$$\sum_{x \in B(x_1,s)} f(x)^2 \sum_{y \in G} a_{xy} |\varphi(x) - \varphi(y)|^2$$
$$\leq c_2 \Big(\frac{s}{R}\Big)^{2\theta} \Big(\sum_{x \in B(x_1,2s)} \sum_{y \in B(x_1,2s)} a_{xy} |f(x) - f(y)|^2 + s^{-\beta} \sum_{x \in B(x_1,2s)} \mu_0(x) f(x)^2 \Big).$$

We also call (1.15) a weighted Sobolev inequality relative to φ, with exponent β and scale R.

We can now state our second main theorem.

Theorem 1.5. *Suppose that (G, E, a) is an infinite connected weighted graph satisfying the (p_0) condition. The following are equivalent:*

(a) There exists $\theta \in (0,1]$ such that (G, E, a) satisfies (VD), $PI(\beta)$ and CS(β).

(b) (G, E, a) satisfies $PHI(\beta)$.

Remarks 1.6. 1. It is obvious that the conditions $PI(\beta)$ and CS(β) are stable, so that Theorem 1.1 is an immediate consequence of Theorem 1.5.

2. Let $2 \leq \beta \leq \beta'$. Then it is easy to see that $PI(\beta)$ implies $PI(\beta')$, while CS(β') implies CS(β). In fact it is easy to check that CS(2) always holds; essentially one can take $\varphi(x) = (2/R) d(x, B(x_0, R)^c)$. Thus $|\varphi(x) - \varphi(y)| \leq 2/R$ if $a_{xy} > 0$, and (1.15) follows easily. In view of this we can regard Theorem 1.5 as an extension of the characterization of PHI(2) due to [Gr] and [SC] to the $\beta > 2$ case: the extra hypothesis CS(2) is always true.

3. It is easy to prove (see Lemma 5.1) that if $PI(\beta)$ holds, then CS(β') cannot hold for any $\beta' > \beta$.

4. If (G_i, E_i, a_i), $i = 1, 2$, are two graphs satisfying PHI(β_i), respectively, with $\beta_1 < \beta_2$, then the product $G = G_1 \times G_2$ satisfies $PI(\beta_2)$. However, since G does not satisfy PHI(β_2) it cannot satisfy CS(β_2) for any θ. Thus the conditions $PI(\beta)$ and CS(β) are independent.

5. We give the corresponding version of CS(β) for a non-discrete space below – see Definition 3.1.

6. It is likely that the results of this paper can be extended to more general time-scale functions F than the case $F(R) = R^\beta$ considered here. For some recent work involving these more general types of space-time scaling, see [HSC] and [T2].

7. The conditions (VD), $PI(\beta)$ and CS(β) are stable under rough isometries – see [HK], Proposition 5.15.

We now give a brief sketch of the proof that (a) implies (b) in Theorem 1.5. Given a harmonic function $u \geq 0$, and writing $f = u^p$, Moser's proof uses a Sobolev inequality to bound $\int_Q |f|^{2+\varepsilon}$ in terms of $\int_Q |\nabla f|^2$. This integral of ∇f is then bounded in terms of $\int_{Q'} |f|^2$: here $Q \subset Q'$ are two cubes of slightly different sizes. Iterating and passing to the limit, one obtains a bound on the L^∞ norm of u. One performs a similar argument for negative powers, and then one links the positive and negative powers.

For spaces with $\beta > 2$ difficulties arise with the initial iteration argument. In order to bound $\int |\nabla f|^2$ in terms of $\int |f|^2$, one needs to perform an integration by parts, and it is here that one needs suitable cut-off functions. Spaces with $\beta > 2$ typically have a large number of holes, and this means that the minimum energy cut-off function φ such that $\varphi = 1$ on a ball $B(x, R/2)$, and $\varphi = 0$ outside $B(x, R)$, will satisfy

$$\int_{B(x,R)} |\nabla\varphi|^2 d\mu_0 \approx R^{-\beta} V_0(x, R) \ll R^{-2} V_0(x, R).$$

(One can use the 'bottlenecks' in the space to find functions with substantially lower energy than the obvious linear function.) The standard Moser iteration argument (and also the method of Davies) uses cut-off functions φ with $||\nabla\varphi||_\infty \approx R^{-2}$, so that one can bound $\int_{B(x,R)} f^2 |\nabla\varphi|^2$ by $R^{-2} \int_{B(x,R)} f^2$. These bounds still hold in the anomalous diffusion case, but are not enough: one needs $R^{-\beta}$ rather than R^{-2} to cancel terms involving R^β which arise from the Poincaré inequality $PI(\beta)$. Thus for spaces with $\beta > 2$, cut-off functions for which one has good enough control of $||\nabla\varphi||_\infty$ do not exist. (This is related to the fact, proved in [K] in some special cases, that in the scaling limit, the energy measure given formally by $|\nabla f|^2 d\mu_0$ is in fact singular with respect to μ_0.)

What we do instead is use CS(β) to find a cut-off function φ and prove that

$$\int_{B(x_0,R/2)} |\nabla f|^2 \le c_1 \int_{B(x_0,R)} |f|^2 d\nu,$$

for the measure $\nu = (1 + R^\beta |\nabla\varphi|^2) d\mu_0$. This measure ν is not very tractable in general – we do not, for example, know if it satisfies volume doubling. However, we can prove a weighted Sobolev inequality linking the $L^{2+\varepsilon}$ norm of f with respect to ν to the L^2 norm of ∇f with respect to μ_0.

In [BB3] we used a version of this argument to prove Harnack inequalities for certain fractal-like domains in $\mathbb{R}^d$. However, the symmetry available in that context played an essential role at several points. In the current paper no symmetry is assumed. Some other differences, although less crucial, are (i) we allow the weaker condition of volume doubling rather than the condition $V(x, R) \approx R^\alpha$ that was present in [BB3], and (ii) we need to circumvent the potentially bad decay of harmonic functions near the boundaries of balls.

In working with graphs one encounters numerous minor difficulties arising from the discrete structure. One possibility would be to deal with these directly, as is done for example in [D1]. However, as in this paper we are concerned with harmonic functions, we can embed the graph G in a connected metric space (called the *cable system* for the graph) and prove the Harnack inequality in that context. It is then easy to obtain the result for the original graph.

In Section 2 we recall the cable system associated with the graph (G, E, a), and derive some needed properties. In Section 3 we prove the equivalence of CS(β) on the graph G and cable system G_C. We prove that implication (b) implies (a) in Theorem 1.5 in Section 4 by using properties of Green functions to build a suitable cutoff function φ. In Section 5 we use the Moser iteration argument outlined above to prove that (a) implies (b).

2. The cable system and preliminaries

We note that the condition (VD) implies that there exists $\alpha_2 > 0$ such that if x, $y \in G$ and $0 < r < R$, then

$$\frac{V_0(x,R)}{V_0(y,r)} \le c_1 \Big(\frac{d(x,y)+R}{r} \Big)^{\alpha_2}. \tag{2.1}$$

We introduce the following *fast volume growth* (FVG) condition: there exists $\alpha_1 > 0$ such that if x, $y \in G$ and $0 < r < R$, then

$$\frac{V_0(x,R)}{V_0(x,r)} \ge c_2 \Big(\frac{R}{r} \Big)^{\alpha_1}. \tag{2.2}$$

Observe that if both (2.1) and (2.2) hold, then $\alpha_1 \le \alpha_2$. Given PHI(β), the condition (FVG) (with $\alpha_1 > \beta$) implies that the Markov chain X is transient and gives good control over the decay of the Green functions.

We now introduce the cable system associated with (G, E, a). Loosely speaking, this is the metric space G_C obtained by replacing each edge in E by a 'cable' of length 1. The associated Markov process behaves like a linear Brownian motion on each cable and when at a vertex picks the edge on which the next excursion will lie according to the conductances a_{xy}. (These processes are sometimes called 'Walsh Brownian motions' – see [W], [BPY].)

To be more precise, we let G_C consist of G together with cables, one for each edge. Each cable is a copy of $(0,1)$. We let $\mu(dx) = a_{xy}\,dx$ on the cable $C(x,y)$ associated with the edge $\{x,y\}$; μ assigns no mass to any vertex. The distance between two points x and y is given as follows: if x and y are on the same cable, the length is just the usual Euclidean distance $|x-y|$. If they are on different cables, then the distance is $\min\{|x-z_x| + d(z_x,z_y) + |z_y - y|\}$, where the minimum is taken over all vertices z_x and z_y such that x is on a cable with one end at z_x and y is on a cable with one end at z_y. We denote this distance by $d(x,y)$ as well, since it generalizes the graph distance. We again let $B(x,r) = \{y \in G_C : d(x,y) < r\}$ denote the ball of radius r and set $V(x,r) = \mu(B(x,r))$.

The following lemma is easy to check.

Lemma 2.1. *The following are equivalent:*

(a) (G,E,a) satisfies (p_0) and (VD) with respect to μ_0.

(b) G_C satisfies (VD) with respect to μ.

Moreover, if $R \ge 1$,

$$V_0(x,R) \approx V(x,R). \tag{2.3}$$

Given a function f on G_C and a point x in the interior of a cable, we define $\nabla f(x)$ as follows. We choose an orientation for the edges of G, and if x is in the oriented edge $\{y_0, y_1\}$, then we set

$$\nabla f(x) = \lim_{z\to x} \frac{f(z)-f(x)}{d(y_0,z)-d(y_0,x)}\,.$$

If $x \in G$, and $\{x,y\}$ is an edge, write $C(x,y)$ for the cable containing x and y and let

$$\nabla_y f(x) = \lim_{z\to x, z\in C(x,y)} \frac{f(z)-f(x)}{d(x,z)}.$$

Note that while the choice of orientation will affect the sign of ∇f, it will not affect the quantities $\nabla_y f$, $|\nabla f(z)|^2$, $\nabla f(z)\nabla g(z)$, or $\nabla^2 f$. We then let

$$\mathcal{E}_Y(f,f) = \tfrac{1}{2} \sum_{\{x,y\}\in E} \int |\nabla f(z)|^2 \mu(dz), \tag{2.4}$$

and let $\mathcal{D}_0(\mathcal{E}_Y)$ be the collection of continuous functions from G_C to $\mathbb{R}$ with compact support such that $\nabla f(y)$ exists at every point of $G_C - G$, $\nabla_y f(x)$ exists at every point $x \in G$ and for every cable $C(x,y)$, and $\|\nabla f\|_\infty < \infty$. Let $\mathcal{D}(\mathcal{E}_Y)$, the domain of $\mathcal{E}_Y$, be the completion of $\mathcal{D}_0(\mathcal{E}_Y)$ with respect to the norm

$$\Big(\sum_{\{x,y\in E\}} \int |f(z)|^2 \mu(dz)\Big)^{1/2} + (\mathcal{E}_Y(f,f))^{1/2}.$$

The *cable process* is the symmetric continuous Markov process Y which corresponds to the Dirichlet form $(\mathcal{E}_Y, \mathcal{D}(\mathcal{E}_Y))$ on $L^2(G_C,\mu)$ – see [FOT]. We denote the infinitesimal generator of Y by $\mathcal{L}$ and its domain by $\mathcal{D}(\mathcal{L})$. If $x \in G_C - G$, then we have $\mathcal{L}f(x) = \frac{1}{2}\nabla^2 f(x)$. To describe $\mathcal{L}f$ at points $x \in G$, we need some additional definitions.

Definition 2.2. Let D be an open domain in G_C. Write $\mathcal{K}(D)$ for the set of functions $f : \overline{D} \to \mathbb{R}$ satisfying the following conditions.

(1) f is continuous on $\overline{D}$.
(2) ∇f exists and is continuous on each cable in D, and $\lim_{x\to z} \nabla f(x)$ exists at each endpoint z of a cable in D.
(3) There exists a discrete subset $\Gamma \subset D$ such that $\nabla^2 f(x)$ exists and is continuous at all $x \in D - (G \cup \Gamma)$. If I is an interval contained in a cable in D and $z \in G \cup \Gamma$ is an endpoint of I, then $\lim_{x\to z, x\in I} \nabla^2 f(x)$ exists.
(4) If $x \in G \cap D$, let x_i, $1 \le i \le m$, be the m neighbors of x in G, and let $p_i = a_{xx_i}/\sum_{j=1}^m a_{xx_j}$. Then the directional gradients $\nabla_{x_i} f(x)$ satisfy the consistency condition

$$\sum_{i=1}^m p_i \nabla_{x_i} f(x) = 0. \tag{2.5}$$

We then have $\mathcal{K}(G_C) \subset \mathcal{D}(\mathcal{L})$, and for $f \in \mathcal{K}(G_C)$

$$\mathcal{L}f(x) = \begin{cases} \frac{1}{2}\nabla^2 f(x), & \text{if } x \in G_C - G, \\ \frac{1}{2}\sum_{i=1}^m p_i \nabla^2_{x_i} f(x) & \text{if } x \in G, \end{cases} \tag{2.6}$$

where we write $\nabla^2_{x_i} f(x) = \lim_{z\to x, z\in C(x,x_i)} \nabla^2 f(x)$.

