COMPARISON OF QUENCHED AND ANNEALED INVARIANCE PRINCIPLES FOR RANDOM CONDUCTANCE MODEL: PART II

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ABSTRACT. We show that there exists an ergodic conductance environment such that the weak (annealed) invariance principle holds for the corresponding continuous time random walk but the quenched invariance principle does not hold. In the present paper we give a proof of the full scaling limit for the weak invariance principle, improving the result in an earlier paper where we obtained a subsequential limit.

1. INTRODUCTION

This article contains the completion of the project started in a previous paper [4], where we proved that there exists an ergodic conductance environment such that the weak (annealed) invariance principle holds for the corresponding continuous time random walk along a subsequence but the quenched invariance principle does not hold. In the present paper we give a proof of the full scaling limit for the weak invariance principle, improving the result in [4]. The improved result is, in a sense, a quantitative form of the invariance principle. The proof consists of several lemmas. Some of them are specific to our model but some of them have the more general character and may serve as technical elements for related projects. Since this paper is a continuation of [4], we start by presenting basic notation and definitions from that paper.

Let $d \geq 2$ and let E_d be the set of all non oriented edges in the d-dimensional integer lattice, that is, $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. Let $\{\mu_e\}_{e \in E_d}$ be a random process with non-negative values, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $\{\mu_e\}_{e \in E_d}$ represents random conductances. We write $\mu_{xy} = \mu_{yx} = \mu_{\{x,y\}}$ and set $\mu_{xy} = 0$ if $\{x, y\} \notin E_d$. Set

$$\mu_x = \sum_y \mu_{xy}, \qquad P(x, y) = \frac{\mu_{xy}}{\mu_x},$$

with the convention that $0/0 = 0$ and $P(x, y) = 0$ if $\{x, y\} \notin E_d$. For a fixed $\omega \in \Omega$, let $X = \{X_t, t \geq 0, P^x_\omega, x \in \mathbb{Z}^d\}$ be the continuous time random walk on $\mathbb{Z}^d$, with transition probabilities $P(x, y) = P_\omega(x, y)$, and exponential waiting times with mean $1/\mu_x$. The corresponding expectation will be denoted E^x_ω. For a fixed $\omega \in \Omega$, the generator $\mathcal{L}$ of X is given by

$$\mathcal{L} f(x) = \sum_y \mu_{xy}(f(y) - f(x)). \tag{1.1}$$

In [3] this is called the *variable speed random walk* (VSRW) among the conductances μ_e. This model, of a reversible (or symmetric) random walk in a random environment, is often called the Random Conductance Model.

Research supported in part by NSF Grant DMS-1206276, by NSERC, Canada, and Trinity College, Cambridge, and by MTA Rényi "Lendulet" Groups and Graphs Research Group.

We are interested in functional Central Limit Theorems (FCLTs) for the process X. Given any process X, for $\varepsilon > 0$, set $X_t^\varepsilon = \varepsilon X_{t/\varepsilon^2}$, $t \geq 0$. Let $\mathcal{D}_T = D([0,T], \mathbb{R}^d)$ denote the Skorokhod space, and let $\mathcal{D}_\infty = D([0,\infty), \mathbb{R}^d)$. Write d_S for the Skorokhod metric and $\mathcal{B}(\mathcal{D}_T)$ for the σ-field of Borel sets in the corresponding topology. Let X be the canonical process on $\mathcal{D}_\infty$ or $\mathcal{D}_T$, P_{BM} be Wiener measure on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ and let E_{BM} be the corresponding expectation. We will write W for a standard Brownian motion. It will be convenient to assume that $\{\mu_e\}_{e\in E_d}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that X is defined on $(\Omega, \mathcal{F}) \times (\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\Omega, \mathcal{F}) \times (\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$. We also define the averaged or annealed measure $\mathbf{P}$ on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$ by

$$\mathbf{P}(G) = \mathbb{E}\, P_\omega^0(G). \tag{1.2}$$

Definition 1.1. For a bounded function F on $\mathcal{D}_T$ and a constant matrix Σ, let $\Psi_\varepsilon^F = E_\omega^0 F(X^\varepsilon)$ and $\Psi_\Sigma^F = E_{\mathrm{BM}} F(\Sigma W)$. We will use I to denote the identity matrix.

(i) We say that the *Quenched Functional CLT* (QFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi_\varepsilon^F \to \Psi_\Sigma^F$ as $\varepsilon \to 0$, with $\mathbb{P}$-probability 1.

(ii) We say that the *Weak Functional CLT* (WFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi_\varepsilon^F \to \Psi_\Sigma^F$ as $\varepsilon \to 0$, in $\mathbb{P}$-probability.

(iii) We say that the *Averaged (or Annealed) Functional CLT* (AFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\mathbb{E}\, \Psi_\varepsilon^F \to \Psi_\Sigma^F$. This is the same as standard weak convergence with respect to the probability measure $\mathbf{P}$.

If we take Σ to be non-random then, since F is bounded, it is immediate that QFCLT $\Rightarrow$ WFCLT. In general for the QFCLT the matrix Σ might depend on the environment $\mu_\cdot(\omega)$. However, if the environment is stationary and ergodic, then Σ is a shift invariant function of the environment, so must be $\mathbb{P}$–a.s. constant. In [9] it is proved that if μ_e is a stationary ergodic environment with $\mathbb{E}\, \mu_e < \infty$ then the WFCLT holds. In [4, Theorem 1.3] it is proved that for the random conductance model the AFCLT and WFCLT are equivalent.

Definition 1.2. We say an environment (μ_e) on $\mathbb{Z}^d$ is *symmetric* if the law of (μ_e) is invariant under symmetries of $\mathbb{Z}^d$.

If (μ_e) is stationary, ergodic and symmetric, and the WFCLT holds with limit ΣW then the limiting covariance matrix $\Sigma^T \Sigma$ must also be invariant under symmetries of $\mathbb{Z}^d$, so must be a constant times the identity.

In a previous paper [4] we proved the following theorem:

Theorem 1.3. *Let $d = 2$ and $p < 1$. There exists a symmetric stationary ergodic environment $\{\mu_e\}_{e\in E_2}$ with $\mathbb{E}(\mu_e^p \vee \mu_e^{-p}) < \infty$ and a sequence $\varepsilon_n \to 0$ such that*
(a) the WFCLT holds for X^{ε_n} with limit W, i.e., for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi_{\varepsilon_n}^F \to \Psi_I^F$ as $n \to \infty$, in $\mathbb{P}$-probability,
but
(b) the QFCLT does not hold for X^{ε_n} with limit ΣW for any Σ.

In this paper we prove that for an environment similar to that in Theorem 1.3 the WFCLT holds for X^ε as $\varepsilon \to 0$, and not just along a subsequence.

Theorem 1.4. *Let $d = 2$ and $p < 1$. There exists a symmetric stationary ergodic environment $\{\mu_e\}_{e\in E_2}$ with $\mathbb{E}(\mu_e^p \vee \mu_e^{-p}) < \infty$ such that*
(a) the WFCLT holds for X^ε with limit W, i.e., for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi_\varepsilon^F \to \Psi_I^F$ as $\varepsilon \to 0$, in $\mathbb{P}$-probability,
but
(b) the QFCLT does not hold for X^ε with limit ΣW for any Σ.

For more remarks on this problem see [4].

Acknowledgment. We are grateful to Emmanuel Rio, Pierre Mathieu, Jean-Dominique Deuschel and Marek Biskup for some very useful discussions.

2. Description of the environment

Here we recall the environment given in [4]. We refer the reader to that paper for proofs of some basic properties.

Let $\Omega = (0,\infty)^{E_2}$, and $\mathcal{F}$ be the Borel σ-algebra defined using the usual product topology. Then every $t \in \mathbb{Z}^2$ defines a transformation $T_t(\omega) = \omega + t$ of Ω. Stationarity and ergodicity of the measures defined below will be understood with respect to these transformations.

All constants (often denoted c_1, c_2, etc.) are assumed to be strictly positive and finite. For a set $A \subset \mathbb{Z}^2$ let $E(A) \subset E_2$ be the set of all edges with both endpoints in A. Let $E_h(A)$ and $E_v(A)$ respectively be the set of horizontal and vertical edges in $E(A)$. Write $x \sim y$ if $\{x, y\}$ is an edge in $\mathbb{Z}^2$. Define the exterior boundary of A by

$$\partial A = \{y \in \mathbb{Z}^2 - A : y \sim x \text{ for some } x \in A\}.$$

Let also

$$\partial_i A = \partial(\mathbb{Z}^2 - A).$$

Define balls in the ℓ^∞ norm by $\mathcal{B}(x,r) = \{y : ||x-y||_\infty \le r\}$; of course this is just the square with center x and side $2r$.

Let $\{a_n\}_{n\ge0}$, $\{\beta_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ be strictly increasing sequences of positive integers growing to infinity with n, with

$$1 = a_0 < b_1 < \beta_1 < a_1 \ll b_2 < \beta_2 < a_2 \ll b_3 \dots$$

We will impose a number of conditions on these sequences in the course of the paper. We collect the main ones here. There is some redundancy in the conditions, for easy reference.

(i) a_n is even for all n.
(ii) For each $n \ge 1$, a_{n-1} divides b_n, and b_n divides β_n and a_n.
(iii) $b_1 \ge 10^{10}$.
(iv) $a_n/\sqrt{2n} \le b_n \le a_n/\sqrt{n}$ for all n, and $b_n \sim a_n/\sqrt{n}$.
(v) $b_{n+1} \ge 2^n b_n$ for all n.
(vi) $b_n > 40a_{n-1}$ for all n.
(vii) b_n is large enough so that the estimates (5.1) and (6.1) of [4] hold.
(viii) $100b_n < \beta_n \le b_n n^{1/4} < 2\beta_n < a_n/10$ for n large enough.

In addition, at various points in the proof, we will assume that a_n is sufficiently much larger than b_{n-1} so that a process $X^{(n-1)}$ defined below is such that for $a \ge a_n$ the rescaled process

$$(a^{-1}X^{(n-1)}_{a^2t}, t \ge 0)$$

is sufficiently close to Brownian motion. We will mark the places in the proof where we impose these extra conditions by ($\clubsuit$) .

We begin our construction by defining a collection of squares in $\mathbb{Z}^2$. Let

$$\begin{aligned} B_n &= [0, a_n]^2, \\ B'_n &= [0, a_n - 1]^2 \cap \mathbb{Z}^2, \\ \mathcal{S}_n(x) &= \{x + a_n y + B'_n : y \in \mathbb{Z}^2\}. \end{aligned}$$

Thus $\mathcal{S}_n(x)$ gives a tiling of $\mathbb{Z}^2$ by disjoint squares of side $a_n - 1$ and period a_n. We say that the tiling $\mathcal{S}_{n-1}(x_{n-1})$ is a refinement of $\mathcal{S}_n(x_n)$ if every square $Q \in \mathcal{S}_n(x_n)$ is a finite union of squares in $\mathcal{S}_{n-1}(x_{n-1})$. It is clear that $\mathcal{S}_{n-1}(x_{n-1})$ is a refinement of $\mathcal{S}_n(x_n)$ if and only if $x_n = x_{n-1} + a_{n-1}y$ for some $y \in \mathbb{Z}^2$.

