BOUNDARIES OF PLANAR GRAPHS, VIA CIRCLE PACKINGS

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ABSTRACT. We provide a geometric representation of the Poisson and Martin boundaries of a transient, bounded degree triangulation of the plane in terms of its circle packing in the unit disc (this packing is unique up to Möbius transformations). More precisely, we show that any bounded harmonic function on the graph is the harmonic extension of some measurable function on the boundary of the disk, and that the space of extremal positive harmonic functions, i.e. the Martin boundary, is homeomorphic to the unit circle.

All our results hold more generally for any "good"-embedding of planar graphs, that is, an embedding in the unit disc with straight lines such that angles are bounded away from 0 and π uniformly, and lengths of adjacent edges are comparable. Furthermore, we show that in a good embedding of a planar graph the probability that a random walk exits a disc through a sufficiently wide arc is at least a constant, and that Brownian motion on such graphs takes time of order r^2 to exit a disc of radius r. These answer a question recently posed by Chelkak [7].

1   Introduction

Given a Markov chain, it is natural to ask what is its "final" behavior, that is, the behavior as the time tends to infinity. For example, consider the lazy simple random walk on a rooted 3-regular tree — the path of the random walk almost surely determines a unique infinite branch of the tree. This branch is determined by the tail σ-field of the random walk and moreover, this σ-field is characterized by the set of such infinite branches. In general, it is more useful to consider the *invariant σ-field* $\mathscr{I}$, that is, all the events that are invariant under the time-shift operator. In the case of lazy Markov chains, these two σ-fields are equivalent [8, 16].

To any invariant event A we can associate a harmonic function h_A on the state space by $h_A(x) = \mathbb{P}_x(A)$, i.e., the probability that A occurs starting the chain from x. (A function h is harmonic if its value at a state is the expected value of h after one step of the chain.) In fact, there is a correspondence between bounded invariant random variables and bounded harmonic functions on the state space

(given such a random variable Y, the function is $h_Y(x) = \mathbb{E}_x(Y)$, see [16, 19]). Thus, the set of bounded harmonic functions on the state space characterizes all the "final" behaviors of the Markov chain.

In this paper we consider reversible Markov chains in discrete time and space (i.e., weighted random walks on a graphs). It is not hard to see that if the chain is recurrent, then there are no non-constant bounded harmonic functions. On the other hand, transience does not guarantee the existence of such functions, as can be seen in the simple random walk on $\mathbb{Z}^3$. However, in the planar case there is such a dichotomy, Benjamini and Schramm [6] proved that if G is a transient, bounded degree planar graph, then G exhibits non-constant bounded harmonic functions.

The proof in [6] relies on the theory of circle packing. Recall that a **circle packing** P of a planar graph G is a set of of circles with disjoint interiors $\{C_v\}_{v \in G}$ such that two circles are tangent if and only if the corresponding vertices form an edge. Koebe's Circle Packing Theorem [17] states that any planar graph has a circle packing, and that for triangulations (graphs where all faces are triangles) the circle packing is essentially unique. Given a circle packing, we embed the graph in $\mathbb{R}^2$, with straight line segments between the corresponding centers of circles for edges. The **carrier** of P, denoted carr(P), is the union of all the polygons corresponding to the faces. He and Schramm [13] provided an insightful connection between the probabilistic notion of recurrence or transience of G and the geometry of carr(P). Their theorem states that if G is a bounded degree one-ended triangulation, then it can be circle packed so that the carrier is either the entire plane or the unit disc U according to whether G is recurrent or transient, respectively. Since we are interested in non-constant bounded harmonic functions, we consider here only the latter case.

Consider a transient, bounded degree, one-ended triangulation G and its circle packing $P = \{C_v\}_{v \in V}$ with carr(P) = U. We identify each vertex v with the center of C_v — it will always be clear from the context if the letter v represents a vertex or a point in $\mathbb{R}^2$. Let $\{X_n\}$ be the simple random walk on G. A principal result of Benjamini and Schramm [6] is that $\lim_{n \to \infty} X_n$ exists and is a point $X_\infty \in \partial U$ almost surely, furthermore, its distribution is non-atomic. This immediately implies that any bounded measurable function $g : \partial U \to \mathbb{R}$ can be extended to a bounded harmonic function $h : V \to \mathbb{R}$ by setting $h(v) = \mathbb{E}_v[g(X_\infty)]$, where for a vertex x we write $\mathbb{E}_x$ for the expectation w.r.t. the random walk started at x. Since the distribution of X_∞ is non-atomic they deduce that a non-constant bounded harmonic function exists.

The main result of this paper is that there are no other bounded harmonic functions, i.e. any bounded harmonic function can be represented this way. Recall that a graph is one-ended if removal of any finite set of vertices leaves only one infinite connected component.

Theorem 1.1. *Let $G = (V, E)$ be a transient bounded degree, one-ended triangulation and let P be a circle packing of G with $\mathrm{carr}(P) = U$. Then for any bounded harmonic function $h : V \to \mathbb{R}$ there exists a bounded measurable function $g : \partial U \to \mathbb{R}$ such that $h(v) = \mathbb{E}_v\big[g(X_\infty)\big]$.*

For a vertex x we write $\mathbb{P}_x$ for the probability measure on $G^{\mathbb{N}}$ of the Markov chain started at x. The measure space $(G^{\mathbb{N}}, \mathscr{I}, \mathbb{P}_x)$ is often called the **Poisson boundary** of the chain. The choice of x does not matter much because the measures $\mathbb{P}_x$ are absolutely continuous with respect to each other. As mentioned before, there is a correspondence between bounded harmonic functions and $L^\infty(G^{\mathbb{N}}, \mathscr{I}, \mathbb{P}_x)$ and for that reason the space of bounded harmonic functions is sometimes also referred to as the Poisson boundary. Theorem 1.1 shows that if we circle pack G in U, then ∂U is a representation of the Poisson boundary. More precisely, let $f : G^{\mathbb{N}} \to \partial U$ be the measurable function (defined $\mathbb{P}_x$-almost everywhere) $f(\{x_n\}) = \lim x_n$ and let $\mathscr{B} \subset \mathscr{I}$ be the pull back σ-algebra on $G^{\mathbb{N}}$. Then $\mathscr{B}$ and $\mathscr{I}$ are in fact equivalent, i.e., for any $A \in \mathscr{I}$ there exists $B \in \mathscr{B}$ such that the measure of $A \triangle B$ is zero.