For D such that $\partial D \cap G = \emptyset$ and for $z \in \partial D$ we define the inward pointing normal derivative

$$\frac{\partial f}{\partial n}(z) = \lim_{y\to z, y\in D} \frac{f(y)-f(z)}{d(y,z)}. \tag{2.7}$$

Lemma 2.3 (Gauss-Green lemma)**.** *Let D be an open subset of G_C satisfying the condition that $\partial D \cap G = \emptyset$, and the intersection of D with any cable consists of finitely many intervals. Let $f \in \mathcal{K}(D)$ and $g \in \mathcal{D}(\mathcal{E}_Y)$. Then*

$$\int_D g(z)\mathcal{L}f(z)d\mu(z) + \int_D \nabla g(z)\nabla f(z)d\mu(z) = -\sum_{\partial D} a_{e(z)} g(z) \frac{\partial f}{\partial n}(z). \tag{2.8}$$

Here $e(z)$ denotes the cable containing z.

Proof. Let $x \in D$ and $\{x,y\}$ be an edge in E. Then if $C(x,y) \subset D$, by standard integration by parts on $[0,1]$ we have

$$\int_{C(x,y)} [g(z)\mathcal{L}f(z) + \nabla g(z)\nabla f(z)]a_{xy}dz = a_{xy}[g(y)\nabla_x f(y) - g(x)\nabla_y f(x)].$$

Similarly, if $C(x,y) \cap \partial D = \{z\}$, then

$$\int_{C(x,y)} [g(z)\mathcal{L}f(z) + \nabla g(z)\nabla f(z)]a_{xy}dz = a_{xy}[g(y)\nabla_x f(y) - g(z)\frac{\partial f}{\partial n}(z)].$$

Summing these identities over all the cables which intersect D and using (2.5), we obtain (2.8). $\square$

Let $B \subset G_C$ and $h : \overline{D} \to \mathbb{R}$. We say h is *harmonic* (with respect to $\mathcal{L}$) if $h \in \mathcal{K}(D)$ and $\mathcal{L}h(x) = 0$ for $x \in D$. The relation between $\mathcal{L}_G$ and $\mathcal{L}$ harmonic functions is given by the following lemma.

Lemma 2.4. *(a) Let $A \subset G$, $h : \overline{A} \to \mathbb{R}$ and h be $\mathcal{L}_G$-harmonic on A. Let B be the open subset of G_C consisting of A and the interior of any cable with at least one endpoint in A. Define $\widetilde{h}$ by setting $\widetilde{h}(x) = h(x)$ for $x \in \overline{A}$, and extend $\widetilde{h}$ to a function on $\overline{B}$ by taking $\widetilde{h}$ to be linear on each cable. Then $\widetilde{h} \in \mathcal{K}(B)$ and $\widetilde{h}$ is $\mathcal{L}$-harmonic on B.*

(b) Let $B \subset G_C$ and let $\widetilde{h} : \overline{B} \to \mathbb{R}$ be $\mathcal{L}$-harmonic. Set $A = \{x \in G : d(x, B^c) > 1\}$. Then $\widetilde{h}|_A$ is $\mathcal{L}_G$-harmonic.

Proof. (a) It is clear that $\widetilde{h}$ is continuous on $\overline{B}$ and that $\nabla^2\widetilde{h} = 0$ on the interior of every cable. So $\widetilde{h}(x_i) = \widetilde{h}(x) + \nabla_{x_i}\widetilde{h}(x)$ for $1 \le i \le m$. Thus if $x \in G$,

$$\mu_0(x)^{-1}\sum_i a_{xx_i}\nabla_{x_i}\widetilde{h}(x) = \mu_0(x)^{-1}\sum_i a_{xx_i}(h(x_i) - h(x)) = \mathcal{L}_G h(x) = 0,$$

so that $\widetilde{h}$ satisfies the consistency condition (2.5) at x. Since $\nabla^2_{x_i}h(x) = 0$ for each i if $x \in G$, we have $\mathcal{L}\widetilde{h}(x) = 0$. Thus $\widetilde{h} \in \mathcal{K}(\overline{B})$, and $\widetilde{h}$ is $\mathcal{L}$-harmonic.

(b) is proved in a similar fashion. $\square$

Corollary 2.5. *(EHI) holds for the $\mathcal{L}_G$-harmonic functions on the graph G if and only if it holds for $\mathcal{L}$-harmonic functions on the cable system G_C.*

Set $\tau_{B(x,r)} = \inf\{t : Y_t \notin B(x,r)\}$.

Lemma 2.6. *If (G,E,a) satisfies (E_β), then $\mathbb{E}^x\tau_{B(x,r)} \approx r^\beta$ for $r \ge 1$.*

Proof. Suppose first that $x \in G$ and r is a positive integer. Let S_i be the times that the cable process Y_t hits successive vertices of G and let $V_i = Y_{S_i}$. Writing $\xi_i = S_i - S_{i-1}$, ξ_i is just the time for a standard one dimensional Brownian motion to move a distance 1. So $\mathbb{E}\xi_i = 1$ and the ξ_i are i.i.d. random variables, which are independent of the process V. Since V_i has the same distribution as the Markov chain X_i we deduce, writing $\tau_V = \min\{n \ge 0 : V_n \in B(x,r)^c\}$,

$$\mathbb{E}^x\tau_{B(x,r)} = \mathbb{E}^x\sum_{i=1}^{\tau_V}\xi_i = \mathbb{E}^x\tau_V \approx r^\beta.$$

The case of general x and r now follows easily. $\square$

We will need the following covering lemma.

Lemma 2.7. *Let $D \subset G_C$ have compact closure, and let $s > 0$. There exists a cover of D by balls $B(x_i, s)$ with $x_i \in D$ such that no point in G_C is in more than M of the $B(x_i, 2s)$. Here M depends only on G_C.*

Proof. Let $x_1 \in D$ and choose $x_2, x_3, \dots$ by letting x_{i+1} be any point in $D - \bigcup_{j=1}^{i} B(x_j, s)$. We do this until we can no longer proceed. Note that the x_i must be at least distance s apart, so the balls $B(x_i, s/2)$ are disjoint. Now suppose y is in m of the balls $B(x_i, 2s)$. Then $B(y, 3s)$ contains m disjoint balls $B(x_i, s/2)$, and using (2.1) we have for each of these

$$\frac{V(y,3s)}{V(x_i, s/2)} \le c_1.$$

Thus

$$V(y, 3s) \ge \sum_{\{i : y \in B(x_i, 2s)\}} V(x_i, s/2) \ge m c_1^{-1} V(y, 3s),$$

and so we can take $M = c_1$. □

We finally remark that under the (p_0) condition any graph satisfies a local Harnack inequality.

Lemma 2.8. *Suppose A is a finite connected set. There exists c_1 depending only on A such that if $h : \overline{A} \to \mathbb{R}$ is nonnegative and harmonic in A, then*

$$h(y) \le c_1 h(x), \qquad x, y \in A.$$

Proof. If $x_i, x_{i+1} \in A$ with $\{x_i, x_{i+1}\} \in E$, then

$$h(x_i) = \sum_z p_{x_i z} h(z) \ge p_{x_i x_{i+1}} h(x_{i+1}) \ge p_0 h(x_{i+1}). \tag{2.9}$$

Since A is connected, if $x, y \in A$, we can find $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ such that each $x_i \in A$ and each $\{x_i, x_{i+1}\} \in E$. Since A is finite, we can take n less than or equal to the cardinality of A. By using (2.9) at most $n-1$ times, we obtain our result. □

3. The cut-off Sobolev inequality

We give the form of the cut-off Sobolev inequality for the cable system. Define the function

$$\psi(r) = \begin{cases} r^\beta, & \text{if } r \ge 1, \\ r^2, & \text{if } r \le 1. \end{cases}$$

Note that we can also write $\psi(r) = r^\beta \vee r^2$ and that $\mathbb{E}^x \tau_{B(x,r)} \approx \psi(r)$.

Definition 3.1. G_C satisfies CS(β) for $\beta \ge 2$ and $\theta \in (0, 1]$ if there exist constants c_1 and c_2 such that the following holds. For every $x_0 \in G_C$, $R > 0$ there exists a function $\varphi (= \varphi_{x_0,R})$ with the following properties:

(a) $\varphi(x) \ge 1$ for $x \in B(x_0, R/2)$.
(b) $\varphi(x) = 0$ for $x \in B(x_0, R)^c$.
(c) $|\varphi(x) - \varphi(y)| \le c_1 (d(x,y)/R)^\theta$ for all x,y.
(d) For any ball $B(x, s)$ with $0 < s \le R$ and $f : B(x, 2s) \to \mathbb{R}$

$$\int_{B(x,s)} f^2 |\nabla \varphi|^2 d\mu \le c_2 (s/R)^{2\theta} \Big(\int_{B(x,2s)} |\nabla f|^2 d\mu + \psi(s)^{-1} \int_{B(x,2s)} f^2 d\mu \Big). \tag{3.1}$$

We now compare CS(β) for the graph G and cable system G_C. We begin with some elementary remarks.

Remarks 3.2. 1. Suppose CS(β) holds for G_C, but with (a) above replaced by

$$\text{(a}'\text{)} \qquad \varphi(x) \geq 1 \text{ for } x \in B(x_0, \delta R),$$

for some $\delta < \frac{1}{2}$. Then an easy covering argument (using (VD)) gives CS(β) with $\delta = \frac{1}{2}$.

2. Let $\lambda > 1$. Suppose that CS(β) holds, except that instead of (3.1) we have

$$\int_{B(x,s)} f^2 |\nabla \varphi|^2 d\mu \leq c_2 (s/R)^{2\theta} \Big(\int_{B(x,\lambda s)} |\nabla f|^2 d\mu + \psi(s)^{-1} \int_{B(x,\lambda s)} f^2 d\mu \Big). \tag{3.2}$$

Then once again it is easy to obtain CS(β) with $\lambda = 2$ by a covering argument.

3. Any operation on the cut-off function φ which reduces $|\nabla \varphi|$ while keeping properties (a), (b) and (c) of Definition 3.1 will generate a new cut-off function which still satisfies (3.1). We can therefore assume that any cut-off function φ satisfies the following:

(a) $0 \leq \varphi \leq 1$.

(b) φ is monotone on each cable.

(c) For each $t \in (0,1)$ the set $\{x : \varphi(x) > t\}$ is connected and contains $B(x_0, R/2)$.

(d) Each connected component A of $\{x : \varphi(x) < t\}$ intersects $B(x_0, R)^c$.

4. We call (3.1) a weighted Sobolev inequality. Note also that in order to prove (3.1) it is enough to consider nonnegative f.

Proposition 3.3. *Suppose CS(β) holds for G. Then it holds for G_C.*

Proof. It is enough to prove this when R and s are both large. Let $x_0 \in G_C$, and choose $x_1 \in G$ with $d(x_1, x_0) \leq 1/2$. Let $\overline{\varphi} : G \to [0,1]$ be the cut-off function for $B(x_1, R)$. Define $\varphi : G_C \to [0,1]$ to be equal to $\overline{\varphi}$ on G and by linear interpolation of $\overline{\varphi}$ on each cable.

Now let $I = B(x_0, s) \subset G_C$, with $s \geq 1$. We may assume $x_0 \in G$: if not, we prove the weighted Sobolev inequality for each of the balls $B(z_i, s)$, where z_i are the endpoints of the cable containing x_0. We may also assume, again by using the remarks above, that $s = n + 1/2$, where $n \in \mathbb{Z}$. Let $f : I^* \to \mathbb{R}_+$, where $I^* = B(x_0, 2s)$.