Take $\mathcal{O}_1$ uniform in B'_1, and for $n \geq 2$ take $\mathcal{O}_n$, conditional on $(\mathcal{O}_1, \dots, \mathcal{O}_{n-1})$, to be uniform in $B'_n \cap (\mathcal{O}_{n-1} + a_{n-1}\mathbb{Z}^2)$. We now define random tilings by letting

$$\mathcal{S}_n = \mathcal{S}_n(\mathcal{O}_n),\ n \geq 1.$$

Let η_n, K_n be positive constants; we will have $\eta_n \ll 1 \ll K_n$. We define conductances on E_2 as follows. Recall that a_n is even, and let $a'_n = \frac{1}{2}a_n$. Let

$$C_n = \{(x, y) \in B_n \cap \mathbb{Z}^2 : y \geq x, x + y \leq a_n\}.$$

We first define conductances $\nu_e^{n,0}$ for $e \in E(C_n)$. Let

$$\begin{aligned} D_n^{00} &= \big\{(a'_n - \beta_n, y), a'_n - 10b_n \leq y \leq a'_n + 10b_n\big\}, \\ D_n^{01} &= \big\{(x, a'_n + 10b_n), (x, a'_n + 10b_n + 1), (x, a'_n - 10b_n), (x, a'_n - 10b_n - 1), \\ &\qquad a'_n - \beta_n - b_n \leq x \leq a'_n - \beta_n + b_n\big\}. \end{aligned}$$

Thus the set $D_n^{00} \cup D_n^{01}$ resembles the letter I (see Fig. 1).

For an edge $e \in E(C_n)$ we set

$$\begin{aligned} \nu_e^{n,0} &= \eta_n \quad \text{if } e \in E_v(D_n^{01}), \\ \nu_e^{n,0} &= K_n \quad \text{if } e \in E(D_n^{00}), \\ \nu_e^{n,0} &= 1 \quad \text{otherwise.} \end{aligned}$$

We then extend $\nu^{n,0}$ by symmetry to $E(B_n)$. More precisely, for $z = (x, y) \in B_n$, let $R_1 z = (y, x)$ and $R_2 z = (a_n - y, a_n - x)$, so that R_1 and R_2 are reflections in the lines $y = x$ and $x + y = a_n$. We define R_i on edges by $R_i(\{x, y\}) = \{R_i x, R_i y\}$ for $x, y \in B_n$. We then extend $\nu^{0,n}$ to $E(B_n)$ so that $\nu_e^{0,n} = \nu_{R_1 e}^{0,n} = \nu_{R_2 e}^{0,n}$ for $e \in E(B_n)$. We define the *obstacle* set D_n^0 by setting

$$D_n^0 = \bigcup_{i=0}^{1} \big(D_n^{0,i} \cup R_1(D_n^{0,i}) \cup R_2(D_n^{0,i}) \cup R_1 R_2(D_n^{0,i})\big).$$

Note that $\nu_e^{n,0} = 1$ for every edge adjacent to the boundary of B_n, or indeed within a distance $a_n/4$ of this boundary. If $e = (x, y)$, we will write $e - z = (x - z, y - z)$. Next we extend $\nu^{n,0}$ to E_2 by periodicity, i.e., $\nu_e^{n,0} = \nu_{e+a_n x}^{n,0}$ for all $x \in \mathbb{Z}^2$. We define the conductances ν^n by translation by $\mathcal{O}_n$, so that

$$\nu_e^n = \nu_{e-\mathcal{O}_n}^{n,0},\ e \in E_2.$$

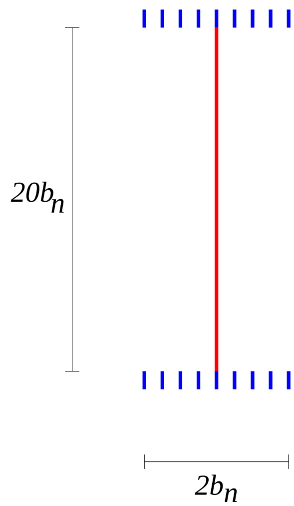

FIGURE 1. The set $D_n^{00} \cup D_n^{01}$ resembles the letter I. Blue edges have very low conductance. The red line represents edges with very high conductance. Drawing not to scale.

We also define the obstacle set at scale n by

$$D_n = \bigcup_{x \in \mathbb{Z}^2} (a_n x + \mathcal{O}_n + D_n^0). \tag{2.1}$$

We will sometimes call the set D_n the set of nth level obstacles.

We define the environment μ_e^n inductively by

$$\mu_e^n = \nu_e^n \quad \text{if } \nu_e^n \neq 1,$$
$$\mu_e^n = \mu_e^{n-1} \quad \text{if } \nu_e^n = 1.$$

Once we have proved the limit exists, we will set

$$\mu_e = \lim_n \mu_e^n. \tag{2.2}$$

Lemma 2.1. *(See* [4, Theorem 3.1]*).*
(a) The environments $(\nu_e^n, e \in E_2)$, $(\mu_e^n, e \in E_2)$ are stationary, symmetric and ergodic.
(b) The limit (2.2) *exists $\mathbb{P}$–a.s.*
(c) The environment $(\mu_e, e \in E_2)$ is stationary, symmetric and ergodic.

Now let

$$\mathcal{L}_n f(x) = \sum_y \mu_{xy}^n (f(y) - f(x)), \tag{2.3}$$

and $X^{(n)}$ be the associated Markov process. Set

$$\eta_n = b_n^{-(1+1/n)},\ n \geq 1. \tag{2.4}$$

From Section 4 of [4] we have:

Theorem 2.2. *For each n there exists a constant K_n, depending on $\eta_1, K_1, \dots \eta_{n-1}, K_{n-1}$, such that the QFCLT holds for $X^{(n)}$ with limit W.*

For each n the process $X^{(n)}$ has invariant measure which is counting measure on $\mathbb{Z}^2$. For $x \in \mathbb{R}^2$ and $a > 0$ write $[xa]$ for the point in $\mathbb{Z}^2$ closest to xa. (We use some procedure to break ties.) We have the following bounds on the transition probabilities of $X^{(n)}$ from [5]. We remark that the constant M_n below is not effective – i.e. the proof does not give any control on its value. Write $k_t(x,y) = (2\pi t)^{-1} \exp(-|x-y|^2/2t)$ for the transition density of Brownian motion in $\mathbb{R}^2$, and

$$p_t^{\omega,n}(x,y) = P_\omega^x(X_t^{(n)} = y)$$

for the transition probabilities for $X^{(n)}$.

Lemma 2.3. *For each $0 < \delta < T$ there exists $M_n = M_n(\delta, T)$ such that for $a \geq M_n$*

$$\frac{1}{2} k_t(x,y) \leq a^2 p_{a^2 t}^{\omega,n}([xa],[ya]) \leq 2k_t(x,y) \text{ for all } \delta \leq t \leq T, |x|, |y| \leq T^2. \tag{2.5}$$

3. Preliminary results

Since a proof of Theorem 1.3(b) was given in [4], all we need to prove is part (a) of Theorem 1.4. The argument consists of several lemmas. We start with some preliminary results on weak convergence of probability measures on the space of càdlàg functions. Recall the definitions of the measures $\mathbb{P}$ and P_ω^0.

Recall that $\mathcal{D} := \mathcal{D}_1 = D([0,1], \mathbb{R}^2)$ denotes the space of càdlàg functions equipped with the Skorokhod metric $\mathrm{d_S}$ defined as follows (see [6, p. 111]). Let Λ be the family of continuous strictly increasing functions λ mapping $[0,1]$ onto itself. In particular, $\lambda(0) = 0$ and $\lambda(1) = 1$. If $x(t), y(t) \in \mathcal{D}$ then

$$\mathrm{d_S}(x,y) = \inf_{\lambda \in \Lambda} \max\Big(\sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} |y(\lambda(t)) - x(t)| \Big).$$

For $x(t) \in \mathcal{D}$, let $\mathrm{Osc}(x, \delta) = \sup\{|x(t) - x(s)| : s,t \in [0,1], |s-t| \leq \delta\}$.

Lemma 3.1. *Suppose that $\sigma : [0,1] \to [0,1]$ is continuous, non-decreasing and $\sigma(0) = 0$ (we do not require that $\sigma(1) = 1$). Suppose that $|\sigma(t) - t| \leq \delta$ for all $t \in [0,1]$. Let $\varepsilon \geq 0$, $\delta_1 > 0$, $x, y \in \mathcal{D}$ with $\mathrm{d_S}(x(\cdot), y(\cdot)) \leq \varepsilon$, and $\mathrm{Osc}(x,\delta) \vee \mathrm{Osc}(y,\delta) \leq \delta_1$. Then $\mathrm{d_S}(x(\sigma(\cdot)), y(\sigma(\cdot))) \leq \varepsilon + 2\delta_1$.*

Proof. For any $\varepsilon_1 > \varepsilon$ there exists $\lambda \in \Lambda$ such that,

$$\max\Big(\sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} |y(\lambda(t)) - x(t)| \Big) \leq \varepsilon_1.$$

We have for λ satisfying the above condition,

$$\begin{aligned}
&\sup_{t \in [0,1]} |y(\sigma(\lambda(t))) - x(\sigma(t))| \\
&\qquad \leq \sup_{t \in [0,1]} \big(|y(\sigma(\lambda(t))) - y(\lambda(t))| + |y(\lambda(t)) - x(t)| + |x(t) - x(\sigma(t))| \big) \\
&\qquad \leq \mathrm{Osc}(y,\delta) + \varepsilon_1 + \mathrm{Osc}(x,\delta) \leq \varepsilon_1 + 2\delta_1.
\end{aligned}$$

Hence,

$$\max\Big(\sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} |y(\sigma(\lambda(t))) - x(\sigma(t))| \Big) \leq \varepsilon_1 + 2\delta_1.$$

Taking infimum over all $\varepsilon_1 > \varepsilon$ we obtain $\mathrm{d_S}(x(\sigma(\cdot)), y(\sigma(\cdot))) \leq \varepsilon + 2\delta_1$. $\square$

Let $\mathbf{d}$ denote the Prokhorov distance between probability measures on a probability space defined as follows (see [6, p. 238]). Recall that $\Omega = (0,\infty)^{E_2}$ and $\mathcal{F}$ is the Borel σ-algebra defined using the usual product topology. We will use measurable spaces $(\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$ and $(\Omega, \mathcal{F}) \times (\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$, for a fixed T (often $T = 1$). Note that $\mathcal{D}_T$ and $\Omega \times \mathcal{D}_T$ are metrizable, with the metrics generating the usual topologies. A ball around a set A with radius ε will be denoted $\mathcal{B}(A, \varepsilon)$ in either space. For probability measures P and Q, $\mathbf{d}(P, Q)$ is the infimum of $\varepsilon > 0$ such that $P(A) \leq Q(\mathcal{B}(A, \varepsilon)) + \varepsilon$ and $Q(A) \leq P(\mathcal{B}(A, \varepsilon)) + \varepsilon$ for all Borel sets A. Convergence in the metric $\mathbf{d}$ is equivalent to the weak convergence of measures. By abuse of notation we will sometimes write arguments of the function $\mathbf{d}(\,\cdot\,,\,\cdot\,)$ as processes rather than their distributions: for example we will write $\mathbf{d}(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, P_{\mathrm{BM}})$. We will use $\mathbf{d}$ for the Prokhorov distance between probability measures on $(\Omega, \mathcal{F}) \times (\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$. We will write $\mathbf{d}_\omega$ for the metric on the space $(\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$. It is straightforward to verify that if, for some processes Y and Z, $\mathbf{d}_\omega(Y, Z) \leq \varepsilon$ for $\mathbb{P}$–a.a. ω, then $\mathbf{d}(Y, Z) \leq \varepsilon$.