The Martin boundary ([10, 20, 26]) is another concept of a boundary of a Markov chain, associated with the space of positive harmonic functions. While the Poisson boundary is naturally defined as a measure space, the Martin boundary is a topological space. It is well known (see chapter 24 of [26]) that the Poisson boundary may be obtained by endowing the Martin boundary with a suitable measure. Hence, in addition to its intrinsic interest, the Martin boundary studied here will provide more information and will yield Theorem 1.1 rather abstractly.

An illustrative example of the difference between the boundaries is the following. Let G be the graph obtained from $\mathbb{Z}^3$ by connecting its root to a disjoint one-sided infinite path. It is possible for a positive harmonic function to diverge only along the path. Thus the Martin boundary will consist of two points (corresponding to the two "infinities" of G), however, since the simple random walk has probability 0 of staying in the infinite path forever, and $\mathbb{Z}^3$ has no non-constant bounded harmonic functions, the Poisson boundary will have all its mass on one of the points.

Let us formally define the Martin boundary. Let x_0 be an arbitrary fixed root of G and $M(x, y)$ be the **Martin kernel**

$$M(x, y) = \frac{G(x, y)}{G(x_0, y)},$$

where $G(x, y) = \mathbb{E}_x[\# \text{ visits to } y]$ is the Green function. For any fixed y, the function $M(\cdot, y)$ is a positive function that is harmonic everywhere except at y. Hence, if for some sequence y_n, such that the graph distance between x_0 and y_n tends to infinity, the functions $M(\cdot, y_n)$ converges pointwise, then the limit is a positive harmonic function on G. The **Martin boundary** is defined to be the set $\mathcal{M}$ of all such limit points, endowed with the pointwise convergence topology.

A positive harmonic function $h : V \to \mathbb{R}$ such that $h(x_0) = 1$ is called **minimal** if for any positive harmonic function g such that $g(x) \leq h(x)$ for all x, then $g = ch$ for some constant $c > 0$. The minimal functions are the extremal points of the convex set of positive harmonic functions, normalized to have $h(x_0) = 1$. By Choquet's Theorem and [26, Theorem 24.8] it follows that any positive harmonic function h can be written as $h = \int g \, d\mu(g)$ for some measure μ depending on h, and supported on the set of minimal harmonic functions. If we normalize so that $h(x_0) = 1$, then μ is a probability measure.

Theorem 1.2. *Let $G = (V, E)$ be a transient bounded degree, one-ended triangulation and let P be a circle packing of G with $\mathrm{carr}(P) = U$. Then*

(1) *For a sequence $y_n \in V$ we have that $M(\cdot, y_n)$ converges pointwise if and only if y_n converges in $\mathbb{R}^2$ (in particular, the limit only depends on $\lim y_n$).*
(2) *If $y_n \to \xi \in \partial U$, then $\lim M(\cdot, y_n)$ is a minimal harmonic function.*
(3) *The map $\xi \mapsto \lim M(\cdot, y_n)$, where $y_n \to \xi$, is a homeomorphism.*

In particular, the Martin boundary is homeomorphic to ∂U.

The limit $\lim_{y \to \xi} M(\cdot, y)$ is denoted by M_ξ. Thus for any positive harmonic function h there is some measure μ on ∂U, so that $h = \int_{\partial U} M_\xi \, d\mu(\xi)$.

A similar characterization of the Poisson boundary of planar graphs in terms of their square tiling was recently obtained by Georgakopoulos [12]. His results allow him to characterize the Poisson boundary for a somewhat more general set of graphs, namely, of bounded degree *uniquely absorbing* planar graphs. The analysis in this paper of random walk via circle packings and other embeddings requires a completely different set of tools and in return allows us to characterize the Martin boundary with no additional cost.

1.1 Good embeddings of planar graphs

A circle packing embedding of a graph G yields an embedding of G in $\mathbb{R}^2$ with straight lines such that no two edges cross. Any embedding satisfying this is called an **embedding with straight lines**. We will prove our results for more general embeddings than the circle packing embedding. The setting below has risen in the study of critical 2D lattice models and was formalized by Chelkak [7]. Let $G = (V, E)$ be an infinite, connected, simple planar graph together with an embedding with straight lines. As before, we identify a vertex v with its image in the embedding. We write $|u - v|$ for the Euclidean distance between points in the plane. For constants $D \in (1, \infty)$ and $\eta > 0$ we say that the embedding is (D, η)-**good** if it satisfies:

(a) **No flat angles.** For any face, all the inner angles are at most $\pi - \eta$. In particular, all faces are convex, there is no outer face and the number of edges in a face is at most $2\pi/\eta$.

(b) **Adjacent edges have comparable lengths.** For any two adjacent edges $e_1 = (u, v)$ and $e_2 = (u, w)$ we have that $|u - w|/|u - v| \in [D^{-1}, D]$.

We say that an embedding is **good** if it has straight lines and it is (D, η)-good for some D, η. A classical lemma of Rodin and Sullivan [21] (known as the *Ring Lemma*) asserts that the ratio between radii of tangent circles in a circle packing of a bounded degree triangulation is bounded above and away from 0. We immediately get the following.