For $x \in I^* \cap G$ define

$$g(x) = V(x, 1/2)^{-1} \int_{B(x,1/2)} f d\mu.$$

Note that $V(x, 1/2) = \mu_0(x)/2$ and that using the (p_0) condition, if $x \sim y$, then $a_{xy} \leq \mu_0(x) \leq p_0^{-1} a_{xy}$. We have

$$\int_{B(x,1/2)} f^2 d\mu = \int_{B(x,1/2)} (f(y) - g(x))^2 d\mu + \tfrac{1}{2} g(x)^2 \mu_0(x).$$

It is elementary to prove the Poincaré inequality for $B(x, 1/2)$, and so

$$\int_{B(x,1/2)} f^2 d\mu \leq c_1 \int_{B(x,1/2)} |\nabla f|^2 d\mu + c_1 g(x)^2 \mu_0(x).$$

Since $\max_{y\sim x}|\overline{\varphi}(y)-\overline{\varphi}(x)|^2 \le \sum_{y\sim x}|\overline{\varphi}(y)-\overline{\varphi}(x)|^2$,

$$\begin{aligned}\int_I f^2|\nabla\varphi|^2 d\mu &= \sum_{x\in I\cap G}\int_{B(x,1/2)} f^2|\nabla\varphi|^2 d\mu\\ &\le c_1\sum_{x\in I\cap G}(\max_{y\sim x}|\overline{\varphi}(y)-\overline{\varphi}(x)|^2)\Big(\int_{B(x,1/2)}|\nabla f|^2 d\mu + g(x)^2\mu_0(x)\Big)\\ &\le c_2R^{-2\theta}\int_{I^*}|\nabla f|^2 d\mu + c_2\sum_{x\in I\cap G}\sum_{y\sim x}a_{xy}|\overline{\varphi}(y)-\overline{\varphi}(x)|^2 g(x)^2.\end{aligned}\tag{3.3}$$

We use the weighted Sobolev inequality for g and $\overline{\varphi}$ to bound the second term in (3.3):

$$\begin{aligned}&c_2\sum_{x\in I\cap G}\sum_{y\sim x}a_{xy}|\overline{\varphi}(y)-\overline{\varphi}(x)|^2 g(x)^2\\ &\qquad\le c_3(s/R)^{2\theta}\Big(\sum_{x\in I^*}\sum_{y\in I^*}a_{xy}|g(x)-g(y)|^2 + s^{-\beta}\sum_{x\in I^*}g(x)^2\mu_0(x)\Big).\end{aligned}\tag{3.4}$$

We now bound the terms in (3.4) by the corresponding expressions involving f. By Jensen's inequality $g(x)^2 \le V(x,1/2)^{-1}\int_{B(x,1/2)} f^2 d\mu$, and so

$$\sum_{x\in I^*} g(x)^2\mu_0(x) \le \int_{I^*} f^2 d\mu.\tag{3.5}$$

If $x, y \in I^*$ with $x \sim y$, then there exist $x' \in B(x,1/2)$, $y' \in B(y,1/2)$ with $|g(x)-g(y)| \le |f(x')-f(y')|$. So, writing dt for the measure on G_C which equals Lebesgue measure on each cable,

$$|g(x)-g(y)| \le \int_{B(x,1/2)\cup B(y,1/2)}|\nabla f(t)|dt \le c_4\Big(\int_{B(x,1/2)\cup B(y,1/2)}|\nabla f(t)|^2 dt\Big)^{1/2}.$$

Writing $A(x) = \int_{B(x,1/2)}|\nabla f|^2 dt$, we have $A(x)\mu_0(x) \le c_5\int_{B(x,1/2)}|\nabla f|^2 d\mu$. So,

$$\begin{aligned}\sum_{x\in I^*\cap G}\sum_{y\in I^*\cap G}a_{xy}|g(x)-g(y)|^2 &\le c_6\sum_{x\in I^*\cap G}\sum_{y\in I^*\cap G}a_{xy}(A(x)+A(y))\\ &= 2c_6\sum_{x\in I^*\cap G}A(x)\mu_0(x)\\ &\le c_7\sum_{x\in I^*\cap G}\int_{B(x,1/2)}|\nabla f|^2 d\mu\\ &\le c_8\int_{I^*}|\nabla f|^2 d\mu.\end{aligned}\tag{3.6}$$

Combining (3.3)–(3.6) we deduce (3.1) for f. □

Proposition 3.4. *Suppose CS(β) holds for G_C. Then it holds for G.*

Proof. Let $x_0 \in G$, and $R \ge 1$. Let $\varphi : G_C \to [0,1]$ be a cut-off function for $B(x_0,R)$. Now let $x_1 \in G$, $1 \le s \le R$ and $I = B(x_1,s)$. Let $g : I^*\cap G \to \mathbb{R}_+$. Extend g to a function $f : I^* \to \mathbb{R}_+$ by linear interpolation on cables. We take f to be constant on any cable with only one end in I^*.

If e is the edge $\{x,y\}$ with $x,y\in I$, write C_e for the cable associated with e, $a_e = a_{xy}$, $g(e)^2=(g(x)^2+g(y)^2)/2$, $|\nabla\varphi(e)|=|\varphi(x)-\varphi(y)|$, and $|\nabla g(e)|=|g(x)-g(y)|$. Let $E_0=\{e=\{x,y\}: x,y\in I\}$,

$$E_1=\{e=\{x,y\}\in E_0 : \tfrac{1}{2}g(x)\le g(y)\le 2g(x)\},$$

and $E_2=E_0-E_1$. If $e\in E_2$, then an elementary computation shows $g(e)^2\le 3|\nabla g(e)|^2$. So by Definition 3.1(c),

$$\sum_{e\in E_2} a_e g(e)^2|\nabla\varphi(e)|^2 \le c_1 R^{-2\theta}\sum_{e\in E_2} a_e|\nabla g(e)|^2. \tag{3.7}$$

If $e\in E_1$ and $g(x)\le g(y)$, then we have $g(e)^2\le 5g(x)^2$, and so

$$a_e g(e)^2|\nabla\varphi(e)|^2\le c_2 g(x)^2\int_{C_e}|\nabla\varphi|^2 d\mu \le c_2\int_{C_e} f^2|\nabla\varphi|^2 d\mu.$$

Hence, using the weighted Sobolev inequality for f and φ,

$$\begin{aligned}\sum_{e\in E_1} a_e g(e)^2|\nabla\varphi(e)|^2 &\le c_3\int_I f^2|\nabla\varphi|^2 d\mu\\ &\le c_4(s/R)^{2\theta}\Big(\int_{I^*}|\nabla f|^2 d\mu + s^{-\beta}\int_{I^*} f^2 d\mu\Big).\end{aligned} \tag{3.8}$$

Note that $|\nabla f|=|g(x)-g(y)|$ on C_e, and so

$$\int_{I^*}|\nabla f|^2 d\mu \le \sum_{x\in I^*}\sum_{y\in I^*} a_{xy}|g(x)-g(y)|^2.$$

We bound $\int_{I^*} f^2 d\mu$ as in the previous result. Combining (3.7), (3.8) with these bounds we obtain the weighted Sobolev inequality for g. □

We will also need the equivalence of the Poincaré inequality for G and for G_C.

Proposition 3.5. *(a) Suppose (G,E,a) satisfies $PI(\beta)$. Suppose the gradient of f is square integrable over $B(x,r)$ and $r>0$. Then, writing $f_B=\mu(B)^{-1}\int_B f d\mu$,*

$$\int_{B(x,r)}|f(z)-f_{B(x,r)}|^2\mu(dz)\le c_1\psi(r)\int_{B(x,r)}|\nabla f(z)|^2\mu(dz). \tag{3.9}$$

(b) Conversely, if (3.9) holds for every x and r with c_1 independent of x and r, then (G,E,a) satisfies $PI(\beta)$.

The proof of Proposition 3.5 is very similar to the proofs of Propositions 3.3 and 3.4. Similarly we have

Proposition 3.6. *G satisfies (R_β) if and only if G_C satisfies (R_β).*

4. Construction of cut-off functions

In this section we will assume that the conditions (a)–(d) of Theorem 1.2 hold and prove that both $PI(\beta)$ and CS(β) follow.

Lemma 4.1. *Suppose (G,E,a) satisfies $PHI(\beta)$. Then (G,E,a) satisfies $PI(\beta)$.*

Proof. Using the estimates (1.12) and (1.13), the Poincaré inequality for G follows by the argument given in [SC]. □

For the remainder of this section we work on the cable system G_C associated with the graph G. If D is a domain in G_C (so $D \subset G_C$, D is connected and relatively open in G_C), write $g_D(x,y)$ for the Green function of Y on D. Then g_D is symmetric and continuous, and we have for $f \in \mathcal{D}(\mathcal{L})$ with support contained in D

$$\mathcal{E}(g_D(x,\cdot), f) = -\int_D g_D(x,y)\mathcal{L}f(y)\mu(dy) = f(x); \tag{4.1}$$

see [FOT], Corollary 1.3.1. We have $g_D(x,y) = 0$ if $y \in \partial D$, so we can extend g_D to $G_C \times G_C$ by taking it to be zero off $D \times D$. We write $g = g_{G_C}$ when this exists. Note that if $x \in G$, $B(x,1) \subset D$, then applying (4.1) to a function f with $f = 1$ on $B(x, 1/2)$ and $f = 0$ off $B(x,1)$, then

$$\tfrac{1}{2}\sum_{y\in G} a_{xy}(g_D(x,x) - g_D(x,y)) = 1. \tag{4.2}$$

The following lemma is a slight generalization of Proposition 4.1 of [GT2] and is proved similarly; note that no geometric assumptions on (G,E) are needed.

Lemma 4.2. *Suppose (G,E,a) satisfies (EHI). Then there exists a constant c_1 such that if $x_0 \in G_C$, $r > 0$, $d(x_0,x) = d(x_0,y) = r$, and $B(x_0, 2r) \subset D$, then*

$$c_1^{-1} g_D(x_0,y) \le g_D(x_0,x) \le c_1 g_D(x_0,y). \tag{4.3}$$

The following result is proved in [GT2] and [T1], but as it has a simple direct proof which does not use (EHI), we present it here.

Lemma 4.3. *Suppose (G,E) satisfies (VD) and (E_β). Then G satisfies (R_β).*

Proof. Write I for $B(x_0,r)$ and I^* for $B(x_0,2r)$. Let f be the function which is 1 on I and zero on $(I^*)^c$ and for which the infimum in (1.10) is attained. f will be bounded between 0 and 1. Then it is well known that

$$f(x) = \int_I g_{I^*}(z,x)e_I(dz), \tag{4.4}$$

where e_I is the capacitary measure for I (relative to g_{I^*}), and also that the effective resistance between ∂I and ∂I^* is exactly $e_I(I)^{-1}$. For $z \in I$ we have,

$$r^\beta \approx \mathbb{E}^z \tau_{I^*} = \int_{I^*} g_{I^*}(z,x)\mu(dx).$$

Using (VD) we have

$$V(x_0,r) = \int_I f(x)\mu(dx) \le \int_{I^*} f(x)\mu(dx) \le V(x_0,2r) \le c_1 V(x_0,r),$$

and also

$$\begin{aligned}\int_{I^*} f(x)\mu(dx) &= \int_{I^*}\Big(\int_I g_{I^*}(z,x)e_I(dz)\Big)\mu(dx)\\ &= \int_I \mathbb{E}^z(\tau_{I^*})\, e_I(dz) \approx e_I(I)r^\beta.\end{aligned}$$

Rearranging the final formula gives the result. □

From now on in this section we assume that G_C satisfies (VD), (EHI) and (E_β) and (FVG) for some $\alpha_1 > \beta$.

Proposition 4.4. *Suppose G_C satisfies (VD), (EHI), (E_β), and also (FVG) for some $\alpha_1 > \beta$.*

(a) If x, $y \in G_C$ with $d(x,y) = r \geq 1$, and $D \subset G_C$ with $B(x, 2r) \subset D$, then

$$g_D(x,y) \approx \frac{r^\beta}{V(x,r)}. \tag{4.5}$$

In particular, (G, E, a) is transient.

(b) If x, $y \in G_C$ with $d(x,y) = r \leq 1$, and $D \subset G_C$ with $B(x,2) \subset D$, then $g_D(x,y) \approx V(x,1)^{-1}$.