We will sometimes write $W(t) = W_t$ and similarly for other processes.

Lemma 3.2. *There exists a function $\rho : (0,\infty) \to (0,\infty)$ such that $\lim_{\delta\downarrow 0} \rho(\delta) = 0$ and the following holds. Suppose that $\delta, \delta' \in (0,1)$ and $\sigma : [0,1] \to [0,1]$ is a non-decreasing stochastic process such that $t - \sigma_t \in [0, \delta]$ for all t, with probability greater than $1 - \delta'$. Suppose that $\{W_t, t \geq 0\}$ has the distribution P_{BM} and $W^*_t = W(\sigma_t)$ for $t \in [0,1]$. Then $\mathbf{d}(\{W^*_t, t \in [0,1]\}, P_{BM}) \leq \rho(\delta) + \delta'$.*

Proof. Suppose that W, W^* and σ are defined on the sample space with a probability measure P. It is easy to see that we can choose $\rho(\delta)$ so that $\lim_{\delta\downarrow 0} \rho(\delta) = 0$ and $P(\mathrm{Osc}(W, \delta) \geq \rho(\delta)) < \rho(\delta)$. Suppose that the event $F := \{\mathrm{Osc}(W, \delta) < \rho(\delta)\} \cap \{\forall t \in [0,1] : t - \sigma_t \in [0, \delta]\}$ holds. Then taking $\lambda(t) = t$,

$$\begin{aligned} \mathrm{d_S}(W, W^*) &\leq \max\Big(\sup_{t\in[0,1]} |\lambda(t) - t|, \sup_{t\in[0,1]} |W(\lambda(t)) - W^*(t)| \Big) \\ &= \sup_{t\in[0,1]} |W(t) - W(\sigma(t))| \leq \mathrm{Osc}(W, \delta) < \rho(\delta). \end{aligned}$$

We see that if F holds and $W \in A \subset \mathcal{D}$ then $W^*(\,\cdot\,) \in \mathcal{B}(A, \rho(\delta))$. Since $P(F^c) \leq \rho(\delta) + \delta'$, we obtain

$$\begin{aligned} &P(W \in A) \\ &\quad \leq P(\{W \in A\} \cap F) + P(F^c) \leq P(\{W^* \in \mathcal{B}(A, \rho(\delta))\} \cap F) + \rho(\delta) + \delta' \\ &\quad \leq P(W^* \in \mathcal{B}(A, \rho(\delta))) + \rho(\delta) + \delta'. \end{aligned}$$

Similarly we have $P(W^* \in A) \leq P(W \in \mathcal{B}(A, \rho(\delta))) + \rho(\delta) + \delta'$, and the lemma follows. □

Lemma 3.3. *Suppose that for some processes X, Y and Z on the interval $[0,1]$ we have $Z = X + Y$ and $P(\sup_{0\leq t\leq 1} |X_t| \leq \delta) \geq 1 - \delta$. Then $\mathbf{d}(\{Z_t, t \in [0,1]\}, \{Y_t, t \in [0,1]\}) \leq \delta$.*

Proof. Suppose that the event $F := \{\sup_{0\leq t\leq 1} |X_t| \leq \delta\}$ holds. Then taking $\lambda(t) = t$,

$$\begin{aligned} \mathrm{d_S}(Z, Y) &\leq \max\Big(\sup_{t\in[0,1]} |\lambda(t) - t|, \sup_{t\in[0,1]} |Z(\lambda(t)) - Y(t)| \Big) \\ &= \sup_{t\in[0,1]} |Z(t) - Y(t)| \leq \delta. \end{aligned}$$

We see that if F holds and $Z \in A \subset \mathcal{D}$ then $Y(\,\cdot\,) \in \mathcal{B}(A,\delta)$. Since $P(F^c) \le \delta$, we obtain

$$\begin{aligned} P(Z \in A) &\le P(\{Z \in A\} \cap F) + P(F^c) \le P(\{Y \in \mathcal{B}(A,\delta)\} \cap F) + \delta \\ &\le P(Y \in \mathcal{B}(A,\delta)) + \delta. \end{aligned}$$

Similarly we have $P(Y \in A) \le P(Z \in \mathcal{B}(A,\delta)) + \delta$, and the lemma follows. □

Recall that the function $e \to \mu_e^n$ is periodic with period a_n. Hence the random field $\{\mu_e^n\}_{e \in E_2}$ takes only finitely many values – this is a much stronger statement than the fact that μ_e^n takes only finitely many values.

By Theorem 2.2 for each $n \ge 1$,

$$\lim_{a\to\infty} \mathbf{d}(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, P_{\mathrm{BM}}) = 0.$$

Thus (♣) we can take a_{n+1} so large that for every ω, $n \ge 1$ and $a \ge a_{n+1}$,

$$\mathbf{d}_\omega(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, P_{\mathrm{BM}}) \le 2^{-n}. \tag{3.1}$$

Let θ denote the usual shift operator for Markov processes, that is, $X_t^{(n)} \circ \theta_s = X_{t+s}^{(n)}$ for all $s, t \ge 0$ (we can and do assume that $X^{(n)}$ is the canonical process on an appropriate probability space). Recall that $\mathcal{B}(x,r) = \{y : ||x-y||_\infty \le r\}$ denote balls in the ℓ^∞ norm in $\mathbb{Z}^2$ (i.e. squares), $a_n' = a_n/2$, $B_n = [0,a_n]^2$ and $u_n = (a_n', a_n')$. Note that u_n is the center of B_n. We choose β_n so that

$$b_n n^{1/8} < \beta_n \le \lfloor b_n n^{1/4} \rfloor < 2\beta_n < a_n/10, \tag{3.2}$$

and we assume that n is large enough so that the above inequalities hold. Let $\mathcal{C}_n = \{u_n + \mathcal{O}_n + a_n\mathbb{Z}^2\}$ be the set of centers of the squares in $\mathcal{S}_n$, and let

$$\mathcal{K}(r) = \bigcup_{z \in \mathcal{C}_n} \mathcal{B}(z,r). \tag{3.3}$$

Now let

$$\begin{aligned} \Gamma_n^1 &= \mathcal{K}(2\beta_n), \\ \Gamma_n^2 &= \mathbb{Z}^2 \setminus \mathcal{K}(4\beta_n). \end{aligned}$$

Now define stopping times as follows.

$$\begin{aligned} S_0^n &= T_0^n = 0, \\ U_k^n &= \inf\{t \ge S_{k-1}^n : X_t^{(n)} \in \Gamma_n^2\}, \qquad k \ge 1, \\ S_k^n &= \inf\{t \ge U_k^n : X_t^{(n)} \in \Gamma_n^1\}, \qquad k \ge 1, \\ V_1^n &= \inf\Big\{t \in \bigcup_{k\ge 1}[U_k^n, S_k^n] : X_t^{(n)} \in X^{(n)}(T_0^n) + a_{n-1}\mathbb{Z}^2\Big\}, \\ T_k^n &= \inf\{t \ge V_k^n : X_t^{(n)} \in \Gamma_n^1\}, \qquad k \ge 1, \\ V_k^n &= V_1^n \circ \theta_{T_{k-1}^n}, \qquad k \ge 2. \end{aligned}$$

Let

$$J = \bigcup_{k=1}^\infty [V_k^n, T_k^n];$$

for $t \in J$ the process $X^{(n)}$ is a distance at least β_n away from any nth level obstacle. Now set for $t \geq 0$,

$$\sigma_t^{n,1} = \int_0^t \mathbf{1}_J(s)ds = \sum_{k=1}^{\infty} \left(T_k^n \wedge t - V_k^n \wedge t\right),$$

$$\sigma_t^{n,2} = t - \sigma_t^{n,1} = \sum_{k=0}^{\infty} \left(V_{k+1}^n \wedge t - T_k^n \wedge t\right).$$

Let $\widehat{\sigma}^{n,j}$ denote the right continuous inverses of these processes, given by

$$\widehat{\sigma}_t^{n,j} = \inf\{s \geq 0 : \sigma_s^{n,j} \geq t\},\ j = 1, 2.$$

Finally let

$$\begin{aligned}
X_t^{n,1} &= X_0^{(n)} + \int_0^t \mathbf{1}_J(s)dX_s^{(n)} \\
&= X_0^{(n)} + \sum_{k=0}^{\infty} \left(X^{(n)}(T_k^n \wedge t) - X^{(n)}(V_k^n \wedge t)\right), \\
\widehat{X}_t^{n,1} &= X_0^{(n)} + X^{n,1}(\widehat{\sigma}_t^{n,1}), \\
X_t^{n,2} &= X_0^{(n)} + \int_0^t \mathbf{1}_{J^c}(s)dX_s^{(n)} \\
&= X_0^{(n)} + \sum_{k=0}^{\infty} \left(X^{(n)}(V_{k+1}^n \wedge t) - X^{(n)}(T_k^n \wedge t)\right), \\
\widehat{X}_t^{n,2} &= X_0^{(n)} + X^{n,2}(\widehat{\sigma}_t^{n,2}).
\end{aligned}$$

The point of this construction is the following. For every fixed ω, the function $e \to \mu_e^{n-1}$ is invariant under the shift by xa_{n-1} for any $x \in \mathbb{Z}^2$, and $X^{(n)}(V_{k+1}^n) = X^{(n)}(T_k^n) + xa_{n-1}$ for some $x \in \mathbb{Z}^2$. It follows that for each $\omega \in \Omega$, we have the following equality of distributions:

$$\{\widehat{X}_t^{n,1}, t \geq 0\} \stackrel{(d)}{=} \{X_t^{(n-1)}, t \geq 0\}. \tag{3.4}$$

The basic idea of the argument which follows is to write $X^{(n)} = X^{n,1} + X^{n.2}$. By Theorem 2.2, or more precisely by (3.1), the process $X^{n,1}$ is close to Brownian motion, so to prove Theorem 1.4 we need to prove that $X^{n,2}$ is small.