Proposition 1.3. *Any circle packing of a bounded degree triangulation is (D, η)-good for some D and η that only depend on the maximum degree.*

In a similar fashion to the circle packing setting, we define the carrier of the embedding of G, denoted by carr(G), to be the union of all the faces of the embedding. Note that if G is a one-ended triangulation, then carr(G) is always an open simply connected set in the plane. Lastly, suppose that the edges of the graph are equipped with positive weights $\{w_e\}_{e \in E}$ and consider the weighted random walk $\{X_n\}$ defined by $P(X_1 = u | X_0 = v) = w_{(v,u)}/w_v$ for any edge (u, v), where $w_v = \sum_{u:u \sim v} w_{(u,v)}$. A function $h : V \to \mathbb{R}$ is harmonic with respect to the weighted graph when

$$h(v) = \sum_{u:u \sim v} \frac{w_{(u,v)}}{w_v} h(u), \tag{1.1}$$

e or in other words, when $h(X_n)$ is a martingale. The general version of Theorems 1.1 and 1.2 is now stated in a straightforward manner.

Theorems 1.1′ and 1.2′. *Let $G = (V, E)$ be a bounded degree planar graph with a good embedding with straight lines such that* $\mathrm{carr}(G) = U$. *Assume that G is equipped with positive edge weights bounded above and away from* 0. *Then the conclusions of Theorems 1.1 and 1.2 hold verbatim.*

Theorems 1.1 and 1.2 are immediate corollaries of this statement together with Proposition 1.3.

1.2   Harmonic measure and exit time of discrete balls

In the following, let $G = (V, E)$ be a planar graph with a good embedding. A discrete domain S is a subset of V along with the induced edges $E(S) = (S \times S) \cap E$. The boundary of S, denoted ∂S is the external vertex boundary, i.e. all vertices not in S with a neighbor in S. For $u \in \mathbb{R}^2$ we denote by $B_{\mathrm{euc}}(u, r)$ the Euclidean ball $\{y \in \mathbb{R}^2 : |u - y| \le r\}$ of radius r centred at u, and the discrete Euclidean ball $V_{\mathrm{euc}}(u, r)$ is the vertex set

$$V_{\mathrm{euc}}(u, r) = V \cap B_{\mathrm{euc}}(u, r).$$

As before, assume that the edges are equipped with positive weights and consider the weighted random walk $\{X_n\}$. For $A \subset V$ let τ_A be the first hitting time of A, that is $\tau_A = \min\{n : X_n \in A\}$ or ∞ if A is never hit. The following two theorems answer a question recently posed by Chelkak [7, Page 9].

Theorem 1.4. *For any positive constants D, η there exists $c = c(D, \eta) > 0$ with the following. Assume that G is a graph with a (D, η)-good embedding, and all edges weights in $[D^{-1}, D]$. Then for any vertex u, any $r \ge 0$ such that $B_{\mathrm{euc}}(u, r) \subset \mathrm{carr}(G)$ and any interval $I \subset \mathbb{R}/(2\pi\mathbb{Z})$ of length $\pi - \eta$ we have*

$$\mathbb{P}_u\big(\arg(X_{T_r} - u) \in I\big) \ge c,$$

where $T_r = \tau_{\partial V_{\mathrm{euc}}(u, r)}$ is the first exit time from $V_{\mathrm{euc}}(u, r)$.

Note that for smaller intervals of arguments the statement above may be false; for example, the left hand side is 0 if $\partial V_{\mathrm{euc}}(u, r)$ contains no vertex in these directions.

For a vertex $u \in V$ we denote its **radius of isolation** by $r_u = \min_{V \setminus \{u\}}\{|u - v|\}$. We use $f \asymp g$ when there is some $C = C(D, \eta)$ so that $C^{-1}g \le f \le Cg$.

Theorem 1.5. *For any positive constants D, η there exists $C = C(D, \eta) \geq 1$ with the following. Assume that G is a graph with a (D, η)-good embedding and all edges weights are in $[D^{-1}, D]$. Then for any vertex u and any $r \geq r_u$ with $B_{\mathrm{euc}}(u, Cr) \subset \mathrm{carr}(G)$ we have*

$$\mathbb{E}_u \sum_{t=0}^{T_r} r_{X_t}^2 \asymp r^2$$

where $T_r = \tau_{\partial V_{\mathrm{euc}}(u,r)}$ is the first exit time from $V_{\mathrm{euc}}(u, r)$.

The reader may wonder why we require $B_{\mathrm{euc}}(x_0, Cr) \subset \mathrm{carr}(G)$, while the theorem only talks about the time to exit the smaller $B_{\mathrm{euc}}(x_0, r)$. This is an artifact of our proof, and the stronger requirement can indeed be removed. This requires showing that it is possible to "extend" the embedding to a good embedding of a larger graph with carrier $\mathbb{R}^2$. This is indeed possible, and we plan to address this in a future paper.

1.3 About the proofs and the organization of the paper

We would like to compare the random walk on a well-embedded graph to Brownian motion, and certainly our results above justify such a comparison. However, the simple random walk on a good embedding can behave rather irregularly. For example, its Euclidean trajectory is not a martingale and can have a local drift. The random walk is also much slower when traversing areas of short edges compared to areas of longer edges. To fix the second problem we could study the variable speed random walk which waits at each vertex an amount of time comparable to r_x^2, or to the area of one of the faces containing the vertex (a good embedding guarantees that all faces sharing a vertex have comparable area). Instead, we use the **cable process** on the graph, which can be thought of as Brownian motion on the embedding (see Section 3). The vertex trajectory of this process has the same distribution as the simple random walk, so the harmonic measures do not change.

A central step in this work is showing that well embedded graphs satisfy volume doubling and a Poincaré inequality with respect to the Euclidean metric (rather than the graph metric). This is done in Section 3. The work of Sturm [25] (which applies in the very general setting of local Dirichlet spaces) then enables us to obtain various corollaries: an elliptic Harnack inequality (Theorem 5.4) and heat kernel estimates (see Theorem 3.5). These already give us enough control to prove Theorems 1.4 and 1.5 in Section 4.