Proof. (a) Write $B_k = B(x, 2^k r)$ for $k \geq 0$. If $B_n \subset D$, then by [GT2], Proposition 4.4,

$$g_D(x,y) \approx \sum_{k=0}^{n-1} R_{\mathrm{eff}}(B_k, B_{k+1}^c).$$

(We can use Lemmas 4.2 and 4.3 to replace the constant M in [GT2] by 2.) So by Lemma 4.3,

$$g_D(x,y) \approx \sum_{k=0}^{n-1} \frac{(2^k r)^\beta}{V(x, 2^k r)} = \frac{r^\beta}{V(x,r)} \sum_{k=0}^{n-1} 2^{k\beta} \frac{V(x,r)}{V(x,2^k r)}.$$

This immediately implies that if $B_2 \subset D$, then $g_D(x,y) \geq 2^\beta r^\beta / V(x,r)$. Since $\alpha_1 > \beta$ the condition (FVG) implies

$$\sum_{k=0}^{n-1} 2^{k\beta} \frac{V(x,r)}{V(x,2^k r)} \leq c_1 \sum_{k=0}^{\infty} 2^{(\beta - \alpha_1)k} \leq c_2,$$

proving the upper bound.

Taking $D = G_C$ we see that $g(x,y) < \infty$, so that G is transient.

(b) Choose z with $d(x,z) = 1$; by (a) $g_D(x,z) \approx V(x,1)^{-1}$. Using the local Harnack inequality and (4.2) completes the proof. □

Let $x_0 \in G_C$. Define

$$Q(h) = Q(x_0, h) = \{y : g(x_0, y) > h\}.$$

Clearly, if $h_1 < h_2$, then $Q(h_2) \subset Q(h_1)$. Since $g(x_0, \cdot)$ is harmonic except at x_0, it is linear on each cable not containing x_0. On the cable containing x_0, it has its maximum at x_0 and is monotonically decreasing as a function of the distance to x_0 on each side of x_0. Therefore the intersection of $Q(h)$ with any cable will consist of at most one interval. Also, by the maximum principle, we have that $Q(h)$ is connected. The next lemma proves that the sets $Q(h)$ can be approximated from within and without by balls.

Given $x_0 \in G_C$, $R > 0$ define

$$h = h(x_0, R) = \sup_{B(x_0,R)^c} g(x_0, \cdot). \tag{4.6}$$

Lemma 4.5. *Suppose G_C satisfies (VD), (E_β), (EHI), and also (FVG) for some $\alpha_1 > \beta$.*

(a) There exists $\sigma \in (2, \infty)$, independent of x_0 and R, such that if $R \geq \sigma$, then

$$B(x_0, \sigma^{-1} R) \subset Q(2h) \subset Q(h) \subset B(x_0, \sigma R). \tag{4.7}$$

(b) Provided that $tR \geq 1$,

$$h(x_0, tR) \leq c_1 t^{\beta-\alpha_1} h(x_0, R).$$

Proof. (a) Note that $Q(h) \subset B(x_0, R)$ from (4.6). Let $t \in (0,1)$ and $y \in B(x_0, tR)$. Then provided $tR \geq 1$, using (4.5) and (FVG)

$$g(x_0, y) \geq c_2 \frac{t^\beta R^\beta}{V(x_0, tR)} \geq c_3 \frac{t^{\beta-\alpha_1} R^\beta}{V(x_0, R)} \geq c_4 t^{\beta-\alpha_1} h.$$

Define σ by $c_4 \sigma^{\alpha_1-\beta} = 2$. Then we deduce that $B(x_0, \sigma^{-1}R) \subset Q(2h)$ provided $R \geq \sigma$.

(b) follows in a similar fashion from (4.5) and (VD). □

Note that if u is harmonic in a domain D and the elliptic Harnack inequality holds, then there exists $\rho < 1$ independent of x_0 and r such that

$$\operatorname{Osc}_{B(x_0,r)} u \leq \rho \operatorname{Osc}_{B(x_0,2r)} u. \tag{4.8}$$

Here $\operatorname{Osc}_A u = \sup_A u - \inf_A u$. This is standard – see [M1], Section 5, for example.

Lemma 4.6. *Let x_0, ρ, and R be as above, let $\theta = -\log\rho/\log 2$, and let h be defined by (4.6). Let $x, y \in B(x_0, R/2\sigma)^c$ with $d(x,y) = s$. Then*

$$|g(x_0, x) - g(x_0, y)| \leq c_1 (s/R)^{2\theta} h(x_0, R).$$

Proof. First note that $g(x_0, x)$ and $g(x_0, y)$ are both bounded by $c_2 h$, so the result is clear if $s \geq R/4\sigma$. So suppose $s < R/4\sigma$, let n (with $n \geq 0$) be the largest integer so that $B(x_0, R/4\sigma) \cap B(x, 2^n s) = \emptyset$, and write $a_k = \operatorname{Osc}_{B(x,2^k s)} g(x_0, \cdot)$. Then $a_n \leq c_2 h$, and using (4.8) we have $a_k \leq \rho a_{k+1}$, where $\rho < 1$ is a constant independent of k and x_0. Since $R/4\sigma < 2^{n+1}s$, then

$$|g(x_0, x) - g(x_0, y)| \leq a_0 \leq c_3 h \rho^n = c_4 h (2^n)^{-\theta} \leq c_5 h (s/R)^\theta.$$

□

Let D be a domain in G_C, and $A \subset D$. Define

$$U(x, A, D) = \int_A g_D(x,y)\mu(dy) = \mathbb{E}^x \int_0^{\tau_D} 1_A(Y_s) ds.$$

Note that U is monotone in A and D: if $A \subset A' \subset D \subset D'$, then

$$U(x, A, D) \leq U(x, A', D) \leq U(x, A', D').$$

Lemma 4.7. *Suppose (VD), (E_β), (EHI), and (FVG) hold. We have*

(*a*) $\qquad U(x, B(x_0, r), B(x_0, r)) \leq c_1 \psi(r)$ *for* $x \in G$,

(*b*) $\qquad U(x, B(x_0, r), B(x_0, 2r)) \geq c_2 \psi(r)$ *for* $x \in B(x_0, r)$.

Proof. The result is easy if $r < 1$, so we take $r \geq 1$. If $x \notin B(x_0, r)$, the left hand side of (a) is 0. If $x \in B(x_0, r)$ we have $\tau_{B(x_0,r)} \leq \tau_{B(x,2r)}$, so

$$U(x, B(x_0, r), B(x_0, r)) = \mathbb{E}^x \tau_{B(x_0,r)} \leq \mathbb{E}^x \tau_{B(x,2r)} \leq c_3 r^\beta,$$

proving (a).

(b) Suppose $x \in B(x_0, r)$. Using Proposition 4.4 we deduce that if $y \in B(x_0, r)$ with $r \geq d(x,y) > r/6$, then $g_{B(x_0,2r)}(x,y) \geq c_4 r^\beta / V(x,r)$. Since for any $x \in$

$B(x_0,r)$ there exists a ball $B(x',r/3) \subset B(x_0,r)$ with $x \notin B(x',r/3)$, it follows that

$$\begin{aligned} U(x,B(x_0,r),B(x_0,2r)) &= \int_{B(x_0,r)} g_{B(x_0,2r)}(x,y)\mu(dy) \\ &\geq \int_{B(x',r/6)} g_{B(x_0,2r)}(x,y)\mu(dy) \geq r^\beta \frac{V(x',r/6)}{V(x,r)} \geq c_5 r^\beta, \end{aligned}$$

where we used (VD) in the final line. $\square$

Lemma 4.8. *Let D be a domain in G_C, and $A \subset D$, and $v = U(\cdot, A, D)$. Suppose that $\partial A \cap G = \emptyset$, and the intersection of ∂A with each cable is a finite set. Then if $f \in \mathcal{D}(\mathcal{E}_Y)$,*

$$\int_D \nabla(f^2 v)\cdot\nabla v\, d\mu = \int_A f^2 v\, d\mu. \tag{4.9}$$

Proof. Note that $v \in \mathcal{K}(D)$ and $\mathcal{L}v = -1_A$. Suppose first $\partial D \cap G = \emptyset$. Then, by Lemma 2.3,

$$\int_D \nabla(f^2 v)\cdot\nabla v\, d\mu = \int_A f^2 v\, d\mu - \sum_{z\in\partial D} a_{e(z)} f^2(z) v(z) \frac{\partial v}{\partial n}(z).$$

As $v(z) = 0$ on ∂D we obtain (4.9). If $\partial D \cap G$ is non-empty, we can now obtain (4.9) by an approximation argument. $\square$

Now let $x_0 \in G_C$ and $R > 0$. We will construct a cut-off function φ for $B(x_0,R)$. If $R \leq c_1$ this is easy, so we assume $R > c_1$. Define h by (4.6), so that, by Proposition 4.4 and Lemma 4.5, $h \approx R^\beta / V(x_0,R)$ and (4.7) holds. Let

$$\begin{aligned} \omega_0(x) &= g(x_0,x), \qquad x \in G_C, \\ \omega(x) &= (2h \wedge g(x_0,x) - h)^+, \qquad x \in G_C. \end{aligned} \tag{4.10}$$

We use the following elementary result.

Lemma 4.9. *Let $x,y,z \geq 0$. If $x \leq c_1(x^{1/2}z^{1/2} + y)$, then $x \leq 2c_1 y + 4c_1^2 z$.*

Proposition 4.10. *Let x_0, R, ω, ω_0 be as above. Let $I = B(x,s)$, with $s \leq R$. Suppose that either*

$$I^* \subset Q(2h) \tag{4.11}$$

or

$$I^* \cap B(x_0, R/2\sigma) = \emptyset. \tag{4.12}$$

Suppose f and its gradient are square integrable over I^*. There exists $c_1 < \infty$ such that

$$\int_I f^2|\nabla\omega|^2 \leq c_1 (s/R)^{2\theta} h^2 \Big(\int_{I^*} |\nabla f|^2 + \psi(s)^{-1} \int_I f^2 \Big). \tag{4.13}$$

Proof. If (4.11) holds, then $\nabla\omega = 0$ on I^*, and the left hand side of (4.13) is 0. So we suppose (4.12) holds. Let $h_1 = h(x_0, R/2\sigma)$; then $h_1 \leq c_2 h$ and $I^* \subset Q(h_1)^c$. Let $v = U(\cdot, I, I^*)$, and write $V_0 = \inf_I v$, $V_1 = \sup_{I^*} v$. By Lemma 4.7 we have

$$c_3\psi(s) \leq V_0 \leq V_1 \leq c_4\psi(s).$$

Fix $x_1 \in I$ and set $\omega_1 = \omega_0 - \omega_0(x_1)$. Let H be a domain such that $\overline{I^*} \subset H \subset \overline{H} \subset Q(2h_1)^c$ and $(\partial H) \cap G = \emptyset$. Since $H \subset Q(2h_1)^c$, then $\omega_0 \leq 2h_1$ on H, and hence

$|\omega_1| \le 2c_2h$ on H. On the other hand, $|\omega_1|$ is bounded on I^* by $\mathrm{Osc}_{I^*}\,\omega_0$. So by Lemma 4.6,

$$L = \sup_{I^*} |\omega_1| \le \mathrm{Osc}_{B(x,2s)}\,\omega_0 \le c_5 h (s/R)^\theta.$$

Write

$$\begin{aligned}
A &= \int_{I^*} f^2 v^2 |\nabla \omega|^2,\\
B &= \int_{I^*} v^2 |\nabla f|^2,\\
C &= \int_I f^2,\\
D &= \int_{I^*} f^2 |\nabla v|^2,\\
E &= \int_{I^*} |\nabla f|^2,\\
F &= \int_H f^2 v^2 |\nabla \omega_1|^2 = \int_{I^*} f^2 v^2 |\nabla \omega_1|^2.
\end{aligned}$$

Observe that

$$\int_I f^2 |\nabla \omega|^2 \le (\inf_I v)^{-2} \int_I f^2 |\nabla \omega|^2 v^2 \le V_0^{-2} A. \tag{4.14}$$

Also, since v is 0 on $(I^*)^c$,

$$\begin{aligned}
A = \int_{I^*} f^2 v^2 |\nabla \omega|^2 &= \int_H f^2 v^2 |\nabla \omega|^2\\
&\le \int_H f^2 v^2 |\nabla \omega_0|^2 = \int_H f^2 v^2 |\nabla \omega_1|^2 = F.
\end{aligned}$$