We state the next lemma at a level of generality greater than what we need in this article. A variant of our lemma is in the book [1] but we could not find a statement that would match perfectly our needs. Consider a finite graph $G = (\mathcal{V}, E)$ and suppose that for any edge $\overline{xy}$, μ_{xy} is a non-negative real number. Assume that $\sum_{y\sim x} \mu_{xy} > 0$ for all x. For $f : \mathcal{V} \to \mathbb{R}$ set

$$\mathcal{E}(f, f) = \sum_{\{x,y\}\in E} \mu_{xy}(f(y) - f(x))^2.$$

Suppose that $A_1, A_2 \subset \mathcal{V}$, $A_1 \cap A_2 = \emptyset$, and let

$$\begin{aligned}
\mathcal{H} &= \{f : \mathcal{V} \to \mathbb{R} \text{ such that } f(x) = 0 \text{ for } x \in A_1, f(y) = 1 \text{ for } y \in A_2\}, \\
\mathbf{r}^{-1} &= \inf\{\mathcal{E}(f, f) : f \in \mathcal{H}\}.
\end{aligned}$$

Thus $\mathbf{r}$ is the effective resistance between A_1 and A_2. Let Z be the continuous time Markov process on $\mathcal{V}$ with the generator $\mathcal{L}$ given by

$$\mathcal{L}f(x) = \sum_y \mu_{xy}(f(y) - f(x)). \tag{3.5}$$

Let $T_i = \inf\{t \geq 0 : Z_t \in A_i\}$ for $i = 1, 2$, and let $Z^{(i)}$ be Z killed at time T_i.

Lemma 3.4. *There exist probability measures ν_1 on A_1 and ν_2 on A_2 such that*

$$E^{\nu_2}T_1 + E^{\nu_1}T_2 = \mathbf{r}|\mathcal{V}|.$$

Moreover, for $i = 1, 2$, ν_i is the capacitary measure of A_i for the process $Z^{(3-i)}$.

Proof. Let $h_{12}(x) = P^x(T_1 < T_2)$. Set $D = \mathcal{V} - A_1$ and recall that $Z^{(i)}$ is Z killed at time T_i. Let G_2 be the Green operator for $Z^{(2)}$, and $g_2(x, y)$ be the density of G_2 with respect to counting measure, so that

$$E^x T_2 = \sum_{y \in \mathcal{V}} g_2(x, y).$$

Note that $g_2(x, y) = g_2(y, x)$. Let e_{12} be the capacitary measure of A_1 for the process $Z^{(2)}$. Then $\mathbf{r}^{-1} = \sum_{z \in A_1} e_{12}(z)$, and

$$h_{12}(x) = \sum_{z \in A_1} e_{12}(z) g_2(z, x).$$

So, if $\nu_1 = \mathbf{r}e_{12}$, then

$$\begin{aligned}\sum_{y \in \mathcal{V}} h_{12}(y) &= \sum_{y \in \mathcal{V}} \sum_{x \in A_1} e_{12}(x) g_2(x, y) \\ &= \mathbf{r}^{-1} \sum_{x \in A_1} \nu_1(x) \sum_{y \in \mathcal{V}} g_2(x, y) \\ &= \mathbf{r}^{-1} \sum_{x \in A_1} \nu_1(x) E^x T_1 = \mathbf{r}^{-1} E^{\nu_1} T_2.\end{aligned}$$

Similarly if $h_{21}(x) = \mathbb{P}^x(T_2 < T_1)$ we obtain $\mathbf{r}^{-1}E^{\nu_2}T_1 = \sum_{y \in \mathcal{V}} h_{21}(y)$, and since $h_{12} + h_{21} = 1$, adding these equalities proves the lemma. $\square$

4. Estimates on the process $X^{n,2}$

In this section we will prove

Proposition 4.1. *For every $\delta > 0$ there exists n_1 such that for all $n \geq n_1$, $u \geq a_n^2$, and ω such that $0 \notin \Gamma_n^1 \setminus \partial_i \Gamma_n^1$,*

$$P_\omega^0 \left(\sigma_u^{n,2}/u \leq \delta, \sup_{0 \leq s \leq u} u^{-1/2} |X_s^{n,2}| \leq \delta \right) \geq 1 - \delta. \tag{4.1}$$

The proof requires a number of steps. We begin with a Harnack inequality.

Lemma 4.2. *Let $1 \leq \lambda \leq 10$. There exist $p_1 > 0$ and $n_1 \geq 1$ with the following properties. (a) Let $x \in \mathbb{Z}^2$, let $B_1 = \mathcal{B}(x, \lambda\beta_n)$ and $B_2 = \mathcal{B}(x, (2/3)\lambda\beta_n)$. Let F be the event that $X^{(n)}$ makes a closed loop around B_2 inside $B_1 - B_2$ before its first exit from B_1. If $n \geq n_1$ and*

$D_n \cap B_1 = \emptyset$ then $P^y_\omega(F) \geq p_1$ for all $y \in B_2$.
(b) Let h be harmonic in B_1. Then

$$\max_{B_2} h \leq p_1^{-1} \min_{B_2} h. \tag{4.2}$$

Proof. (a) Using (♣) and (3.1) we can make a Brownian approximation to $\beta_n^{-1} X^{(n)}_\cdot$ which is good enough so that this estimate holds.
(b) Let $y \in B_1$ be such that $h(y) = \max_{z \in B_2} h(z)$. Then by the maximum principle there exists a connected path γ from y to $\partial_i B_1$ with $h(w) \geq h(y)$ for all $w \in \gamma$. Now let $y' \in B_2$. On the event F the process $X^{(n)}$ must hit γ, and so we have

$$h(y') \geq P^{y'}_\omega(F) \min_\gamma h \geq p_1 h(y),$$

proving (4.2). □

Lemma 4.3. *For some n_1 and c_1, for all $n \geq n_1$, $k \geq 1$, and ω such that $0 \notin \Gamma^1_n \setminus \partial_i \Gamma^1_n$,*

$$E^0_\omega(U^n_k - S^n_{k-1} \mid \mathcal{F}_{S^n_{k-1}}) \leq c_1 \beta_n^2. \tag{4.3}$$

Proof. Assume that ω is such that $0 \notin \Gamma^1_n \setminus \partial_i \Gamma^1_n$. By the strong Markov property applied at S^n_{k-1} for $k > 1$, it is enough to prove the Lemma for $k = 1$, that is that $E^x_\omega(U^n_1) \leq c_1 \beta_n^2$ for all $x \notin \Gamma^1_n \setminus \partial_i \Gamma^1_n$. Let

$$\begin{aligned}
\mathcal{V} &= \mathcal{B}(u_n + \mathcal{O}_n, 4\beta_n + 1),\\
A_1 &= \partial_i \mathcal{B}(u_n + \mathcal{O}_n, (3/2)\beta_n),\\
A_2 &= \partial_i \mathcal{V},\\
A_3 &= \partial_i \mathcal{B}(u_n + \mathcal{O}_n, 2\beta_n)\\
T_i &= \inf\{t \geq 0 : X^{(n)}_t \in A_i\}, \qquad i = 1, 2, 3.
\end{aligned}$$

Let Z be the continuous time Markov chain defined on $\mathcal{V}$ by (3.5), relative to the environment μ^n. Note that the transition probabilities from x to one of its neighbors are the same for Z and $X^{(n)}$ if x is in the interior of $\mathcal{V}$, i.e., $x \notin \partial_i \mathcal{V} \cup (\mathbb{Z}^2 \setminus \mathcal{V})$. Note also that Z and $X^{(n-1)}$ have the same transition probabilities in the region between A_1 and A_3. The expectations and probabilities in this proof will refer to Z. By Lemma 3.4, there exists a probability measure ν_1 on A_1 such that $E^{\nu_1} T_2 \leq \mathbf{r}|\mathcal{V}|$. We have $|\mathcal{V}| \leq c_2 \beta_n^2$.

To estimate $\mathbf{r}$ note that by the choice of the constants η_{n-1} and K_{n-1} in Theorem 2.2, the resistance (with respect to μ^{n-1}_e) between two opposite sides of any square in $\mathcal{S}_{n-1}$ will be 1. It follows that the resistance between two opposite sides of any square side β_n which is a union of squares in $\mathcal{S}_{n-1}$ will also be 1. So, using Thompson's principle as in [2] we deduce that $\mathbf{r} \leq c_3$.

So, by Lemma 3.4 we have

$$E^{\nu_1} T_2 \leq c_4 \beta_n^2. \tag{4.4}$$

We have for some c_5, $p_1 > 0$ all n and $x \in \mathcal{V} \setminus \mathcal{B}(u_n + \mathcal{O}_n, (3/2)\beta_n)$,

$$P^x_\omega(T_1 \wedge T_2 \leq c_5 \beta_n^2) > p_1,$$

because an analogous estimate holds for Brownian motion and (♣) we have (3.1). This and a standard argument based on the strong Markov property imply that for $x \in A_3$,

$$E^x_\omega(T_1 \wedge T_2) \leq c_6 \beta_n^2.$$

Now for $y \in A_1$ and $x \in \mathcal{V}$ set

$$\nu_3^x(y) = P_\omega^x(X^{(n)}(T_1 \wedge T_2) = y).$$

(Note that there exist x with $\sum_{y \in A_1} \nu_3^x(y) < 1$.) We obtain for $n \geq n_2$ and $x \in A_3$,

$$\begin{aligned} E_\omega^x(T_2) &= E_\omega^x(T_1 \wedge T_2) + E_\omega^x((T_2 - T_1)\mathbf{1}_{T_1 < T_2}) \\ &= E_\omega^x(T_1 \wedge T_2) + E^{\nu_3^x} T_2 \leq c_6 \beta_n^2 + E_\omega^{\nu_3^x} T_2. \end{aligned} \tag{4.5}$$

For $y \in A_1$ the function $x \to \nu_3^x(y)$ is harmonic in $\mathcal{V} \setminus A_1$. So we can apply the Harnack inequality Lemma 4.2 to deduce that there exists c_7 such that

$$\nu_3^x(y) \leq c_7 \nu_3^{x'}(y) \text{ for all } x, x' \in A_3, y \in A_1. \tag{4.6}$$