To prove Theorems 1.1 and 1.2 we require a boundary Harnack inequality (see Theorem 5.5). Roughly speaking, this states that two positive harmonic functions
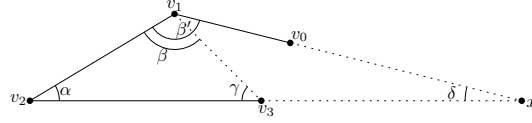
Figure 1. No acute angles: In a good embedding, the angle α must be at least $D^{-1}\sin(\eta/2)$.

that vanish on most of the boundary of the domain do so in a uniform way. In our setting, the boundary Harnack inequality is a consequence of the volume doubling and Poincaré inequality, as shown in [18], following an argument of Aikawa [1] that originates in the work of Bass and Burdzy [5]. Given the boundary Harnack inequality, it is possible to prove Theorem 1.1 by constructing an explicit coupling between two random walks starting at two different points conditioned to converge to some $\xi \in \partial U$ (by conditioning that the random walk is swallowed in a small neighborhood and taking a weak limit) so that with probability 1 their traces coincides after a finite number of steps. This coupling is constructed by showing that for any annulus of constant aspect ratio around ξ the conditioned random walks have a positive chance to meet.

We do not use this proof approach and instead use the more succinct approach of Aikawa [1]. His argument (following Jerison and Kenig [15]) shows how the characterization of the Martin boundary of Brownian motion on a uniform domain follows from the boundary Harnack principle. Our argument in Section 5 is very similar to [1] except for the complication that our process is not a martingale. Thus, a separate argument is necessary to show the convergence of the random walk to the boundary and that the distribution of the limit is non-atomic.

2   Preliminaries

We begin with some geometric consequences of having a good embedding. In this section we assume that we are given a (D, η)-good embedding of a graph G.

Lemma 2.1 (No acute angles). *The angle between any two adjacent edges is at least $D^{-1}\sin(\eta/2)$.*

Proof. Let α be the angle between three consecutive vertices on a face v_1, v_2, v_3 such that the edge $[v_1, v_2]$ is not longer than the edge $[v_2, v_3]$. By convexity, the triangle v_1, v_2, v_3 is contained in the face. Let β, γ be the angles $\angle v_2 v_1 v_3$ and

$\angle v_2 v_3 v_1$, respectively. By our assumption we learn that $\beta \geq \gamma$, hence $\gamma \leq \pi/2$. See Figure 1.

If $\alpha \geq \eta/2$, then we are done since $\eta/2 \geq D^{-1} \sin(\eta/2)$. Otherwise, let v_0 be the vertex before v_1 on the face (if the face is a triangle, then $v_0 = v_3$). Let β' be the angle $\angle v_0 v_1 v_2$ so that $\beta' \geq \beta$, and by (b) we have $\beta' \leq \pi - \eta$. Let $x \in \mathbb{R}^2$ be the meeting point of the ray emanating from v_1 towards v_0 and the ray emanating from v_2 towards v_3 (since $\alpha + \beta' < \pi$ these rays must intersect and the intersection point x must be on the same side of the infinite line through v_1, v_2 as v_0 and v_3). Let δ be the angle $\angle v_1 x v_2$. We have that $\delta = \pi - \beta' - \alpha$ hence $\eta/2 \leq \delta \leq \gamma \leq \pi/2$. Hence, by the law of sines

$$\alpha \geq \sin(\alpha) = \frac{|v_1 x| \sin(\delta)}{|v_1 v_2|} \geq \frac{|v_1 v_0| \sin(\delta)}{|v_1 v_2|} \geq D^{-1} \sin(\eta/2),$$

where $|v_1 x| \geq |v_1 v_0|$ since by convexity v_0 and v_1 are on the same side of the infinite line passing through v_2 and v_3. ∎

Lemma 2.2 (Sausage lemma)**.** *There exists $c = c(D, \eta) > 0$ such that if e, f are non adjacent edges then $d(e, f) \geq c|e|$, where $|e|$ and $d(\cdot, \cdot)$ are Euclidean length and distance. In particular, any vertex $u \in V \setminus \{v, w\}$ is of distance at least $c|e_0|$ from e_0.*

Proof. See Figure 2. Let v_1, v_2 be two consecutive neighbors of v. Because the angle $\angle v_1 v v_2$ is at most $\pi - \eta$, one of the angles $\angle v_1 v_2 v$ or $\angle v_2 v_1 v$ is at least $\eta/2$ and by (c) both $|v v_1|$ and $|v v_2|$ are at least $D^{-1}|e_0|$. Hence, the distance between v and the line through v_1 and v_2 is at least $D^{-1} \sin(\eta/2)|e_0|$. We conclude that there are no points of X inside a ball around v of radius $D^{-1} \sin(\eta/2)|e_0|$ except for the edges emanating from v and the same holds for w.

Next, consider one of the two faces containing e_0 and let v_1 be the neighbor of v in the face that is not w and similarly w_1 be the neighbor of w in the face that is not v (if the face is a triangle, then $w_1 = v_1$). By Lemma 2.1 the angles $\angle v_1 v w$ and $\angle v w w_1$ are at least $D^{-1} \sin(\eta/2)$ and by condition (b) these angles are at most $\pi - \eta$. Hence by condition (c), the face contains a trapezoid in which e_0 is a base and the two sides are sub-intervals containing v and w of the edges (v, v_1) and (w, w_1), respectively and of height at least $D^{-1}|e_0| \sin(D^{-1} \sin(\eta/2))$. ∎

Corollary 2.3. *There exists $c = c(D, \eta) > 0$ such that for any edge e and vertex $u \notin e$ and $r > 0$ we have that if e intersects $B_{\mathrm{euc}}(u, cr)$, then e is contained in $B_{\mathrm{euc}}(u, r)$.*