We work first on bounding F. By Lemma 2.3,

$$\begin{aligned}
F &= \int_H f^2 v^2 \nabla \omega_1 \cdot \nabla \omega_1 = \int_H \nabla(\omega_1 f^2 v^2) \cdot \nabla \omega_1 - \int_H \omega_1 \nabla (f^2 v^2) \cdot \nabla \omega_1\\
&= -\int_H \omega_1 f^2 v^2 \mathcal{L} \omega_1 - \sum_{\partial H} a_{e(\cdot)} \omega_1 f^2 v^2 \frac{\partial \omega_1}{\partial n} - \int_H \omega_1 \nabla (f^2 v^2) \cdot \nabla \omega_1.
\end{aligned}$$

As $H \cap Q(2h_1) = \emptyset$, then $\mathcal{L}\omega_1 = 0$ in H, while $v = 0$ on ∂H. Hence

$$F = -\int \omega_1 \nabla (f^2 v^2) \cdot \nabla \omega_1.$$

We then proceed to bound F by writing

$$\begin{aligned}
F &\le \Big| \int_H \omega_1 \nabla \omega_1 \cdot \Big(2 f^2 v \nabla v + 2 f v^2 \nabla f \Big) \Big|\\
&\le 2 \Big| \int_H f v \omega_1 \nabla \omega_1 \cdot (f \nabla v) \Big| + 2 \Big| \int_H f v \omega_1 \nabla \omega_1 \cdot (v \nabla f) \Big|\\
&\le 2 \Big(\int_H f^2 v^2 \omega_1^2 |\nabla \omega_1|^2 \Big)^{1/2} \Big[\Big(\int_H f^2 |\nabla v|^2 \Big)^{1/2} + \Big(\int_H v^2 |\nabla f|^2 \Big)^{1/2} \Big]\\
&= 2 (B^{1/2} + D^{1/2}) \Big(\int_H \omega_1^2 f^2 v^2 |\nabla \omega_1|^2 \Big)^{1/2}.
\end{aligned}$$

Then as $v = 0$ on $(I^*)^c$ and $|\omega_1| \le L$ on I^*,

$$F \le 2(B^{1/2} + D^{1/2})LF^{1/2},$$

and dividing both sides by $F^{1/2}$,

$$F \le c_6 L^2 (B^{1/2} + D^{1/2})^2 \le c_7 L^2 (B + D). \tag{4.15}$$

We now bound D. We have, using Lemma 4.8

$$\begin{aligned} D &= \int_H f^2 \nabla v \cdot \nabla v = \int_H \nabla(f^2 v) \cdot \nabla v - \int_H 2fv \nabla f \cdot \nabla v \\ &= \int_I f^2 v - \int_H 2fv\nabla f \cdot \nabla v \\ &\le \int_I f^2 v + c_8 \Big(\int_{I^*} f^2 |\nabla v|^2 \Big)^{1/2} \Big(\int_{I^*} v^2 |\nabla f|^2 \Big)^{1/2} \\ &\le c_9 V_1 C + c_9 B^{1/2} D^{1/2}. \end{aligned}$$

Therefore by Lemma 4.9

$$D \le c_{10}(B + c_{11} V_1 C).$$

Since

$$B \le V_1^2 \int_{I^*} |\nabla f|^2 = V_1^2 E,$$

we obtain

$$A \le F \le c_{12} L^2 (B + D) \le c_{13} L^2 (V_1 C + V_1^2 E).$$

We have V_1/V_0 bounded independently of s. Since $L \le c_{14} h(s/R)^\theta$, our result follows using (4.14). □

Corollary 4.11. *Let (G, E, a) satisfy $PHI(\beta)$ and FVG for some $\alpha_1 > \beta$. Then G and G_C satisfy CS(β) for some $\theta > 0$.*

Proof. Let $x_0 \in G_C$ and $R > 0$, let ω be given by (4.10), and let $\varphi = 1 \wedge (h(x_0, R)^{-1}\omega)$. It is clear that $\varphi = 1$ on $B(x_0, R/\sigma)$ and that $\varphi = 0$ outside $B(x_0, R)$. By Lemma 4.6 φ satisfies Definition 3.1(c).

It remains to prove that φ satisfies the weighted Sobolev inequality (3.1). Let $I = B(x, s)$ with $s \le R$. If I^* satisfies either (4.11) or (4.12), then, using Proposition 4.10, we are done. If I^* fails to satisfy both, then I^* must intersect both $B(x_0, R/2\sigma)$ and $B(x_0, R/\sigma)^c$, and so $s \ge R/8\sigma$.

We use Lemma 2.7 to cover I with balls $B_i = B(x_i, c_1 R)$, where $c_1 \in (0, 1/4\sigma)$ has been chosen small enough so that each B_i^* satisfies at least one of (4.11) or (4.12). We can then apply (4.13) with I replaced by each ball B_i: writing $s' = c_1 R$ we have

$$\int_{B_i} f^2 |\nabla \varphi|^2 \le c_2 (s'/R)^{2\theta} \Big(\int_{B_i^*} |\nabla f|^2 + \psi(s')^{-1} \int_{B_i^*} f^2 \Big).$$

We then sum over i. Since no point of I^* is in more than M (not depending on x_0, R or h) of the B_i^*, and $s/c_1 \le s' \le s$, we obtain (3.1) for I. □

It remains to remove the hypothesis (FVG), which we do by a trick involving products.

Theorem 4.12. *Suppose (G, E, a) satisfies $PHI(\beta)$. Then there exists $\theta > 0$ such that G satisfies CS(β).*

Proof. Choose $\alpha_1 > \beta$. By Theorem 2 of [B1] there exists a graph (G', E', a') satisfying (p_0), (EHI) and (E_β) and with $V'(x,r) \approx r^{\alpha_1}$, and $a'_{x'y'}$ equal to 0 or 1. We may also assume that $\{x', x'\} \in E'$ for all $x' \in G'$. Write $p'_n(x', y')$ for the transition density of the random walk X' on (G', E'). Since $\{y', y'\} \in E'$, we have $p'_{n+1}(x', y') \geq c_1 p'_n(x', y')$ for some $c_1 > 0$. By Theorem 1.2 (G', E', a') satisfies PHI(β), and also $p'_n(x', y')$ satisfies the estimates (1.12) and (1.13). Further, the lower bound (1.13) holds for p'_n, and not just $p'_n + p'_{n+1}$.

We now define the product graph $\widetilde{G} = G \times G'$, and say that $\{(x, y), (x', y')\} \in \widetilde{E}$ if and only if $\{x, x'\} \in E$ and $\{y, y'\} \in E'$. Define weights on $(\widetilde{G}, \widetilde{E})$ by

$$\widetilde{a}_{(x,x'),(y,y')} = \begin{cases} a_{xy} a'_{x'y'} & \text{if } \{x, x'\}, \{y, y'\} \in E', \\ 0 & \text{otherwise.} \end{cases}$$

The graph $\widetilde{G}$ is connected, since G' is not bipartite.

Write $\widetilde{V}$ and $\widetilde{d}$ for the volume and distance, respectively, on $\widetilde{G}$. It is easy to check that

$$\widetilde{V}((x, x'), R) \approx V(x, R) V'(x', R) \approx R^{\alpha_1} V(x, R),$$

so that $(\widetilde{G}, \widetilde{E})$ satisfies (VD) and (FVG) with the exponent $\alpha_1 > \beta$. Because the weights on $(\widetilde{G}, \widetilde{E})$ are given as products, the Markov chain on $\widetilde{G} = G \times G'$ is given by $\widetilde{X} = (X, X')$, where X is the Markov chain on G with weights a and X' is the Markov chain on G'; moreover X and X' are independent. It follows that the transition densities for $\widetilde{X}$ are given as a product: we have $\widetilde{p}_n((x, x'), (y, y')) = p_n(x, y) p'_n(x', y')$. From Theorem 1.2 we deduce that

$$\begin{aligned} \widetilde{p}_n((x, x'), (y, y')) &\leq c_2 V(x, n^{1/\beta})^{-1} V'(x', n^{1/\beta})^{-1} \\ &\qquad \times \exp\Big(- c_3 \Big(\frac{d(x, y)^\beta}{n} \Big)^{1/(\beta-1)} - c_3 \Big(\frac{d'(x', y')^\beta}{n} \Big)^{1/(\beta-1)} \Big) \\ &\leq c_4 \widetilde{V}((x, x'), n^{1/\beta})^{-1} \exp\Big(- c_5 \Big(\frac{\widetilde{d}((x, x'), (y, y'))^\beta}{n} \Big)^{1/(\beta-1)} \Big). \end{aligned}$$

Similarly we have if $n \geq \widetilde{d}((x, x'), (y, y')) \vee 1$,

$$\begin{aligned} \widetilde{p}_n((x, x'), &(y, y')) + \widetilde{p}_{n+1}((x, x'), (y, y')) \\ &= p_n(x, y) p'_n(x', y') + p_{n+1}(x, y) p'_{n+1}(x', y') \\ &\geq c_6 (p_n(x, y) + p_{n+1}(x, y)) p'_n(x', y') \\ &\geq c_7 \widetilde{V}((x, x'), n^{1/\beta})^{-1} \exp\Big(- c_8 \Big(\frac{\widetilde{d}((x, x'), (y, y'))^\beta}{n} \Big)^{1/(\beta-1)} \Big). \end{aligned}$$

Thus $(\widetilde{G}, \widetilde{E}, \widetilde{a})$ satisfies (1.12) and (1.13), and so by Theorem 1.2, it satisfies PHI(β).

Therefore by Corollary 4.11 $\widetilde{G}_C$ satisfies CS(β) for some $\theta \in (0, 1)$. Hence by Proposition 3.4, $\widetilde{G}$ also satisfies CS(β). We now show that this implies that G satisfies CS(β).

Fix a point $0' \in G'$. Let $R \geq c_9$, and $x_0 \in G$. Using Remark 3.2 we can find a cut-off function $\widetilde{\varphi}$ which is Hölder continuous of order θ and such that $\varphi = 0$ outside $(B(x_0, R) \times B'(0', R))^c$ and $\widetilde{\varphi} = 1$ in $B(x_0, R/2) \times B'(0', R/2)$. Let $A' = B'(0', R/3)$, and let

$$\varphi(x) = \mu'_0(A')^{-1} \sum_{x' \in A'} \widetilde{\varphi}(x, x') \mu'_0(x').$$

It is clear that φ satisfies conditions (a)–(c) for a cut-off function for $B(x_0, R)$. To verify (d), let $c_{10} \le s \le R/3$, $I = B(x, s)$ for some $x \in G$, and $f : I^* \to \mathbb{R}$. Extend f to $\widetilde{f}$ on $I^* \times G'$ by setting $\widetilde{f}(x, x') = f(x)$. Using Lemma 2.7 we can cover A' by balls $J_i = B'(x'_i, s)$, $i = 1, \dots, n$, so that no point is in more than M of the sets J_i^*. We have

$$\begin{aligned}
&\sum_{x \in I} f(x)^2 \sum_{y \sim x} a_{xy} |\varphi(x) - \varphi(y)|^2 \\
&\quad \le \mu'_0(A')^{-1} \sum_{x \in I} \sum_{y \sim x} \sum_{x' \in A'} a_{xy} f(x)^2 |\widetilde{\varphi}(x, x') - \widetilde{\varphi}(y, x')|^2 \mu'_0(x') \\
&\quad \le \mu'_0(A')^{-1} \sum_{x \in I} \sum_{y \sim x} \sum_{x' \in A'} \sum_{y' \sim x'} a_{xy} a'_{x'y'} \widetilde{f}(x, x')^2 |\widetilde{\varphi}(x, x') - \widetilde{\varphi}(y, y')|^2 \mu'_0(x') \\
&\quad \le \mu'_0(A')^{-1} \sum_{i=1}^{n} \sum_{(x,x') \in I \times J_i} \sum_{(y,y') \sim (x,x')} a_{xy} a'_{x'y'} \widetilde{f}(x, x')^2 |\widetilde{\varphi}(x, x') - \widetilde{\varphi}(y, y')|^2 \mu'_0(x').
\end{aligned}$$

Since $V'(x, r) \approx r^{\alpha_1}$ and a'_{xy} is 0 or 1, there exists c_{11} such that $\mu'_0(x') \le c_{11}$ for all x'. For each i we use the weighted Sobolev inequality for $\widetilde{\varphi}$ (where we use Remark 3.2 to replace balls in $\widetilde{G}$ by products of balls):

$$\begin{aligned}
&\sum_{(x,x') \in I \times J_i} \sum_{(y,y') \sim (x,x')} a_{xy} a'_{x'y'} \widetilde{f}(x, x')^2 |\widetilde{\varphi}(x, x') - \varphi(y, y')|^2 \\
&\qquad \le c_{12} (s/R)^{2\theta} \Big(\sum_{(x,x') \in I^* \times J_i^*} \sum_{(y,y') \sim (x,x')} a_{xy} a'_{x'y'} |\widetilde{f}(x, x') - \widetilde{f}(y, y')|^2 \\
&\qquad\qquad + s^{-\beta} \sum_{(x,x') \in I^* \times J_i^*} \widetilde{f}(x, x')^2 \mu_0(x) \mu'_0(x') \Big) \\
&\qquad = c_{12} (s/R)^{2\theta} \mu'_0(A') \Big(\sum_{x \in I^*} \sum_{y \sim x} a_{xy} |f(x) - f(y)|^2 + s^{-\beta} \sum_{x \in I^*} f(x)^2 \mu_0(x) \Big).
\end{aligned}$$

Summing over i we obtain

$$\begin{aligned}
&\sum_{x \in I} f(x)^2 \sum_{y \sim x} a_{xy} |\varphi(x) - \varphi(y)|^2 \\
&\qquad \le c_{13} M (s/R)^{2\theta} \Big(\sum_{x \in I^*} \sum_{y \sim x} a_{xy} |f(x) - f(y)|^2 + s^{-\beta} \sum_{x \in I^*} f(x)^2 \mu_0(x) \Big),
\end{aligned}$$

which is the weighted Sobolev inequality for φ. □

5. Sobolev inequalities and elliptic Harnack inequality

In this section we will prove that the conditions (VD), $PI(\beta)$ and CS(β) imply the elliptic Harnack inequality. We begin with the connection between these conditions and the effective resistance of annuli.