The measure ν_1 is the hitting distribution on A_1 for the process Z starting with ν_2 (see [1, Chap. 3, p. 45]). So for any $x' \in A_3$,

$$\begin{aligned} \nu_1(y) = P_0^{\nu_2}(Z_{T_1} = y) &= \sum_{x \in A_3} P_0^{\nu_2}(Z_{T_1} = x) P_\omega^x(Z_{T_1} = y) \\ &\geq \sum_{x \in A_3} P_0^{\nu_2}(Z_{T_1} = x) P_\omega^x(Z_{T_1 \wedge T_2} = y) \geq \min_{x \in A_3} \nu_3^x(y) \geq c_7^{-1} \nu_3^{x'}(y). \end{aligned}$$

Hence for any $x \in A_3$,

$$E_\omega^{\nu_3^x} T_2 \leq c_7 E_\omega^{\nu_1} T_2 \leq c_8 \beta_n^2,$$

and combining this with (4.5) completes the proof. □

Let

$$R_n^y = \inf \left\{ t \geq 0 : X_t^{(n)} \in (y + a_{n-1}\mathbb{Z}^2) \cup \Gamma_n^1 \right\}.$$

Lemma 4.4. *There exist $c_1 > 0$ and $p_1 < 1$ such that for all $x, y \in \mathbb{Z}^2$,*

$$P_\omega^x \left(R_n^y \geq c_1 b_n^2 \right) \leq p_1, \tag{4.7}$$

$$P_\omega^x \left(\sup_{0 \leq t \leq R_n^y} |x - X_t^{(n)}| \geq c_1 b_n \right) \leq p_1. \tag{4.8}$$

Proof. Recall that the family $\{\mu_{x+\cdot}^{n-1}\}_{x \in \mathbb{Z}^2}$ of translates of the environment $\mu_\cdot^{n-1}$ contains only a finite number of distinct elements. Since each square in $\mathcal{S}_{n-1}$ contains one point in $(y + a_{n-1}\mathbb{Z}^2)$, if b_n/a_{n-1} is sufficiently large (♣) then using the transition density estimates (2.5) as well as (3.1), we obtain (4.7) and (4.8). □

Lemma 4.5. *For some n_1 and c_1, for all $n \geq n_1$, $k \geq 1$, and ω such that $0 \notin \Gamma_n^1 \setminus \partial_i \Gamma_n^1$,*

$$E_\omega^0(V_k^n - T_{k-1}^n \mid \mathcal{F}_{T_{k-1}^n}) \leq c_1 b_n^2 n^{1/2}. \tag{4.9}$$

Proof. Assume that ω is such that $0 \notin \Gamma_n^1 \setminus \partial_i \Gamma_n^1$. Let

$$\widehat{R}_k^n = \inf \left\{ t \geq U_k^n : X_t^{(n)} \in (X^{(n)}(T_0^n) + a_{n-1}\mathbb{Z}^2) \cup \Gamma_n^1 \right\}.$$

Let $F_k = \{\widehat{R}_k^n < S_k^n\}$ and $G_k = \bigcap_{j=1}^k F_j^c$. Since $b_n n^{1/8} < \beta_n$ for large n, we obtain from (4.8) and definitions of $\Gamma_n^1, \Gamma_n^2, U_k^n$ and S_k^n that there exists $p_2 > 0$ such that for $x \in \Gamma_n^2$,

$$P_\omega^x(F_k \mid \mathcal{F}_{U_k^n}) > p_2.$$

Hence,

$$P^x_\omega(G_k) < (1-p_2)^k. \tag{4.10}$$

Note that if F_k occurs then $V^n_1 \le \widehat{R}^n_k$. We have, using (4.3), (4.7) and (4.10),

$$\begin{aligned}
E^0_\omega(V^n_1 - T^n_0) &\le \sum_{k=1}^\infty E^0_\omega((U^n_k - S^n_{k-1})\mathbf{1}_{G_{k-1}}) + \sum_{k=1}^\infty E^0_\omega((\widehat{R}^n_k - U^n_k)\mathbf{1}_{G_{k-1}}) \\
&\le \sum_{k=1}^\infty c_2\beta_n^2(1-p_2)^{k-1} + \sum_{k=1}^\infty c_3 b_n^2(1-p_2)^{k-1} \\
&\le c_4\beta_n^2 \le c_5 b_n^2 n^{1/2}.
\end{aligned}$$

This proves the lemma for $k=1$. The general case is obtained by applying this estimate to the process shifted by T^n_{k-1}; in other words, by using the strong Markov property. □

Lemma 4.6. *For every $\delta > 0$ there exists n_1 such that for all $n \ge n_1$, $u \ge a_n^2$, and ω such that $0 \notin \Gamma^1_n \setminus \partial_i\Gamma^1_n$,*

$$P^0_\omega\left(\sigma^{n,2}_u / u \le \delta\right) \ge 1 - \delta/2. \tag{4.11}$$

Proof. Assume that ω is such that $0 \notin \Gamma^1_n \setminus \partial_i\Gamma^1_n$. Fix an arbitrarily small $\delta > 0$, consider $u \ge a_n^2$ and let $j_* = \lceil u/(b_n^2 n^{5/8})\rceil$. Then (4.9) implies that for some c_1 and n_2, all $n \ge n_2$, $u \ge a_n^2$,

$$E^0_\omega\left(\frac{1}{j_*}\sum_{j=1}^{j_*} V^n_j - T^n_{j-1}\right) \le c_1 b_n^2 n^{1/2}.$$

Hence, for some n_3, all $n \ge n_3$, $u \ge a_n^2$,

$$P^0_\omega\left(\frac{1}{j_*}\sum_{j=1}^{j_*} V^n_j - T^n_{j-1} \ge \delta b_n^2 n^{9/16}\right) \le \delta/8,$$

and, since $j_*\delta b_n^2 n^{9/16} \le \delta u$,

$$P^0_\omega\left(\sum_{j=1}^{j_*} V^n_j - T^n_{j-1} \ge \delta u\right) \le \delta/8. \tag{4.12}$$

Recall $\mathcal{K}(r)$ from (3.3). Let

$$\begin{aligned}
\widehat{V}^n_k &= \inf\{t \ge V^n_k : X^{(n)}_t \in \mathbb{Z}^2 \setminus \mathcal{K}(b_n n^{3/8})\} \wedge T^n_k, \qquad k \ge 1, \\
\widetilde{V}^n_k &= \inf\{t \ge \widehat{V}^n_k : |X^{(n)}_t - X^{(n)}(\widehat{V}^n_k)| \ge (1/2) b_n n^{3/8}\}, \qquad k \ge 1.
\end{aligned}$$

We can use estimates for Brownian hitting probabilities (♣) to see that for some c_2, c_3 and n_4, all $n \ge n_4$, k,

$$P^0_\omega(\widehat{V}^n_k < T^n_k \mid \mathcal{F}_{V^n_k}) \ge c_2 \frac{\log(4\beta_n) - \log(2\beta_n)}{\log(2b_n n^{3/8}) - \log(2\beta_n)} \ge c_3/\log n. \tag{4.13}$$

There exist (♣) c_4 and n_5, such that for all $n \ge n_5$, $k \ge 2$,

$$\begin{aligned}
&P^0_\omega(T^n_k - V^n_k \ge c_4 b_n^2 n^{3/4} \mid \widehat{V}^n_k < T^n_k, \mathcal{F}_{\widehat{V}^n_k}) \\
&\qquad \ge P^0_\omega(\widetilde{V}^n_k - \widehat{V}^n_k \ge c_4 b_n^2 n^{3/4} \mid \widehat{V}^n_k < T^n_k, \mathcal{F}_{\widehat{V}^n_k}) \ge 3/4.
\end{aligned}$$

This and (4.13) imply that the sequence $\{T_k^n - V_k^n\}_{k\geq 2}$ is stochastically minorized by a sequence of i.i.d. random variables which take value $c_4 b_n^2 n^{3/4}$ with probability $c_3/\log n$ and they take value 0 otherwise. This implies that for some n_6, all $n \geq n_6$, $u \geq a_n^2$,

$$P_\omega^0\left(\frac{1}{j_*}\sum_{j=2}^{j_*} T_j^n - V_j^n \leq b_n^2 n^{3/4}/\log^2 n\right) \leq \delta/4$$

and, because $j_* b_n^2 n^{3/4}/\log^2 n \geq u$ assuming n_6 is large enough,

$$P_\omega^0\left(\sum_{j=2}^{j_*} T_j^n - V_j^n \leq u\right) \leq \delta/4.$$

We combine this with (4.12) and the definition of $\sigma_u^{n,2}$ to obtain for some n_7, all $n \geq n_7$, $u \geq a_n^2$,

$$P_\omega^0(\sigma_u^{n,2}/u \leq \delta) \geq 1 - 3\delta/8. \tag{4.14}$$

This completes the proof of the lemma. □

Let $Y_k^n = (Y_{k,1}^n, Y_{k,2}^n) = X^{(n)}(V_{k+1}^n) - X^{(n)}(T_k^n)$. Set $\bar{Y}_k^n = \sup_{T_k^n \leq t \leq V_{k+1}^n} |X^{(n)}(t) - X^{(n)}(T_k^n)|$. For $x \in \mathbb{Z}^2$, let $\Pi_n(x) \in B_n' - u_n + \mathcal{O}_n$ be the unique point with the property that $x - \Pi_n(x) = a_n y$ for some $y \in \mathbb{Z}^2$.

We next estimate the variance of $X^{n,2}(V_{m+1}^n) = \sum_{k=0}^m Y_k^n$.

Lemma 4.7. *There exist c_1, c_2 and n_1 such that for all $n \geq n_1$, $k \geq 0$, $j = 1, 2$, and ω,*

$$E_\omega^0 |Y_{k,j}^n| \leq E_\omega^0 |Y_k^n| \leq E_\omega^0 |\bar{Y}_k^n| \leq c_1 \beta_n, \tag{4.15}$$

$$\operatorname{Var} Y_{k,j}^n \leq \operatorname{Var} \bar{Y}_k^n \leq c_2 \beta_n^2, \qquad \textit{under } P_\omega^x. \tag{4.16}$$

Proof. Let

$$\mathcal{X}_k^{(n)}(t) = X_t^{(n)} + \Pi_n(X^{(n)}(T_k^n)) - X^{(n)}(T_k^n), \qquad t \in [T_k^n, V_{k+1}^n], \tag{4.17}$$

and note that

$$Y_k^n = (Y_{k,1}^n, Y_{k,2}^n) = \mathcal{X}_k^{(n)}(V_{k+1}^n) - \mathcal{X}_k^{(n)}(T_k^n).$$

It follows from the definition that we have $\sup_{S_{k-1}^n \leq t \leq U_k^n} |X^{(n)}(t) - X^{(n)}(S_{k-1}^n)| \leq 16\beta_n$, a.s. This, (4.8) and the definition of V_{k+1}^n imply that $|\bar{Y}_k^n|$ is stochastically majorized by an exponential random variable with mean $c_3\beta_n$. This easily implies the lemma. □

Next we will estimate the covariance of $Y_{k,1}^n$ and $Y_{j,1}^n$ for $j \neq k$.