Lemma 2.4. *There exists a constant $C = C(D, \eta) < \infty$ such that if an edge e is contained in a face f, then $\mathrm{diam}(f) \leq C|e|$, and $|e|^2 \leq C \mathfrak{Leb}(f)$, where $\mathfrak{Leb}(f)$ is the usual Lebesgue area measure in $\mathbb{R}^2$.*
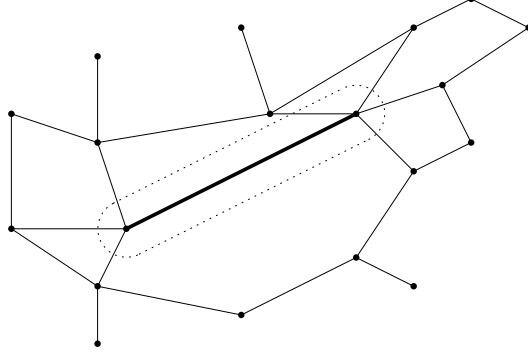
Figure 2. The sausage lemma: no edge can intersect the marked "sausage" with width $c|e|$.

Proof. Since external angles in a polygon add up to 2π, the number of sides of a face is at most $2\pi/\eta$, and since consecutive sides have length ratio at most D, any two sides of a face have ratio at most C, and the diameter of the face is at most some constant times the shortest edge.

The relation to the area of f follows from Lemma 2.1 and that edges adjacent to e have comparable lengths. ∎

Most of our arguments will take place in the metric space (X, d_0) defined as follows. For an edge $(u, v) \in E$, write $[u, v]$ for the closed line segment in the plane from u to v. We put

$$X = \bigcup_{(u,v)\in E} [u, v],$$

and let d_0 be the shortest path distance in X. For $x \in X$ and $r > 0$ we write $B_{d_0}(x, r)$ for the ball $\{y \in X : d_0(x, y) \le r\}$. An idea that we frequently use throughout is to take a curve in $\mathbb{R}^2$ with some useful properties and modify it slightly to get a curve in X with similar properties.

Proposition 2.5. *There exists a constant $C_1 = C_1(D, \eta)$ such that for any $x, y \in X$ we have*

$$|x - y| \le d_0(x, y) \le C_1|x - y|.$$

Proof. Since $d_0(x, y)$ is the Euclidean length of the shortest path in X between x and y the inequality $|x - y| \le d_0(x, y)$ is obvious.

To prove the other inequality, we first prove the assertion for x and y that are on the same face. If x and y are on the same edge then $d_0(x, y) = |x - y|$. If x and y are on two different edges that share a vertex v, then since the angle at v is bounded away from 0 (by Lemma 2.1) we deduce by the law of sines on the

triangle x, v, y that $d_0(x, y) \leq |x - v| + |y - v| \leq C|x - y|$. Lastly, when x and y are on two edges of the same face not sharing a vertex, Lemma 2.2 immediately gives that $|x - y|$ is at least $c|e|$, where e is some edge that contains x. Lemma 2.4 gives that $|e|$ is at least a constant multiple times the diameter of the face and the assertion follows.

Finally, when x and y are not on the same face let $[x, y]$ be the straight segment connecting x and y and let $x = x_0, x_1, \ldots, x_k = y$ be the points on $[x, y]$ where the segment intersects X, so that x_i and x_{i+1} are on the boundary of some face for all $i = 0, \ldots, k - 1$. Then $d_0(x_i, x_i + 1) \leq C|x_i - x_{i+1}|$ and summing over i finishes the proof of the lemma. ∎

Consequently, if $\mathrm{carr}(G) = U$ then the completion of (X, d_0) is $\overline{X} = X \cup \partial U$ with the topology induced from $\mathbb{R}^2$. A similar statement holds for any domain with rectifiable Jordan boundary.

Lemma 2.6 (X is inner uniform)**.** *Assume that G has a (D, η)-good embedding and that $\mathrm{carr}(G) = U$. There exist constants $C = C(D, \eta) < \infty$ and $c = c(D, \eta) > 0$ such that for any $\xi_1, \xi_2 \in \partial U$ with $\xi_1 \neq \xi_2$ there exists a continuous curve $\Gamma : [0, L] \to \overline{X}$ such that the following holds:*

(1) Γ is parametrized by length, that is, $\mathrm{length}(\Gamma[0, t]) = t$ for all $t \in [0, L]$.
(2) $\Gamma(0) = \xi_1$ and $\Gamma(L) = \xi_2$,
(3) $L \leq C d_0(\xi_1, \xi_2)$.
(4) For any $t \in (0, L)$ we have

$$d_0(\Gamma(t), \partial U) \geq c \min(t, L - t).$$

Proof. Consider a circle orthogonal to U through ξ_1, ξ_2, and the continuous curve γ which is the arc from ξ_1 to ξ_2 in that circle. Let $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ be the set $\gamma \cap X$ with the order induced by γ. Two consecutive points x_i and x_{i+1} are on the same face, so write Γ_i for a piecewise linear curve on the boundary of that face connecting x_i to x_{i+1} in the shorter way according to d_0. Let Γ be the concatenation of $(\Gamma_i)_{\{-\infty < i < \infty\}}$, parametrized by arc length (which we shall see below is finite). Note that Γ might not be a simple curve, which does not cause any difficulty. Thus (1) holds. Let us show that Γ satisfies requirements (2)–(4).

We first note that by Lemma 2.2 we have that there exists $C > 0$ such that for any face f

$$\max_{z \in \partial f} d_0(z, \partial U) \leq C \min_{z \in \partial f} d_0(z, \partial U), \tag{2.1}$$

where by $z \in \partial f$ we mean that $z \in X$ is on one of the edges encompassing f. Now, it is clear that $x_k \to \xi_1$ when $k \to -\infty$ and $x_k \to \xi_2$ when $k \to \infty$, so by (2.1) we get that Γ satisfies requirement (2).