Lemma 5.1. *Let G_C be the cable system of the graph G which satisfies (p_0) and (VD).*

(a) If $PI(\beta)$ holds, then for any $x_0 \in G_C$, $R \ge 1$,

$$R_{\text{eff}}(B(x_0, R), B(x_0, 2R)^c) \le c_1 \frac{R^\beta}{V(x_0, R)}.$$

(b) If CS(β) holds, then for any $x_0 \in G_C$, $R \geq 1$,

$$R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c) \geq c_2 \frac{R^\beta}{V(x_0,2R)}.$$

(c) If G_C satisfies CS(β_1) and $PI(\beta_2)$, then $\beta_1 \leq \beta_2$.

Proof. (a) Let f be the function which attains the minimum on the right hand side of (1.10) when $A = B(x_0,R)$ and $B = B(x_0,2R)^c$. Let

$$\overline{f} = \int_{B(x_0,3R)} f d\mu / \mu(B(x_0,3R)).$$

Choose y_0 so that $d(x_0,y_0) = 5R/2$. Then by (2.1) we have $V(y_0,R/2) \geq c_3 V(x_0,R)$. Depending on whether $\overline{f} \geq 1/2$ or $\overline{f} < 1/2$, $|f - \overline{f}| \geq 1/2$ on either $B(x_0,R)$ or $B(y_0,R/2)$, and then using $PI(\beta)$ we have

$$\begin{aligned} V(x_0,R) &\leq c_4 \int_{B(x_0,3R)} (f-\overline{f})^2 d\mu \leq c_5 R^\beta \int_{B(x_0,3R)} |\nabla f|^2 d\mu \\ &= c_5 R^\beta R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c)^{-1}. \end{aligned}$$

(b) Let φ be a cut-off function for $B(x_0,R)$ given by CS(β). Then taking $f \equiv 1$ and $I = B(x_0,R)$ in (3.1) we obtain

$$R_{\text{eff}}(B(x_0,R/2), B(x_0,R)^c)^{-1} \leq \int_I |\nabla\varphi|^2 d\mu \leq c_6 R^{-\beta} \int_{I^*} f^2 d\mu \leq c_7 \frac{V(x_0,R)}{R^\beta}.$$

(c) is immediate from (a) and (b). □

Let us now assume G_C satisfies (VD), $PI(\beta)$ and CS(β). Using CS(β) we will obtain weighted Sobolev and Poincaré inequalities which we can then use to drive the Moser iteration.

Fix $x_0 \in G_C$, and let $R \geq 1$. Let φ be a cut-off function given by CS(β). Write

$$\gamma = 1 + R^\beta |\nabla\varphi|^2.$$

Proposition 5.2. *Let $I = B(x,s)$ with $s \leq R$. Suppose f and its gradient are square integrable over I^*. Let $f_A = \mu(A)^{-1}\int_A f$.*

(a) We have

$$\int_I f^2\gamma \leq c_1 (s/R)^{2\theta} R^\beta \Big(\int_{I^*} |\nabla f|^2 + \psi(s)^{-1} \int_{I^*} f^2 \Big). \tag{5.1}$$

(b) We have

$$\int_I (f - f_{I^*})^2 \gamma \leq c_2 (s/R)^{2\theta} R^\beta \int_{I^*} |\nabla f|^2. \tag{5.2}$$

(c) If $J \subset I$, then

$$\int_J f^2\gamma \leq c_3 \Big(R^\beta (s/R)^{2\theta} \Big) \int_{I^*} |\nabla f|^2 + \mu(J)^{-1} \Big(\int_J |f|\gamma \Big)^2.$$

(d) We have

$$\int_{B(x_0,R)} \gamma \leq c_4 V(x_0,R).$$

Proof. (a) Using the definition of γ and (3.1),

$$\begin{aligned}\int_I f^2\gamma &= \int_I f^2 + R^\beta \int_I f^2|\nabla\varphi|^2 \\ &\le \int_I f^2 + c_5(s/R)^{2\theta}R^\beta \int_{I^*}|\nabla f|^2 + c_5(s/R)^{2\theta}R^\beta\psi(s)^{-1}\int_{I^*} f^2.\end{aligned}$$

Since $\beta \ge 2 \ge 2\theta$ and $s \le R$, this implies (a).

For (b), applying (5.1) to $f - f_{I^*}$ we have

$$\int_I (f-f_{I^*})^2\gamma \le c_6(s/R)^{2\theta}R^\beta\Big(\int_{I^*}|\nabla f|^2 + \psi(s)^{-1}\int_{I^*}(f-f_{I^*})^2\Big). \tag{5.3}$$

Using $PI(\beta)$ applied to the ball I^* we have

$$\int_{I^*}(f-f_{I^*})^2 \le c_7\psi(s)\int_{I^*}|\nabla f|^2. \tag{5.4}$$

Substituting (5.4) into (5.3) gives (5.2).

(c) Now let $b = \int_J f\gamma / \int_J \gamma$. Then

$$\begin{aligned}\int_J f^2\gamma &= \int_J (f-b)^2\gamma + b^2\int_J\gamma \\ &\le \int_J (f-f_{I^*})^2\gamma + \Big(\int_J\gamma\Big)^{-1}\Big(\int_J f\gamma\Big)^2. \\ &\le \int_I (f-f_{I^*})^2\gamma + \mu(J)^{-1}\Big(\int_J |f|\gamma\Big)^2.\end{aligned} \tag{5.5}$$

Using (5.2) to bound the first term in (5.5) completes the proof of (c).

(d) follows from (a) by taking $s = R$ and $f = 1$ and using (VD). □

Our next result is a weighted Nash inequality. For any set $J \subset G_C$ set

$$J^s = \{y : d(y,J) \le s\}.$$

Proposition 5.3. *Let $s \le R$ and $J \subset B(x_0,R)$ be a finite union of balls of radius s. Suppose the gradient of f is square integrable over J^s and $\int_{J^s} f^2\gamma < \infty$. There exist $c_1 < \infty$ and $\alpha_3 \in (0,1)$ such that*

$$\begin{aligned}&\mu(J)^{-1}\int_J f^2\gamma \\ &\quad\le c_1\Big[R^\beta\mu(J)^{-1}\int_{J^s}|\nabla f|^2 + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2\gamma\Big]^{1-\alpha_3}\Big[\mu(J)^{-1}\int_J |f|\gamma\Big]^{2\alpha_3}.\end{aligned}$$

Proof. Suppose that $0 < t < s$. Using Lemma 2.7 we can cover J by balls $B(x_i,t)$ with $x_i \in J$ so that any point of J^s is in at most M of the balls $B(x_i,2t)$. Set $B_i = B(x_i,t)\cap J$, $B_i^* = B(x_i,2t)$. Then $\bigcup_i B_i = J$, $\bigcup_i B_i^* \subset J^s$, and $\sum \mu(B_i^*) \le M\mu(J^s)$.

As J is a union of balls, for each i there exists y_i so that $d(x_i,y_i) = t/2$ and $B(y_i,t/2) \subset J$. Then by (2.1)

$$\frac{\mu(J)}{\mu(B_i)} \le \frac{\mu(J)}{\mu(B(y_i,t/2))} \le c_2\left(\frac{R}{t}\right)^{\alpha_2}. \tag{5.6}$$

By Proposition 5.2(c), and (5.6)

$$\begin{aligned}\int_J f^2\gamma &\le \sum_i \int_{B_i} f^2\gamma\\ &\le c_3(t/R)^{2\theta}R^\beta \sum_i \int_{B_i^*}|\nabla f|^2 + \sum_i \frac{1}{\mu(B_i)}\Big(\int_{B_i}|f|\gamma\Big)^2\\ &\le c_4(t/R)^{2\theta}R^\beta M\int_{J^*}|\nabla f|^2 + c_5(R/t)^{\alpha_2}\mu(J)^{-1}\Big(\sum_i\int_{B_i}|f|\gamma\Big)^2\\ &\le c_6(t/R)^{2\theta}R^\beta \int_{J^*}|\nabla f|^2 + c_7(R/t)^{\alpha_2}\mu(J)^{-1}\Big(\int_J|f|\gamma\Big)^2.\end{aligned}$$

Hence

$$\mu(J)^{-1}\int_J f^2\gamma \le c_8[(t/R)^{2\theta}A + (R/t)^{\alpha_2}B], \tag{5.7}$$

where

$$A = \Big[R^\beta\mu(J)^{-1}\int_{J^*}|\nabla f|^2 + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2\gamma\Big],\quad B = \Big[\mu(J)^{-1}\int_J |f|\gamma\Big]^2.$$

If $t \ge s$, (5.7) is obvious.

We choose t so that the two terms on the right hand side of (5.7) are equal. Thus $(t/R)^{2\theta+\alpha_2} = B/A$, and substituting this into (5.7) completes the proof, with $\alpha_3 = 2\theta/(2\theta+\alpha_2)$. Note that if $\theta = 1$ and $\alpha_2 = d$, we obtain the powers in the standard Nash inequality. □

We now use this to derive a weighted Sobolev inequality.