Lemma 4.8. *There exist c_1, c_2 and n_1 such that for all $n \geq n_1$, $j < k-1$ and ω such that $0 \notin \Gamma_n^1 \setminus \partial_i \Gamma_n^1$, under P_ω^0,*

$$\operatorname{Cov}(Y_{j,1}^n, Y_{k,1}^n) \leq c_1 e^{-c_2(k-j)} \beta_n^2. \tag{4.18}$$

Proof. Assume that ω is such that $0 \notin \Gamma_n^1 \setminus \partial_i \Gamma_n^1$. Let

$$\begin{aligned}
\Gamma_n^3 &= \Gamma_n^1 \cap \mathcal{B}(u_n + \mathcal{O}_n, a_n/2) = \mathcal{B}(u_n + \mathcal{O}_n, 2\beta_n), \\
\Gamma_n^4 &= \partial_i \mathcal{B}(u_n + \mathcal{O}_n, 3\beta_n), \\
\tau(A) &= \inf\{t \geq 0 : \mathcal{X}_0^{(n)}(t) \in A\}.
\end{aligned}$$

Suppose that $x, v \in \Gamma_n^3$ and $y \in \Gamma_n^4$. By the Harnack inequality proved in Lemma 4.2,

$$\frac{P_\omega^x(\mathcal{X}_0^{(n)}(\tau(\Gamma_n^4)) = y)}{P_\omega^v(\mathcal{X}_0^{(n)}(\tau(\Gamma_n^4)) = y)} \geq c_3. \tag{4.19}$$

Let $\mathcal{T}_k^n$ have the same meaning as T_k^n but relative to the process $\mathcal{X}_k^{(n)}$ rather than $X^{(n)}$. We obtain from (4.19) and the strong Markov property applied at $\tau(\Gamma_n^4)$ that, for any $x, v, y \in \Gamma_n^3$ we have

$$\frac{P_\omega^x(\mathcal{X}_0^{(n)}(\mathcal{T}_1^n) = y)}{P_\omega^v(\mathcal{X}_0^{(n)}(\mathcal{T}_1^n) = y)} \geq c_3.$$

Recall that $T_0^n = 0$. The last estimate implies that, for $x, v, y \in \Gamma_n^3$,

$$\frac{P_\omega(\mathcal{X}_1^{(n)}(T_1^n) = y \mid \mathcal{X}_0^{(n)}(T_0^n) = x)}{P_\omega(\mathcal{X}_1^{(n)}(T_1^n) = y \mid \mathcal{X}_0^{(n)}(T_0^n) = v)} \geq c_3.$$

Since the process $X^{(n)}$ is time-homogeneous, this shows that for $x, v, y \in \Gamma_n^3$ and all k,

$$\frac{P_\omega(\mathcal{X}_{k+1}^{(n)}(T_{k+1}^n) = y \mid \mathcal{X}_k^{(n)}(T_k^n) = x)}{P_\omega(\mathcal{X}_{k+1}^{(n)}(T_{k+1}^n) = y \mid \mathcal{X}_k^{(n)}(T_k^n) = v)} \geq c_3. \tag{4.20}$$

We now apply Lemma 6.1 of [8] (see Lemma 1 of [7] for a better presentation of the same estimate) to see that (4.20) implies that there exist constants C_k, $k \geq 1$, such that for every k and all $x, v, y \in \Gamma_n^3$,

$$\frac{P_\omega^x(\mathcal{X}_k^{(n)}(T_k^n) = y)}{P_\omega^v(\mathcal{X}_k^{(n)}(T_k^n) = y)} \geq C_k.$$

Moreover, $C_k \in (0,1)$, C_k's depend only on c_3, and $1 - C_k \leq e^{-c_4 k}$ for some $c_4 > 0$ and all k. By time homogeneity of $X^{(n)}$, for $m \leq j < k$ and all $x, v, y, z \in \Gamma_n^3$,

$$\frac{P_\omega^z(\mathcal{X}_k^{(n)}(T_k^n) = y \mid \mathcal{X}_j^{(n)}(T_j^n) = x)}{P_\omega^z(\mathcal{X}_k^{(n)}(T_k^n) = y \mid \mathcal{X}_j^{(n)}(T_j^n) = v)} \geq C_{k-j},$$

and, by the strong Markov property applied at T_j^n,

$$\frac{P_\omega^z(\mathcal{X}_k^{(n)}(T_k^n) = y \mid \mathcal{X}_j^{(n)}(T_j^n) = x)}{P_\omega^z(\mathcal{X}_k^{(n)}(T_k^n) = y \mid \mathcal{X}_m^{(n)}(T_m^n) = v)} \geq C_{k-j}.$$

This and (4.15) imply that for $j < k-1$ and $x \in \mathbb{Z}^2$,

$$\begin{aligned} |E_\omega^x(Y_{k,1}^n - E_\omega^x Y_{k,1}^n \mid \mathcal{F}_{T_{j+1}^n})| &= |E_\omega^x(Y_{k,1}^n \mid \mathcal{F}_{T_{j+1}^n}) - E_\omega^x Y_{k,1}^n| \\ &\leq (1 - C_{k-j-1}) \sup_{y \in \mathbb{Z}^2} E_\omega^y |Y_{k,1}^n| \\ &\leq e^{-c_4(k-j-1)} c_5 \beta_n \leq c_6 e^{-c_4(k-j)} \beta_n. \end{aligned} \tag{4.21}$$

Hence for $j < k-1$,

$$\begin{aligned}
\mathrm{Cov}(Y^n_{j,1}, Y^n_{k,1}) &= E^x_\omega((Y^n_{j,1} - E^x_\omega Y^n_{j,1})(Y^n_{k,1} - E^x_\omega Y^n_{k,1})) \\
&= E^x_\omega(E^x_\omega((Y^n_{j,1} - E^x_\omega Y^n_{j,1})(Y^n_{k,1} - E^x_\omega Y^n_{k,1}) \mid \mathcal{F}_{T^n_{j+1}})) \\
&= E^x_\omega((Y^n_{j,1} - E^x_\omega Y^n_{j,1}) E^x_\omega(Y^n_{k,1} - E^x_\omega Y^n_{k,1} \mid \mathcal{F}_{T^n_{j+1}})) \\
&\le E^x_\omega(|Y^n_{j,1} - E^x_\omega Y^n_{j,1}| \cdot |E^x_\omega(Y^n_{k,1} - E^x_\omega Y^n_{k,1} \mid \mathcal{F}_{T^n_{j+1}})|) \\
&\le 2E^x_\omega |Y^n_{j,1}| c_6 e^{-c_4(k-j)} \beta_n \\
&\le c_7 e^{-c_4(k-j)} \beta_n^2.
\end{aligned}$$

□

Proof of Proposition 4.1. Assume that ω is such that $0 \notin \Gamma^1_n \setminus \partial_i \Gamma^1_n$. We combine (4.18) and (4.16) to see that for some c_1 and c_2 and all $m \ge 1$, we have under P^0_ω,

$$\begin{aligned}
\mathrm{Var}\left(\sum_{k=0}^m Y^n_{k,1}\right) &= \sum_{j=0}^m \sum_{k=0}^m \mathrm{Cov}(Y^n_{j,1}, Y^n_{k,1}) \\
&\le \sum_{j=0}^m \sum_{k=0}^m c_1 e^{-c_3(k-j)} \beta_n^2 \le c_2 m \beta_n^2.
\end{aligned} \tag{4.22}$$

For fixed n and ω, the process $\{\mathcal{X}^{(n)}_k(T^n_k), k \ge 1\}$ is Markov with a finite state space and one communicating class, so it has a unique stationary distribution. We will call it $\mathbf{p}(n)$. We will argue that $E^{\mathbf{p}(n)}_\omega Y^n_{k,1} = 0$. Since $X^{(n)}$ and $X^{(n-1)}$ satisfy the quenched invariance principle and they are random walks among symmetric (in distribution) conductances, they have zero means. Recall that $X^{(n)} = X^{n,1} + X^{n,2}$ and $\widehat{X}^{n,1}$ has the same distribution as $X^{(n-1)}$. It follows that for some $c_4 > 0$ and $c_5 < 1/4$ and all large t, we have

$$P^{\mathbf{p}(n)}_\omega\left(\sup_{1\le s\le t} |\widehat{X}^{n,1}_s| \ge c_4\sqrt{t}\right) = P^{\mathbf{p}(n)}_\omega\left(\sup_{1\le s\le t} |X^{(n-1)}_s| \ge c_4\sqrt{t}\right) < c_5.$$

Since $\widehat{X}^{n,1}_t = X^{n,1}(\widehat{\sigma}^{n,1}_t)$ and $\widehat{\sigma}^{n,1}_t \ge t$, the last estimate implies that

$$P^{\mathbf{p}(n)}_\omega\left(\sup_{1\le s\le t} |X^{n,1}_s| \ge c_4\sqrt{t}\right) < c_5.$$

We also have for some $c_6 > 0$ and $c_7 < 1/4$, and all large t,

$$P^{\mathbf{p}(n)}_\omega\left(\sup_{1\le s\le t} |X^{(n)}_s| \ge c_6\sqrt{t}\right) < c_7.$$

Since $X^{n,2} = X^{(n)} - X^{n,1}$, we obtain for some $c_8 > 0$ and $c_9 < 1/2$ and all large t,

$$P^{\mathbf{p}(n)}_\omega\left(\sup_{1\le s\le t} |X^{n,2}_s| \ge c_8\sqrt{t}\right) < c_9.$$

This shows that $X^{n,2}$ does not have a linear drift. It is clear from the law of large numbers that $\liminf_{t\to\infty} \sigma^{n,2}_t / t > 0$, so $\widehat{X}^{n,2}$ does not have a linear drift either. We conclude that $E^{\mathbf{p}(n)}_\omega Y^n_{k,1} = 0$.