Next, we have that $\text{length}(\Gamma_i) = d_0(x_i, x_{i+1})$ since x_i and x_{i+1} are on the same face f, and since the shortest curve between two points on the boundary of a convex face f that does not enter the face is along its boundary. Thus we have $\text{length}(\Gamma) = \sum_i d_0(x_i, x_{i+1}) \leq C|\xi_1 - \xi_2|$ by Proposition 2.5, so requirement (3) holds. Lastly, (4) holds immediately for the points x_k, and by (2.1) we obtain this for any point on Γ. ∎

3   The Cable process

It will be convenient for us to obtain useful estimates using results of Sturm [25]. For that we need to introduce the *cable process* which can be thought of as Brownian motion on the embedding of G. An intuitive description of the process is as follows: Let x be a vertex and $e_1, \ldots, e_k$ the edges emanating from it, and let $\{W_t\}_{t \geq 0}$ be standard Brownian motion. It is well known that W_t can be decomposed into countably many excursions in which $W_t \neq 0$. For each such excursion we choose the edge e_i with probability proportional to $w_{e_i}|e_i|$ for $i = 1, \ldots, k$ and embed the excursion on the edge e_i. We stop when we hit one of the neighbors $x_1, \ldots, x_k$ of x, and continue from this neighbor using the strong Markov property. Note that it is possible for this process to "explode", or visit infinitely many vertices in finite time, and indeed this does happen almost surely in the transient setting.

Before defining the process formally, let us state two useful properties that will make the connection to the discrete time weighted random walk evident. Denote by Z_t the process and let T be the hitting time of $\{x_1, \ldots, x_k\}$. Then for $1 \leq i \leq k$ we have (see [11, Theorem 2.1])

$$\mathbb{P}_x(Z_T = x_i) = \frac{w_{e_i}}{\sum_{i=1}^{k} w_{e_i}}, \tag{3.1}$$

that is, the process Z_t observed on vertices has the same trace as the simple random walk (we ignore the uncountably many times it visits each x before proceeding to one of its neighbors). Also, in our setting, there exists a constant $C = C(D, \eta) > 0$ such that (see [11, Theorem 2.2])

$$C^{-1} \leq \frac{\mathbb{E}_x T}{r_x^2} \leq C, \tag{3.2}$$

where r_x is the length of the shortest edge touching x (and so is comparable to the length of any edge touching x). Intuitively, the process Z_t behaves like the variable speed random walk that waits roughly r_x^2 time at vertex x before proceeding.

The construction based on excursions can be made precise. However, following [25] we give instead an analytic definition of the cable process using the setting of local Dirichlet spaces. Let (X, d_0) be the metric space defined in Section 2. We say that a function f on X is differentiable if it is differentiable w.r.t. the length measure on every edge. The derivative f' depends on the direction, and only makes sense if we fix a direction for every edge. However $f'g'$ is well defined for differentiable f, g and does not depend on choosing a direction on the edges. For an edge (u, v) write dx for Lebesgue measure on $[u, v]$ and define

$$m(dx) = \sum_{(u,v) \in E} |u - v| dx,$$

$$d\Gamma(f, g)(dx) = \sum_{(u,v) \in E} 1_{[u,v]}(x) f'(x) g'(x) |u - v| dx,$$

$$\mathcal{E}(f, g) = \sum_{(u,v)} \int_{[u,v]} f'(x) g'(x) |u - v| dx = \int_X d\Gamma(f, g).$$

The space $(X, m, \mathcal{E})$ has an intrinsic metric associated with it (see [25]), where the distance between x, y is given by

$$\sup \{ f(x) - f(y) : f \in C^1(X) \text{ and } |f'| \le 1 \}.$$

In our case it is clear that this metric coincides with d_0 defined above. We will show that this space is doubling and has a weak Poincaré inequality.

3.1 Doubling

Lemma 3.1. *Let X be a good embedding of some graph. There exists a integer $M = M(D, \eta) > 0$ such that for any $x \in X$ and any $r > 0$ there exists $x_1, \ldots, x_M \in X$ such that*

$$B_{d_0}(x, 2r) \subset \bigcup_{i=1}^{M} B_{d_0}(x_i, r).$$

Proof. This is an easy consequence of the equivalence between d_0 and the Euclidean metric. We have that $B_{d_0}(x, 2r) \subset B_{\text{euc}}(x, 2r)$. Let C_1 be the constant from Proposition 2.5. The Euclidean ball can be covered by M Euclidean balls $B_{\text{euc}}(y_i, r/2C_1)$ of radius $r/2C_1$ for some $M \asymp C_1^2$. For each i, if $B_{\text{euc}}(y_i, r/2C_1)$ intersects X and x_i is an arbitrary point in the intersection then $B_{\text{euc}}(y_i, r/2C_1) \cap$

$X \subset B_{d_0}(x_i, r)$. Otherwise we ignore this ball. So the collection of balls $B_{d_0}(x_i, r)$ covers $B_{d_0}(x, 2r)$. ∎

Recall the radius of isolation r_u defined for $u \in V$ as the distance to the nearest vertex $v \neq u$. We extend this to $x \in X$ by letting r_x be the length of the edge containing x when $x \in X \setminus V$.

Lemma 3.2. *For any $x \in X$ and $r > 0$ such that $B_{\mathrm{euc}}(x, 2r) \subset \mathrm{carr}(G)$ we have $m(B_{d_0}(x, 2r)) \asymp r \cdot (r \vee r_x)$. In particular, there exists a constant $C > 0$ such that*

$$m(B_{d_0}(x, 2r)) \le C m(B_{d_0}(x, r)).$$

Proof. First we assume that x is a vertex. When $r \le r_x$ we have that $m(B_{d_0}(x, r)) \asymp r_x r$, because the degrees of G are bounded and adjacent edges have comparable length. So it suffices to prove that $m(B_{d_0}(x, r)) \asymp r^2$.