Theorem 5.4. *Let s, J, and f be as in Proposition 5.3. There exist $\kappa > 1$ and $c_1 < \infty$ such that*

$$\Big(\mu(J)^{-1}\int_J |f|^{2\kappa}\gamma\Big)^{1/\kappa} \le c_1\Big[R^\beta\mu(J)^{-1}\int_{J^s}|\nabla f|^2 + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2\gamma\Big]. \tag{5.8}$$

Proof. The proof is similar to the one in [BB3], but we correct some algebraic errors in the proof there. Since $|\nabla(f^+)| \le |\nabla f|$ a.e. and $|f| \le f^+ + f^-$, it suffices to consider nonnegative f. Write

$$\begin{aligned}A_0(f) &= \mu(J)^{-1}R^\beta\int_{J^s}|\nabla f|^2 + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2\gamma,\\ B_0(f) &= \Big(\mu(J)^{-1}\int_J|f|\gamma\Big)^2.\end{aligned}$$

Multiplying f by $A_0(f)^{-1/2}$, it is enough to prove

$$\int_J |f|^{2\kappa}\gamma \le c_1\mu(J) \qquad \text{if } A_0(f) = 1. \tag{5.9}$$

Set

$$p_n = \mu(J)^{-1}\int_{\{f\ge 2^n\}\cap J}\gamma.$$

Let $f_n = (f\wedge 2^{n+1}) - (f\wedge 2^n)$; note that $f_n \le 2^n$, that $f_n = 2^n$ on $J\cap\{f \ge 2^{n+1}\}$, and that $f_n = 0$ on $\{f < 2^n\}$. Therefore

$$\mu(J)^{-1}\int_J f_n\gamma = \mu(J)^{-1}\int_{\{f\ge 2^n\}\cap J} f_n\gamma \le \mu(J)^{-1}\int_{\{f\ge 2^n\}\cap J} 2^n\gamma = 2^n p_n, \tag{5.10}$$

while

$$\mu(J)^{-1}\int_J f_n^2\gamma \ge \mu(J)^{-1}\int_{\{f\ge 2^{n+1}\}\cap J} f_n^2\gamma = 2^{2n}p_{n+1}. \tag{5.11}$$

Since $\int_J f_n^2\gamma \le \int_J f^2\gamma$ and $\int_{J^*}|\nabla f_n|^2 \le \int_{J^*}|\nabla f|^2$, we have $A_0(f_n)\le A_0(f)$. As $s\le R$, from (5.11) we deduce

$$p_n \le c_2 2^{-2(n-1)}(s/R)^{-2\theta}\mu(J)^{-1}\int_J f_{n-1}^2\gamma \le c_3 2^{-2n}A_0(f_{n-1}) \le c_4 2^{-2n}. \tag{5.12}$$

Applying Proposition 5.3 to f_n we have, using the fact that $A_0(f)\le 1$ and $s\le R$,

$$\mu(J)^{-1}\int_J f_n^2\gamma \le c_5 B_0(f_n)^{\alpha_3}.$$

Using this, (5.10), and (5.11) we obtain

$$2^{2n}p_{n+1} \le c_6\mu(J)^{-1}\int_J f_n^2\gamma \le c_7\Big(\mu(J)^{-1}\int_J f_n\gamma\Big)^{2\alpha_3} \le c_8(2^n p_n)^{2\alpha_3}. \tag{5.13}$$

By (5.12) $2^{2n}p_n$ is bounded. Using (5.12), (5.13), and the fact that $\alpha_3<1$,

$$2^{2n}p_{n+1}\le c_8(2^np_n)^{2\alpha_3}\le c_9 2^{-2\alpha_3 n}.$$

Hence

$$p_n \le c_{10}2^{-n(2+2\alpha_3)}.$$

Since

$$\mu(J)^{-1}\int_J |f|^{2\kappa}\gamma \le c_{11}\sum_{n=0}^{\infty}2^{2n\kappa}p_n,$$

we deduce (5.9) if $\kappa\in(1,1+\alpha_3)$. □

The following result is proved exactly as in Moser [M1], Lemma 4.

Proposition 5.5. *Let $D\subset G_C$ be a domain in G_C, and suppose u is $\mathcal{L}$-harmonic in D, $v=u^k$, where $k\in\mathbb{R}$, $k\ne 1/2$, and η is supported in D. Suppose the gradient of η is square integrable over D. Then*

$$\int_D \eta^2|\nabla v|^2 \le c_1\Big(\frac{2k}{2k-1}\Big)^2\int_D|\nabla\eta|^2v^2.$$

Corollary 5.6. *Let D, u, k, v be as above. Suppose $\eta>0$ on D, $\eta=0$ on ∂D, η is continuous on $\overline{D}$, and the gradient of η is square integrable over D. Then*

$$\int_D \eta^2|\nabla v|^2d\mu \le c_1\Big(\frac{2k}{2k-1}\Big)^2\int_D|\nabla\eta|^2v^2d\mu.$$

Proof. Since η is continuous on $\overline{D}$, 0 on ∂D and positive in D, then $\eta_\varepsilon=(\eta-\varepsilon)^+$ will be 0 in a neighborhood of ∂D. Applying Proposition 5.5 we obtain

$$\int_D \eta_\varepsilon^2|\nabla v|^2 \le c_2\Big(\frac{2k}{2k-1}\Big)^2\int_D|\nabla\eta_\varepsilon|^2v^2 \le c_2\Big(\frac{2k}{2k-1}\Big)^2\int_D|\nabla\eta|^2v^2.$$

Now let $\varepsilon\to 0$. □

Let u be $\mathcal{L}$-harmonic and nonnegative in $B(x_0,4R)$. By looking at $u+\varepsilon$ and letting $\varepsilon\downarrow 0$ we may without loss of generality suppose u is strictly positive. The usual Harnack inequality in $\mathbb{R}$, combined with the local Harnack inequality for graphs, implies that u and u^{-1} are continuous and bounded (by a constant depending on u, x, r, and R) on any ball $\overline{B(x,r)}$ in $B(x_0,4R)$. By Proposition 5.5, the gradient of powers of u will be square integrable over bounded subsets of G_C. We take φ to be a cut-off function for $B(x_0, R)$.

We will need the following estimate.

Proposition 5.7. *Let $w = \log u$. There exists c_1 such that*

$$\int_{B(x_0,2R)} |\nabla w|^2 \le c_1 \frac{V(x_0, R)}{R^\beta}.$$

Proof. Again, this is essentially Moser's proof. Let $\varphi_1(x)$ be a cut-off function given by CS(β) for the ball $B(x_0, 4R)$. So

$$\int_{B(x_0,2R)} |\nabla w|^2 \le c_2 \int \varphi_1^2 |\nabla w|^2.$$

By Lemma 2.3

$$\begin{aligned} 0 &= \int \frac{\varphi_1^2}{u} \mathcal{L} u \, d\mu = -\int \nabla(\varphi_1^2/u) \cdot \nabla u \, d\mu \\ &= -\int \Big(2\frac{\varphi_1}{u} \nabla\varphi_1 \cdot \nabla u - \frac{\varphi_1^2}{u^2} |\nabla u|^2 \Big) d\mu \\ &= -2 \int \varphi_1 \nabla\varphi_1 \cdot \nabla w \, d\mu + \int \varphi_1^2 |\nabla w|^2 d\mu. \end{aligned}$$

So

$$\int \varphi_1^2 |\nabla w|^2 d\mu \le c_3 \Big| \int \varphi_1 \nabla\varphi_1 \cdot \nabla w \, d\mu \Big| \le c_3 \Big(\int |\nabla\varphi_1|^2 d\mu \Big)^{1/2} \Big(\int \varphi_1^2 |\nabla w|^2 d\mu \Big)^{1/2}.$$

Dividing and squaring,

$$\int \varphi_1^2 |\nabla w|^2 d\mu \le c_4 \int |\nabla \varphi_1|^2 d\mu.$$

Finally, using CS(β) in $B(x_0, 4R)$ with $f = 1$ and (VD) we deduce that

$$\int_{B(x_0,4R)} |\nabla\varphi_1|^2 d\mu \le c_5 R^{-\beta} V(x_0, R).$$

$\square$

Proposition 5.8. *Let v be either u or u^{-1}. There exists c_1 such that if $B(x, 2r) \subset B(x_0, 4R)$ and $0 < q < 2$, then*

$$\sup_{B(x,r/2)} v^{2q} \le c_1 V(x, 2r)^{-1} \int_{B(x,2r)} (r^\beta |\nabla v^q|^2 + v^{2q}) d\mu.$$

Proof. If $r < 1$ this follows easily from the local Harnack inequality and the linearity of u on each cable, so suppose $r \ge 1$. Let φ_0 be a (regularized) cut-off function given by CS(β) for $B(x, r)$. Let $h_n = 1 - 2^{-n}$, $0 \le n \le \infty$, so that $0 = h_0 < h_\infty = 1$. For $k \ge 0$ set

$$\begin{aligned} \varphi_k(x) &= (\varphi_0(x) - h_k)^+, \\ \gamma_0(x) &= 1 + r^\beta |\nabla\varphi_0(x)|^2. \end{aligned}$$

Set $A_k = \{x : \varphi_0(x) > h_k\}$, and note that $B(x, r/2) \subset A_{n_0} \subset A_0 \subset B(x, r)$ for every n_0. We therefore have, writing V for $V(x, r)$,

$$c_2 V \le \mu(A_k) \le V, \quad k \ge 0.$$

The Hölder condition on φ_0 given by Definition 3.1(c) implies that if $x \in A_{k+1}$ and $y \in A_k^c$, then $d(x,y) \geq c_3 r 2^{-k/\theta}$. Set $s_k = \frac{1}{2} c_3 r 2^{-k/\theta}$, and note that $\varphi_k > c_4 2^{-k}$ on $A_{k+1}^{s_k}$. Let $\{B_i\}$ be a cover of A_{k+1} by balls of radius $s_k/2$, and let $J_{k+1} = \bigcup B_i$. Write $J'_{k+1} = J_{k+1}^{s_k/2}$ and note that $A_{k+1} \subset J_{k+1} \subset J'_{k+1} \subset A_{k+1}^{s_k}$.

From Theorem 5.4 with $f = v^p$ and s replaced by $s_k/2$,

$$\begin{aligned}\Big(V^{-1}\int_{A_{k+1}} f^{2\kappa}\gamma_0 d\mu\Big)^{1/\kappa} &\leq \Big(V^{-1}\int_{J_{k+1}} f^{2\kappa}\gamma_0 d\mu\Big)^{1/\kappa} \\ &\leq c_5 V^{-1}\Big[r^\beta \int_{J'_{k+1}} |\nabla f|^2 d\mu + (r/s_k)^{2\theta}\int_{J'_{k+1}} f^2\gamma_0 d\mu\Big] \\ &\leq c_6 V^{-1}\Big[r^\beta \int_{A_{k+1}^{s_k}} |\nabla f|^2 d\mu + 2^{2k}\int_{A_k} f^2\gamma_0 d\mu\Big].\end{aligned} \tag{5.14}$$

We now control the first term in (5.14) using Corollary 5.6.

$$\begin{aligned} r^\beta \int_{A_{k+1}^{s_k}} |\nabla f|^2 &\leq r^\beta (c_7 2^{-k})^{-2}\int_{A_{k+1}^{s_k}} \varphi_k^2 |\nabla f|^2 \\ &\leq c_8 2^{2k} r^\beta \int_{A_k} \varphi_k^2|\nabla f|^2 \\ &\leq c_9 2^{2k} r^\beta \Big(\frac{2p}{2p-1}\Big)^2 \int_{A_k} f^2 |\nabla\varphi_k|^2 \\ &\leq c_{10} 2^{2k}\Big(\frac{2p}{2p-1}\Big)^2\int_{A_k} f^2\gamma_0.\end{aligned}$$

We therefore deduce that

$$\Big(V^{-1}\int_{A_{k+1}} f^{2\kappa}\gamma_0\Big)^{1/\kappa} \leq c_{11}\Big(\frac{2p}{2p-1}\Big)^2 2^{2k} V^{-1}\int_{A_k} f^2\gamma_0. \tag{5.15}$$

Choose $q' > 0$ such that $\inf_{m\in\mathbb{Z}} |q'\kappa^m - \frac{1}{2}| \geq c_{12} > 0$. Suppose first that $q_0 = q'\kappa^{-i}$ for some i. Let $p_n = 2q_0\kappa^n$ for $n \geq 0$, and write

$$\Psi_k = \Big[\mu(A_k)^{-1}\int_{A_k} v^{p_k}\gamma_0\Big]^{1/p_k}.$$

Note that $p_{k+1}/2\kappa = p_k/2$. Applying (5.15) to $f = v^{p_{k+1}/(2\kappa)} = v^{p_k/2}$ we have

$$\begin{aligned}\Psi_{k+1}^{p_{k+1}/\kappa} &= \Big(\mu(A_{k+1})^{-1}\int_{A_{k+1}} v^{p_{k+1}}\gamma_0\Big)^{1/\kappa} \\ &\leq c_{13} 2^{2k}\Big(\mu(A_k)^{-1}\int_{A_k} v^{p_k}\gamma_0\Big) = c_{13}2^{2k}\Psi_k^{p_k},\end{aligned}$$

or

$$\Psi_{k+1} \leq \Big(c_{13}2^{2k}\Big)^{1/p_k}\Psi_k.$$

Hence for every m

$$\log\Psi_m \leq \log\Psi_0 + \sum_{k=1}^m p_k^{-1}\log(c_{13}2^{2k}). \tag{5.16}$$

As the sum in (5.16) converges and $\sup_{B(x,r/2)} v \leq \limsup_{m\to\infty}\Psi_m$, we have

$$\sup_{B(x,r/2)} v \leq c_{14}\Psi_0 \leq c_{15}\Big(V^{-1}\int_{B(x,r)} v^{2q_0}\gamma_0\Big)^{1/(2q_0)}. \tag{5.17}$$

Now let $q \in (0,2)$. We can take $q_0 = q'\kappa^{-i} < q$. Then by Hölder's inequality and Proposition 5.2(d)

$$V^{-1}\int_{B(x,r)} v^{2q_0}\gamma_0 \le \Big(V^{-1}\int_{B(x,r)} v^{2q}\gamma_0\Big)^{q_0/q}\Big(V^{-1}\int_{B(x,r)}\gamma_0\Big)^{1-q_0/q}$$
$$\le c_{16}\Big(V^{-1}\int_{B(x,r)} v^{2q}\gamma_0\Big)^{q_0/q}.$$

Thus

$$\sup_{B(x,r/2)} v^{2q} \le c_{17}V^{-1}\int_{B(x,r)} v^{2q}\gamma_0.$$

By Proposition 5.2(a) and (VD) this implies

$$\sup_{B(x,r/2)} v^{2q} \le c_{18}V(x,2r)^{-1}\int_{B(x,2r)} (r^\beta|\nabla v^q|^2 + v^{2q}).$$

□

We now follow the ideas of Moser [M3] to link the L^∞ norms of u and u^{-1}. Recall that φ is a cut-off function for $B(x_0,R)$, and let

$$\gamma = 1 + R^\beta|\nabla\varphi^2|.$$

We define

$$Q(t) = \{x : \varphi(x) > t\}, \quad 0 < t < 1,$$

and write $Q(1)$ for the interior of $\{x : \varphi(x) \ge 1\}$.