Now suppose that $X_0^{(n)}$ does not necessarily have the distribution $\mathbf{p}(n)$. The fact that $E_\omega^{\mathbf{p}(n)} Y_{k,1}^n = 0$ and a calculation similar to that in (4.21) imply that,

$$|E_\omega^0 Y_{k,1}^n| \le c_{10} e^{-c_{11} k} \beta_n.$$

Let c_{12} be the constant denoted c_1 in (4.15). The last estimate and (4.15) imply that for some c_{13} and all $m \ge 1$,

$$\begin{aligned} \left| E_\omega^0 \sum_{k=0}^m Y_{k,1}^n \right| &\le \sum_{k\ge 0} |E_\omega^0 Y_{k,1}^n| + \sup_{k\ge 1} E_\omega^0 |\bar{Y}_k^n| \\ &\le \sum_{k \ge 0} c_{10} e^{-c_{11} k} \beta_n + c_{12}\beta_n \le c_{13}\beta_n. \end{aligned} \tag{4.23}$$

All estimates that we derived for $Y_{k,1}^n$'s apply to $Y_{k,2}^n$'s as well, by symmetry.

Note that $|X^{(n)}(U_{k+1}^n) - X^{(n)}(T_k^n)| \ge \beta_n/2$. We have $V_{k+1}^n - T_k^n \ge U_{k+1}^n - T_k^n$ so we can assume (♣) that b_n/a_{n-1} is so large that for some $p_1 > 0$ and n_2, for all $n \ge n_2$ and $k \ge 1$,

$$P_\omega^x (V_{k+1}^n - T_k^n \ge \beta_n^2 \mid \mathcal{F}_{T_k^n}) \ge p_1.$$

Let $\mathcal{V}_m$ be a binomial random variable with parameters m and p_1. We see that $\sigma^{n,2}(V_m^n) = \sum_{k=0}^m V_{k+1}^n - T_k^n$ is stochastically minorized by $\beta_n^2 \mathcal{V}_m$.

Recall that $u \ge a_n^2$. Let m_1 be the smallest integer such that

$$P_\omega^0 (V_{m_1}^n \le u) < \delta/4. \tag{4.24}$$

Then

$$P_\omega^0 (V_{m_1 - 1}^n \le u) \ge \delta/4. \tag{4.25}$$

Since δ in (4.14) can be arbitrarily small, we have for for some n_3 and all $n \ge n_3$,

$$P_\omega^0 (\sigma_u^{n,2}/u \le \delta^4) \ge 1 - \delta/8. \tag{4.26}$$

The following estimate follows from the fact that $\sigma^{n,2}(V_{m_1-1}^n)$ is stochastically minorized by $\beta_n^2 \mathcal{V}_{m_1-1}$, and from (4.25)-(4.26),

$$\begin{aligned} P_\omega^0(\beta_n^2 \mathcal{V}_{m_1-1} \le \delta^4 u) &\ge P_\omega^0(\sigma^{n,2}(V_{m_1-1}^n) \le \delta^4 u) \\ &\ge P_\omega^0(\sigma_u^{n,2} \le \delta^4 u, V_{m_1-1}^n \le u) \ge \delta/8. \end{aligned}$$

This implies that for some c_{14}, we have $m_1 \le c_{14}\delta^3 u/\beta_n^2$. In other words, $u \ge m_1 \beta_n^2/(c_{14}\delta^3)$. Note that for a fixed δ, we have for large n, (♣) $u^{1/2}\delta/4 - c_{13}\beta_n \ge u^{1/2}\delta/8$. These observations, (4.22), (4.23) and the Chebyshev inequality imply that for $m \le m_1$,

$$
\begin{aligned}
(4.27)\qquad & P^0_\omega\left(u^{-1/2}\left(\left|\sum_{k=0}^m Y^n_{k,1}\right| + \left|\sum_{k=0}^m Y^n_{k,2}\right|\right) \ge \delta/2\right) \\
&\le P^0_\omega\left(\left|\sum_{k=0}^m Y^n_{k,1}\right| \ge u^{1/2}\delta/4\right) + P^0_\omega\left(\left|\sum_{k=0}^m Y^n_{k,2}\right| \ge u^{1/2}\delta/4\right) \\
&\le P^0_\omega\left(\left|\sum_{k=0}^m Y^n_{k,1} - E^0_\omega \sum_{k=0}^m Y^n_{k,1}\right| \ge u^{1/2}\delta/4 - c_{13}\beta_n\right) \\
&\qquad + P^0_\omega\left(\left|\sum_{k=0}^m Y^n_{k,2} - E^0_\omega \sum_{k=0}^m Y^n_{k,2}\right| \ge u^{1/2}\delta/4 - c_{13}\beta_n\right) \\
&\le \frac{\mathrm{Var}\left(\sum_{k=0}^m Y^n_{k,1}\right)}{u\delta^2/64} + \frac{\mathrm{Var}\left(\sum_{k=0}^m Y^n_{k,2}\right)}{u\delta^2/64} \\
&\le \frac{2c_2 m_1 \beta_n^2}{(c_{14}^{-1}\delta^{-3} m_1 \beta_n^2)\delta^2/64} \le c_{15}\delta.
\end{aligned}
$$

Let $M = \min\{m \ge 1 : u^{-1/2}\left(\left|\sum_{k=0}^m Y^n_{k,1}\right| + \left|\sum_{k=0}^m Y^n_{k,2}\right|\right) \ge \delta\}$. By the strong Markov property applied at M and (4.27),

(4.28)

$$
\begin{aligned}
& P^0_\omega\left(\sup_{1\le m\le m_1} u^{-1/2}\left(\left|\sum_{k=0}^m Y^n_{k,1}\right| + \left|\sum_{k=0}^m Y^n_{k,2}\right|\right) \ge \delta,\ u^{-1/2}\left(\left|\sum_{k=0}^{m_1} Y^n_{k,1}\right| + \left|\sum_{k=0}^{m_1} Y^n_{k,2}\right|\right) \le \delta/2\right) \\
&\le P^0_\omega\left(u^{-1/2}\left(\left|\sum_{k=0}^{m_1-M} Y^n_{k,1}\right| + \left|\sum_{k=0}^{m_1-M} Y^n_{k,2}\right|\right) \ge \delta/2 \mid M < m_1\right) \le c_{15}\delta.
\end{aligned}
$$

Recall that $u \ge m_1\beta_n^2/(c_{14}\delta^3)$. For a fixed δ and large n, (♣) $u^{1/2}\delta - 2c_{12}\beta_n \ge u^{1/2}\delta/2$. It follows from this, (4.15) and (4.16) that

$$
\begin{aligned}
(4.29)\qquad P^0_\omega\left(\exists k \le m_1 : |\bar{Y}^n_k| \ge u^{1/2}\delta\right) &\le m_1 \sup_{k\le m_1} P^0_\omega\left(|\bar{Y}^n_k| \ge u^{1/2}\delta\right) \\
&\le m_1 \sup_{k\le m_1} P^0_\omega\left(|\bar{Y}^n_k| - E^0_\omega|\bar{Y}^n_k| \ge u^{1/2}\delta - c_{12}\beta_n\right) \\
&\le m_1 \frac{c_{11}\beta_n^2}{u\delta^2/4} \le m_1 \frac{c_{11}\beta_n^2}{(c_{14}^{-1}\delta^{-3} m_1\beta_n^2)\delta^2} \le c_{16}\delta.
\end{aligned}
$$

We use (4.24), (4.27), (4.28) and (4.29) to obtain

$$
\begin{aligned}
&P^0_\omega\left(\sup_{0\le s\le u} u^{-1/2}|X^{n,2}_s| \ge 2\delta\right)\\
&\le P^0_\omega(V^n_{m_1}\le u) + P^0_\omega\left(u^{-1/2}\left(\left|\sum_{k=0}^{m_1} Y^n_{k,1}\right| + \left|\sum_{k=0}^{m_1} Y^n_{k,2}\right|\right)\ge \delta/2\right)\\
&+ P^0_\omega\left(\sup_{1\le m\le m_1} u^{-1/2}\left(\left|\sum_{k=0}^{m} Y^n_{k,1}\right| + \left|\sum_{k=0}^{m} Y^n_{k,2}\right|\right)\ge \delta,\ u^{-1/2}\left(\left|\sum_{k=0}^{m_1} Y^n_{k,1}\right| + \left|\sum_{k=0}^{m_1} Y^n_{k,2}\right|\right)\le \delta/2\right)\\
&+ P^0_\omega\left(\exists k\le m_1 : |\bar Y^n_k| \ge u^{1/2}\delta\right)\\
&\le \delta/4 + c_{15}\delta + c_{15}\delta + c_{16}\delta.
\end{aligned}
$$

Since $\delta > 0$ is arbitrarily small, this implies that for every $\delta > 0$, some n_3 and all $n \ge n_3$,

$$P^0_\omega\left(\sup_{0\le s\le u} u^{-1/2}|X^{n,2}_s| \ge \delta\right) \le \delta/2.$$

This and (4.11) yield the proposition. □

Recall from (1.2) the definition of the averaged measure $\mathbf{P}$.

Lemma 4.9. *For every $\delta > 0$ there exists n_1 such that for all $n \ge n_1$ and $u \ge a_n^2$,*

$$\mathbf{P}\left(\sigma^{n,2}_u/u \le \delta, \sup_{0\le s\le u} u^{-1/2}|X^{n,2}_s| \le \delta\right) \ge 1-\delta. \tag{4.30}$$

Proof. By Proposition 4.1 applied to $\delta/2$ in place of δ, for every $\delta > 0$ there exists n_2 such that for all $n \ge n_2$, $u \ge a_n^2$, and ω such that $0 \notin \Gamma^1_n \setminus \partial_i\Gamma^1_n$,

$$P^0_\omega\left(\sigma^{n,2}_u/u \le \delta, \sup_{0\le s\le u} u^{-1/2}|X^{n,2}_s| \le \delta\right) \ge 1-\delta/2. \tag{4.31}$$

Let $|A|$ denote the cardinality of $A \subset \mathbb{Z}^2$. Since $|\Gamma^1_n| \le 25\beta_n^2 \le 25a_n^2 n^{-1/2} = 25n^{-1/2}|B'_n|$, the definitions of $\mathcal{O}_n$ and Γ^1_n imply that $\mathbf{P}(0 \in \Gamma^1_n \setminus \partial_i\Gamma^1_n) < \delta/2$ for some $n_3 \ge n_2$ and all $n \ge n_3$. This and (4.31) imply (4.30). □

In the following lemma and its proof, when we write the Prokhorov distance between processes such as $\{(1/a)X^{(n-1)}_{ta^2}, t \in [0,1]\}$, we always assume that they are distributed according to $\mathbf{P}$.