By Lemma 2.4, for an edge e we have $m(e) = |e|^2 \le C\mathcal{L}eb(f)$ where f is a face containing e. Since each face has a bounded degree, and since faces intersecting $B_{d_0}(x, r)$ are fully contained in $B_{\mathrm{euc}}(x, Cr)$ we find $m(B_{d_0}(x, r)) \le Cr^2$. This immediately gives that $m(B_{d_0}(x, r)) \le Cr^2$ (where $C = C(D, \eta) < \infty$). Similarly, by Corollary 2.3 and Proposition 2.5, all the faces that intersect $B_{\mathrm{euc}}(x, cr)$ are entirely contained in $B_{\mathrm{euc}}(x, r)$ giving the required lower bound. The argument showing this when x is not a vertex is similar, and we omit the details. ∎

3.2 Poincaré inequality

The strong Poincaré inequality states that for any differentiable f on $B = B_{d_0}(x, r)$ we have

$$\int_B |f(x) - \bar{f}|^2 \, m(dx) \le Cr^2 \int_B |f'(x)|^2 \, m(dx), \tag{3.3}$$

where $\bar{f} = \frac{1}{m(B)} \int_B f \, m(dx)$ is the mean of f. A well-known technique due to Jerison (see [14, Section 5] and also [23, Section 5.3] for a simpler proof) shows that for spaces satisfying the doubling property, this follows from the weak Poincaré inequality which we now prove.

Theorem 3.3 (Weak Poincaré inequality)**.** *There exist positive constants $C = C(D, \eta)$ and $C' = C'(D, \eta)$ such that for any $x_0 \in X$ and $r > 0$ with $B_{\mathrm{euc}}(x_0, Cr) \subset \mathrm{carr}(G)$, and all f differentiable on $B_{d_0}(x, Cr)$ we have*

$$\int_{B_{d_0}(x_0, r)} |f(x) - \bar{f}|^2 \, m(dx) \le C'r^2 \int_{B_{d_0}(x_0, Cr)} |f'(x)|^2 \, m(dx).$$

We begin with the following lemma.

Lemma 3.4. *Let x have law $\frac{m(dx)}{m(B)}$ on some set $B \subset X$, and conditioned on x let $\hat{x}$ be uniform in the union of the two faces incident to the edge containing x. Then there is some constant $C = C(D, \eta)$ so that the law of $\hat{x}$ is bounded by $\frac{C\mathcal{L}eb}{m(B)}$, where $\mathcal{L}eb$ is the usual Lebesgue measure on $\mathbb{R}^2$.*

Proof. For an edge e of X we have that $\mathbb{P}(x \in e) = \frac{m(e \cap B)}{m(B)} \leq \frac{|e|^2}{m(B)}$ (with equality holding when $e \subset B$). If e is incident to some face f then the conditional contribution to the density of $\hat{x}$ in f is at most $1/\mathcal{L}eb(f)$, and so the density on a face f surrounded by edges $e_1, \ldots, e_k$ is at most $\frac{1}{m(B)} \sum \frac{|e_i|^2}{\mathcal{L}eb(f)}$. The number of edges surrounding a face is at most $2\pi/\eta$, and the square of each is comparable to the area of f, giving the claim. $\blacksquare$

Proof of Theorem 3.3. We let $B = B_{d_0}(x_0, r)$. Let x, y be independent points chosen in B with law $\frac{m(dx)}{m(B)}$. We start with the simple identity

$$\int_B |f(x) - \bar{f}|^2 m(dx) = \frac{m(B)}{2} \mathbb{E}|f(x) - f(y)|^2,$$

that follows from expanding. Let $\gamma = \gamma_{xy}$ be some (possibly random) path in X between x and y, then $f(y) - f(x) = \int_\gamma f'(z) dz$, where dz is the length element along γ. Applying Cauchy-Schwartz gives

$$|f(x) - f(y)|^2 \leq |\gamma| \int_\gamma |f'(z)|^2 dz,$$

where $|\gamma|$ denotes the length of γ.

We apply this to a path constructed as follows. Let $\hat{x}$ (resp. $\hat{y}$) be uniformly chosen in the union of the two faces of X incident to x (resp. to y). The straight line segment $\hat{x}\hat{y}$ begins at a face containing x, ends at a face containing y, and possibly passes through some other faces in between. As in the proof of Proposition 2.5 we can approximate this line segment by a path γ_{xy} in X, which stays in the boundaries of faces crossed by the line segment.

We first observe that due to Lemma 2.2 we have that $\hat{x}, \hat{y} \in B_0 := \{u \in \mathbb{R}^2 : |u - x_0| < C_0 r\}$ for some $C_0 = C_0(D, \eta) \geq 1$, and $B_0 \subset \text{carr}(G)$ by our assumptions. By increasing C_0, we can guarantee that γ_{xy} also does not leave B_0, and as in Proposition 2.5, we have $|\gamma_{xy}| \leq Cr$. We shall see below that $\mathbb{P}(z \in \gamma_{xy}) \leq \frac{C\rho_z}{r}$ where ρ_z is the length of the edge containing z (we neglect the measure 0 set of

vertices). Given that we conclude the proof as follows.

$$\int_B |f(x) - \bar{f}|^2 m(dx) \le C m(B) \mathbb{E}\left[r \int_\gamma |f'(z)|^2 dz \right]$$

$$\le C r m(B) \int_{z \in B_0} \frac{C \rho_z}{r} |f'(z)|^2 dz$$

$$\le C r^2 \int_{B_0} |f'(z)|^2 m(dz),$$

since $\rho_z dz = m(dz)$, and $m(B) \le C r^2$.