Corollary 5.9. *Let $1 > s > t > 0$. There exists $\zeta_1 > 2$ such that if $0 < q < \frac{1}{3}$,*

$$\sup_{Q(s)} v^{2q} \le c_1(s-t)^{-\zeta_1}V(x_0,R)^{-1}\int_{Q(t)} v^{2q}\gamma. \tag{5.18}$$

Proof. By the maximum principle the supremum of v^{2q} in $\overline{Q(s)}$ is attained at a point $x' \in \partial Q(s)$. Let $\eta = \frac{1}{4}(s-t)$, $s' = s - 2\eta$. By the Hölder continuity of φ the sets $Q(s)$ and $Q(s')^c$ are separated by a distance of at least $\xi = c_2R(s-t)^{1/\theta}$, so that $B(x',\xi) \subset Q(s')$. By Proposition 5.8,

$$\sup_{B(x',\xi/4)} v^{2q} \le c_3\xi^\beta V(x',\xi)^{-1}\int_{B(x',\xi)} |\nabla v^q|^2 + c_3V(x',\xi)^{-1}\int_{B(x',\xi)} v^{2q}. \tag{5.19}$$

Note that by (2.1) we have

$$\frac{V(x_0,R)}{V(x',\xi)} \le c_4\Big(\frac{d(x',x_0)+R}{\xi}\Big)^{\alpha_2} \le c_5(s-t)^{-\alpha_2/\theta}.$$

Using (5.19),

$$\sup_{Q(s)} v^{2q} \le c_6\xi^\beta V(x',\xi)^{-1}\int_{Q(s')} |\nabla v^q|^2 + c_6V(x',\xi)^{-1}\int_{Q(s')} v^{2q}.$$

Let

$$\varphi_{st} = (s\wedge\varphi - t)^+$$

and observe that $|\nabla\varphi_{st}| \le |\nabla\varphi|$. Now $\varphi_{st} \ge c_7(s-t)$ on $Q(s')$, so we have, using Corollary 5.6,

$$\begin{aligned}\int_{Q(s')} |\nabla v^q|^2 &\le c_7(s-t)^{-2}\int_{Q(s')} |\nabla v^q|^2\varphi_{st}^2\\ &\le c_7(s-t)^{-2}\int_{Q(t)} |\nabla v^q|^2\varphi_{st}^2\\ &\le c_8(s-t)^{-2}\int_{Q(t)} |\nabla \varphi_{st}|^2 v^{2q}\\ &\le c_9(s-t)^{-2}R^{-\beta}\int_{Q(t)} v^{2q}\gamma.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{Q(s)} v^{2q} &\le c_{10}(\xi/R)^\beta(s-t)^{-2}V(x',\xi)^{-1}\int_{Q(t)} v^{2q}\gamma + c_{10}V(x',\xi)^{-1}\int_{Q(t)} v^{2q}\\ &\le c_{11}V(x',\xi)^{-1}(s-t)^{-2}\int_{Q(t)} v^{2q}\gamma\\ &\le c_{12}V(x_0,R)^{-1}(s-t)^{-2-\alpha_2/\theta}\int_{Q(t)} v^{2q}\gamma.\end{aligned}$$

So taking $\zeta_1 = 2+\alpha_2/\theta$ we obtain (5.18). □

Now let $w = \log u$, and write $\overline{w} = V(x_0,R)^{-1}\int_{B(x_0,R)} w\,d\mu$.

Corollary 5.10. *Let $1 \ge s > t > 0$. Then*

$$\int_{\{|w-\overline{w}|>A\}\cap Q(s)} \gamma \le \frac{c_1 V(x_0,R)}{A^2}.$$

Proof. By Chebyshev's inequality, Proposition 5.2(b) and Proposition 5.7

$$\begin{aligned}A^2\int_{\{|w-\overline{w}|>A\}\cap Q(s)} \gamma &\le \int_{\{|w-\overline{w}|>A\}\cap Q(s)} |w-\overline{w}|^2\gamma\\ &\le \int_{Q(s)} |w-\overline{w}|^2\gamma\\ &\le \int_{B(x_0,R)} |w-\overline{w}|^2\gamma\\ &\le c_2R^\beta\int_{B(x_0,2R)} |\nabla w|^2 \le c_3V(x_0,R).\end{aligned}$$

□

Without loss of generality, we multiply u by a constant so that

$$V(x_0,R)^{-1}\int_{B(x_0,R)} \log v = \overline{w} = 0.$$

Recall that v is either u or u^{-1} and define $\Phi(t) = \sup_{\overline{Q}(t)} \log v$.

Lemma 5.11. *Let $1 \ge s > t \ge \frac{1}{2}$. Then*

$$\Phi(s) \le \tfrac{3}{4}\Phi(t) + c_1(s-t)^{-\zeta_1}. \tag{5.20}$$

Proof. Fix t and write Φ for $\Phi(t)$. Let $c_2 > e$ satisfy $c_2 = 6\log c_2$. If $\Phi(t) \le c_2$, then

$$\Phi(s) \le \Phi(t) \le \tfrac{3}{4}\Phi(t) + \tfrac{1}{4}c_2,$$

so that (5.20) holds provided $c_1 2^{-\zeta_1} \ge c_2/4$.

Now suppose $\Phi > c_2$. From Proposition 5.2(d) we have $\int_{Q(t)} \gamma \le c_3 V(x_0, R)$. By Corollary 5.10 and the fact that $v^p \le e^{p\Phi}$ on $Q(t)$,

$$\begin{aligned}\int_{Q(t)} v^{2p}\gamma &= \int_{Q(t)\cap\{\log v \ge \Phi/2\}} v^{2p}\gamma + \int_{Q(t)\cap\{\log v < \Phi/2\}} v^{2p}\gamma \\ &\le e^{2p\Phi}\int_{Q(t)\cap\{\log v \ge \Phi/2\}} \gamma + e^{p\Phi}\int_{Q(t)\cap\{\log v < \Phi/2\}} \gamma \\ &\le \frac{4c_4 e^{2p\Phi}}{\Phi^2} V(x_0, R) + e^{p\Phi}\int_{Q(t)} \gamma \\ &\le c_5\Big(\frac{e^{2p\Phi}}{\Phi^2} + e^{p\Phi}\Big)V(x_0, R).\end{aligned}$$

Let $p = \frac{2}{\Phi}\log\Phi$, so that $e^{p\Phi} = \Phi^2$. As $\Phi > c_2$ we have $p < (2/c_2)\log c_2 = \frac{1}{3}$. So

$$V(x_0, R)^{-1}\int_{Q(t)} v^{2p}\gamma \le c_5 e^{p\Phi}\Big(1 + \frac{e^{p\Phi}}{\Phi^2}\Big) = 2c_5 e^{p\Phi}.$$

Therefore by Corollary 5.9,

$$\begin{aligned}\Phi(s) &= \frac{1}{2p}\log[\sup_{Q(s)} v^{2p}] \\ &\le \frac{1}{2p}\log\Big[c_6(s-t)^{-\zeta_1}V(x_0, R)^{-1}\int_{Q(t)} v^{2p}\gamma\Big] \\ &\le \frac{1}{2p}\log\Big[c_7(s-t)^{-\zeta_1}e^{p\Phi}\Big]. \\ &= \Big[1 + \frac{\log(c_7(s-t)^{-\zeta_1})}{2\log\Phi}\Big]\frac{\Phi}{2}.\end{aligned} \tag{5.21}$$

Without loss of generality we may take c_7 larger than c_2. If $\Phi(t) \ge c_7(s-t)^{-\zeta_1}$, then by (5.21) $\Phi(s) \le \frac{3}{4}\Phi(t)$, and (5.20) is satisfied. If, on the other hand, $\Phi(t) \le c_7(s-t)^{-\zeta_1}$, then since $\Phi(s) \le \Phi(t)$, we have (5.20) satisfied with $c_1 = c_7$. □

Theorem 5.12. *There exists c_1 such that if u is nonnegative and $\mathcal{L}'$-harmonic in $B(x_0, 4R)$, then*

$$\sup_{B(x_0, R/2)} u \le c_1 \inf_{B(x_0, R/2)} u.$$

Proof. Since u is linear on each cable and $B(x_0, 4R) \cap G$ is a finite set, then u is continuous and bounded in $B(x_0, R)$. We need to show we can bound the ratio of the supremum of u to the infimum of u in $B(x_0, R/2)$ by a constant not depending on u. Multiplying u by a constant we can assume $\int_{B(x_0,R)} \log u = 0$. First let $v = u$.

Choose $t_j = 1/(j+1)$, so that $t_0 = 1$ and $t_i \downarrow 0$. Then by Lemma 5.11,

$$\begin{aligned}\Phi(t_0) &\le \tfrac{3}{4}\Phi(t_1) + c_2(t_0 - t_1)^{-\zeta_1} \\ &\le (\tfrac{3}{4})^2\Phi(t_2) + c_2(t_0 - t_1)^{-\zeta_1} + \tfrac{3}{4}c_2(t_1 - t_2)^{-\zeta_1} \\ &\le \cdots \\ &\le (\tfrac{3}{4})^n\Phi(t_n) + \sum_{i=1}^{n}(\tfrac{3}{4})^{i-1}c_2(t_{i-1} - t_i)^{-\zeta_1},\end{aligned}$$

for any $n \ge 0$. Since $\Phi(t_n) \le \sup_{B(x_0,R)} \log v < \infty$, and

$$\sum_{i=1}^{\infty}(\tfrac{3}{4})^{i-1}c_2(t_{i-1} - t_i)^{-\zeta_1} = c_3 < \infty,$$

we obtain

$$\sup_{B(x_0,R/2)} \log u \le c_3.$$

Now let $v = u^{-1}$; $\log v = -\log u$ so we still have $\int_{B(x_0,R)} \log v = 0$. The same argument as above now implies $\sup_{B(x_0,R/2)} \log v \le c_3$, or

$$\inf_{B(x_0,R/2)} \log u \ge -c_3.$$

Combining we deduce

$$e^{-c_3} \le \inf_{B(x_0,R/2)} u \le \sup_{B(x_0,R/2)} u \le e^{c_3},$$

which is what we wanted to prove. □

Proof of Theorem 1.5. First we show that (a) implies (b). By Propositions 3.3 and 3.5, $PI(\beta)$ and CS(β) hold for G_C. The elliptic Harnack inequality (EHI) for G_C then follows from Theorem 5.12 by a covering argument. By Corollary 2.5, (EHI) holds for G. Lemma 5.1 shows that G_C satisfies (R_β), and hence by Proposition 3.6, G does also. Hence (G, E, a) satisfies condition (d) of Theorem 1.2, and (b) follows from that theorem.

Now suppose that (b) holds. Then (G, E, a) satisfies $PI(\beta)$ by Lemma 4.1, and CS(β) by Theorem 4.12. Thus (b) implies (a). □

Proof of Theorem 1.1. The conditions $PI(\beta)$ and CS(β) are obviously stable. So the conclusion of Theorem 1.1 follows from Lemma 2.1 and Theorem 1.5. □

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