Lemma 4.10. *There exists a function $\rho^* : (0,\infty) \to (0,\infty)$ with $\lim_{\delta\downarrow 0}\rho^*(\delta) = 0$ and a sequence $\{a_n\}$ with the following properties,*

$$\mathbf{d}(\{(1/a)X^{(n-1)}_{ta^2}, t \in [0,1]\}, P_{BM}) \le 2^{-n}, \qquad a \ge a_n. \tag{4.32}$$

Moreover, suppose that for $\delta < 1/2$ and all $u \ge a_n^2$,

$$\mathbf{P}\left(\sigma^{n,2}_u/u \le \delta, \sup_{0\le s\le u} u^{-1/2}|X^{n,2}_s| \le \delta\right) \ge 1-\delta. \tag{4.33}$$

Then $\mathbf{d}(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, P_{BM}) \le 2^{-n} + \rho^*(\delta)$, for all $a \ge a_n$.

Proof. Formula (4.32) is special case of (3.1).

Fix some $a \geq a_n$. We will apply (4.33) with $u = a^2$. Note that on the event in (4.33) we have

$$1 - \sigma^{n,1}_{a^2}/a^2 = u/u - \sigma^{n,1}_u/u = \sigma^{n,2}_u/u \leq \delta. \tag{4.34}$$

The function $t \to \sigma^{n,1}_{ta^2}/a^2$ is Lipschitz with the constant 1 and $\sigma^{n,1}_{ta^2}/a^2 \leq t$ so (4.34) implies for $t \in [0,1]$,

$$t - \sigma^{n,1}_{ta^2}/a^2 \leq 1 - \sigma^{n,1}_{a^2}/a^2 \leq \delta. \tag{4.35}$$

Recall the function $\rho(\delta)$ from the proof of Lemma 3.2, such that $P_{\rm BM}(\mathrm{Osc}(W,\delta) \geq \rho(\delta)) < \rho(\delta)$ and $\lim_{\delta\downarrow 0} \rho(\delta) = 0$. By (4.35), we can apply Lemma 3.2 with $\sigma_t = \sigma^{n,1}_{ta^2}/a^2$. Recall that $W^*(t) = W(\sigma_t)$. By the definition of $\widehat{X}^{n,1}$,

$$\begin{aligned}
&\mathbf{d}(\{(1/a)X^{n,1}_{ta^2}, t \in [0,1]\}, P_{\rm BM}) \\
&\leq \mathbf{d}(\{(1/a)X^{n,1}_{t/a^2}, t \in [0,1]\}, \{W^*_t, t \in [0,1]\}) + \mathbf{d}(\{W^*_t, t \in [0,1]\}, P_{\rm BM}) \\
&\leq \mathbf{d}(\{(1/a)X^{n,1}_{ta^2}, t \in [0,1]\}, \{W^*_t, t \in [0,1]\}) + \rho(\delta) + \delta \\
&= \mathbf{d}(\{(1/a)\widehat{X}^{n,1}(\sigma^{n,1}_{ta^2}), t \in [0,1]\}, \{W(\sigma^{n,1}_{ta^2}/a^2), t \in [0,1]\}) + \rho(\delta) + \delta. \qquad (4.36)
\end{aligned}$$

Recall from (3.4) that for a fixed $\omega \in \Omega$, the distribution of $\{\widehat{X}^{n,1}_t, t \geq 0\}$ is the same as that of $\{X^{n-1}_t, t \geq 0\}$. In view of Theorem 2.2, we can make a_n so large (♣) that $\mathbb{P}(\mathrm{Osc}(\widehat{X}^{n,1}, \delta) \geq 2\rho(\delta)) < 2\rho(\delta)$. This, Lemma 3.1 and the definition of the Prokhorov distance imply that

$$\begin{aligned}
&\mathbf{d}(\{(1/a)\widehat{X}^{n,1}(\sigma^{n,1}_{ta^2}),\, t \in [0,1]\}, \{W(\sigma^{n,1}_{ta^2}/a^2), t \in [0,1]\}) \\
&\qquad \leq \mathbf{d}(\{(1/a)\widehat{X}^{n,1}_{ta^2}, t \in [0,1]\}, \{W_t, t \in [0,1]\}) + 4\rho(\delta) \\
&\qquad = \mathbf{d}(\{(1/a)X^{(n-1)}_{ta^2}, t \in [0,1]\}, \{W_t, t \in [0,1]\}) + 4\rho(\delta) \\
&\qquad \leq 2^{-n} + 4\rho(\delta).
\end{aligned}$$

In the final two lines line we used (3.4) and (4.32).

Combining the estimates above, since $P^0_\omega\left(\sup_{0\leq s\leq u} u^{-1/2}|X^{n,2}_s| \leq \delta\right) \geq 1 - \delta$ and $X^{(n)} = X^{n,1} + X^{n,2}$, Lemma 3.3 shows that

$$\begin{aligned}
&\mathbf{d}(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, P_{\rm BM}) \\
&\leq \mathbf{d}(\{(1/a)X^{(n)}_{ta^2}, t \in [0,1]\}, \{(1/a)X^{n,1}_{ta^2}, t \in [0,1]\}) \\
&\qquad + \mathbf{d}(\{(1/a)X^{n,1}_{ta^2}, t \in [0,1]\}, P_{\rm BM}) \\
&\leq \delta + 2^{-n} + 5\rho(\delta) + \delta.
\end{aligned}$$

We conclude that the lemma holds if we take $\rho^*(\delta) = 5\rho(\delta) + 2\delta$. □

Proof of Theorem 1.4. Choose an arbitrarily small $\varepsilon > 0$. We will show that there exists a_* such that for every $a \geq a_*$,

$$\mathbf{d}(\{(1/a)X_{ta^2}, t \in [0,1]\}, P_{\rm BM}) \leq \varepsilon. \tag{4.37}$$

Recall ρ^* from Lemma 4.10. Let n_1 be such that $2^{-n_1} \leq \varepsilon/4$ and let $\delta > 0$ be so small that $2^{-n_1} + \rho^*(\delta) < \varepsilon/2$. Let n_2 be defined as n_1 in Lemma 4.9, relative to this δ. Then, according to Lemma 4.10,

$$\mathbf{d}(\{(1/a)X^n_{ta^2}, t \in [0,1]\}, P_{\mathrm{BM}}) \le 2^{-n} + \rho^*(\delta) < \varepsilon/2, \tag{4.38}$$

for all $n \ge n_3 := n_1 \vee n_2$ and $a \ge a_n$.

For a set K let $\mathcal{B}(K,r) = \{z : \mathrm{dist}(z,K) < r\}$ and recall the definition of D_n given in (2.1). Let

$$\begin{aligned}
F_1 &= \{0 \in \mathcal{B}(D_{n+1}, a_{n+1}/\log(n+1))\},\\
F_2 &= \{0 \notin \mathcal{B}(D_{n+1}, a_{n+1}/\log(n+1))\} \cap \{\exists t \in [0, a^2_{n+1}] : X^{(n)}_t \in D_{n+1}\},\\
G^k_1 &= \{0 \in \mathcal{B}(D_k, b_k/k)\}, \qquad k > n+1,\\
G^k_2 &= \{0 \notin \mathcal{B}(D_k, b_k/k)\} \cap \{\exists t \in [0, a^2_{n+1}] : X^{(n)}_t \in D_k\}, \qquad k > n+1.
\end{aligned}$$

The area of $\mathcal{B}(D_{n+1}, a_{n+1}/\log(n+1))$ is bounded by $c_1(a_{n+1}/\log(n+1))^2$ so

$$\mathbb{P}(F_1) \le c_1(a_{n+1}/\log(n+1))^2/a^2_{n+1} = c_1/\log^2(n+1). \tag{4.39}$$

We choose $n_4 > n_3$ such that $c_1/\log^2(n+1) < \varepsilon/8$ for $n \ge n_4$.

Note that D_{n+1} is a subset of a square with side $4\beta_{n+1} \le 4a_{n+1}n^{-1/4}$. This easily implies that there exists $n_5 \ge n_4$ such that for $n \ge n_5$,

$$P_{\mathrm{BM}}\left(\exists t \in [0, a^2_{n+1}] : W(t) \in D_{n+1} \mid 0 \notin \mathcal{B}(D_{n+1}, a_{n+1}/\log(n+1))\right) \le \varepsilon/16.$$

We can assume (♣) that a_{n+1}/a_n is so large that for some $n_6 \ge n_5$ and all $n \ge n_6$,

$$\mathbb{P}(F_2) \le \mathbb{P}\left(\exists t \in [0, a^2_{n+1}] : X^{(n)}_t \in D_{n+1} \mid 0 \notin \mathcal{B}(D_{n+1}, a_{n+1}/\log(n+1))\right) \tag{4.40}$$

$$\le \varepsilon/8. \tag{4.41}$$

The area of $\mathcal{B}(D_k, b_k/k)$ is bounded by $c_2 b^2_k/k$ so

$$\mathbb{P}(G^k_1) \le (c_2 b^2_k/k)/a^2_k \le c_3(b^2_k/k)/(kb^2_k) = c_3/k^2. \tag{4.42}$$

We let $n_7 > n_6$ be so large that $\sum_{k \ge n_7} c_3/k^2 < \varepsilon/8$. For all $k > n+1 \ge n_7+1$, we make b_k/k so large (♣) that

$$\mathbb{P}(G^k_2) \le \mathbb{P}\left(\sup_{t \in [0, a^2_{n+1}]} |X^n_t| \ge b_k/k\right) \le c_3/k^2. \tag{4.43}$$

We combine (4.39), (4.40), (4.42) and (4.43) to see that for $n \ge n_7$,

$$\begin{aligned}
&\mathbb{P}(\exists t \in [0, a^2_{n+1}]\ \exists k \ge n+1 : X^{(n)}_t \in D_k)\\
&\quad\le \mathbb{P}(F_1) + \mathbb{P}(F_2) + \sum_{k>n+1} \mathbb{P}(G^k_1) + \sum_{k>n+1} \mathbb{P}(G^k_2)\\
&\quad\le \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 = \varepsilon/2.
\end{aligned} \tag{4.44}$$

Let $R_{n+1} = \inf\{t \ge 0 : X_t \in \bigcup_{k \ge n+1} \mathcal{D}_k\}$. It is standard to construct X and $X^{(n)}$ on a common probability space so that $X_t = X^n_t$ for all $t \in [0, R_{n+1})$. This and (4.44) imply that for $n \ge n_7$ and all $a \in [a_n, a_{n+1}]$ we have

$$P(\exists t \in [0,1] : (1/a)X_{ta^2} \ne (1/a)X^{(n)}_{ta^2}) \le \varepsilon/2.$$

We combine this with (4.38) to see that for all $a \ge a_{n_6}$,

$$\mathbf{d}(\{(1/a)X_{ta^2}, t \in [0,1]\}, P_{\mathrm{BM}}) \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We conclude that (4.37) holds with $a_* = a_{n_7}$.

This completes the proof of AFCLT. The WFCLT then follows from Theorem 2.13 of [4]. $\square$

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