To bound the probability that $z \in \gamma_{xy}$, note that the faces incident to z are contained in $\{u \in \mathbb{R}^2 : |u - z| \le C_1 \rho_z\}$. Let A be the event that the segment $\hat{x}\hat{y}$ intersects a face incident to z, and A' the event that the segment passes within distance $C_1 \rho_z$ of z, so that $A \subset A'$. Let $\hat{m}$ be the law of $\hat{x}$ and $\hat{y}$. By Lemma 3.4 we have that

$$\hat{m} \le \frac{C}{m(B)} \mathfrak{L}eb \le \frac{C_2}{r^2} \mathfrak{L}eb,$$

and $\operatorname{supp} \hat{m} \subset B_0 \subset \{|\hat{x} - z| \le 2 C_0 r\}$. We have now

$$\mathbb{P}(z \in \gamma_{xy}) \le \mathbb{E} 1_A \le \mathbb{E} 1_{A'} = \iint 1_{A'} d\hat{m} \times d\hat{m}$$

$$\le \int_{|\hat{x} - z| \le 2 C_0 r} \int_{|\hat{y} - z| \le 2 C_0 r} 1_{A'} \left(\frac{C_2}{r^2} \right)^2 d\hat{x} d\hat{y}.$$

By scaling and translating this is $(C_2/2C_0)^2$ times the probability that the segment between two uniform points $u, v \in U$ passes within $C_1 \rho_z / r$ of the origin. For such u, v, the distance between the segment and the origin is a continuous random variable with finite density at 0, so the distance is at most $C_1 \rho_z / r$ with probability at most $C \rho_z / r$. ∎

3.3   Heat kernel estimates

Finally, we are able to deduce estimates for the heat kernel of the cable process on X. Let $q_t(x, y)$ denote the heat kernel for the Markov process $\{Z_t\}_{t \ge 0}$ associated with $(X, m, \mathcal{E})$, that is, $q_t(x, \cdot)$ is the density (with respect to m) of Z_t conditioned on $Z_0 = x$. For a set $A \subset X$ we let $q_t^A(x, y)$ denote the heat kernel for the process killed when it exits A.

Theorem 3.5. *There exists constants c, C depending only on D, η such that for any $x_0 \in X$ and $r > 0$ such that $B_{\text{euc}}(x_0, Cr) \subset \text{carr}(G)$ we have that for any $t \le r^2$ and $x, y \in X \cap B_{\text{euc}}(x_0, \sqrt{t})$*

$$q_t^A(x, y) \ge \frac{c}{m(B_{\text{euc}}(x_0, \sqrt{t}))},$$

where $A = X \cap B_{\text{euc}}(x_0, Cr)$.

Proof. This is obtained by combining Theorem 3.5 of [25] with (3.3) and Lemma 3.2 (giving parabolic Harnack inequality), and then appealing to Theorem 3.2 in [4] (the assertion that (c) implies (b) is what we use with the function $\tau(t) = t^2$). Finally, using Proposition 2.5 to move from balls in d_0 to Euclidean balls. ∎

4   Harmonic measure and exit time of discrete discs

In this section we prove Theorems 1.4 and 1.5. Let $G = (V, E)$ be a planar graph with a (D, η)-good embedding and let (X, d_0) be the associated metric space. We consider the cable process $(X, d_0, m, \mathscr{E})$ on G defined in Section 3. We slightly abuse notation, and use τ_A to denote the hitting time of A by the cable process, i.e. $\tau_A = \inf\{t : Z_t \in A\}$. Recall that the restriction of Z_t to V is the simple random walk, and so when $A \subset V$, the law of X_{τ_A} is the same for the cable process and for the simple random walk.

For $u \in \mathbb{R}^2$, radius $r > 0$ and an interval of angles $I \subset \mathbb{R}/(2\pi\mathbb{Z})$ let $\text{Cone}(u, r, I)$ denote the intersection of X and the cone of radius r centered at u with opening angles I, that is,

$$\text{Cone}(u, r, I) = \{v \in X : |v - u| \le r \text{ and } \arg(v - u) \in I\}.$$

A **wide cone** is a cone where $|I| \ge \pi - \eta$. By definition, if u is a vertex in a good embedding then there is an edge containing u entering every fat cone with tip at u.

Lemma 4.1. *For any vertex $u \in V$ and any wide cone $A = \text{Cone}(u, r, I)$ such that $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ we have*

$$m(A) \asymp r(r \vee r_u)$$

Proof. The case $r \le r_u$ is easy and we omit the details. Assume $r \ge r_u$ and write A' for the Euclidean cone $A' = \{x \in \mathbb{R}^2 : |x - u| \le r \text{ and } \arg(x - u) \in I\}$. For any face f, we will show that $m(A \cap \partial f) \ge c\mathcal{L}eb(A' \cap f)$ for some $c(D, \eta)$. This implies the lower bound, since summing over all faces gives $m(A) \ge cr^2$.

Let $\ell = \text{diam}(A' \cap f) \le \text{diam}(f)$, and note that every edge of f has length at least $c\ell$ for some c. Consider the circle $C_s = \{z : |z - u| = s\}$. We have that the

length $|C_s \cap f|$ is at most $C\ell$, and is non-zero for s in some interval J. Integrating over s gives

$$\mathfrak{L}eb(A' \cap f) = \int_J |C_s \cap f| \le C\ell |J|.$$

We next argue that ∂f must cross inside A any circle C_s that intersects f. To see this, note that we can construct a path from u taking only edges with directions in I until we exit A after finitely many steps (since $A \subset \text{carr}(G)$). The face f is restricted to one side of the path, and so ∂f intersects $A \cap C_s$. It now follows that the length of $A \cap \partial f$ is at least $|J|$, and since the length of edges of f is at least $c\ell$ we get $m(A \cap \partial f) \ge c\ell |J|$, and the lower bound follows.

For the upper bound, we prove only the case $r \ge r_u$, as the other is immediate. By Lemma 2.2 every edge intersecting A has length at most Cr, and all incident faces are contained in $B_{\text{euc}}(u, C'r)$. For any such edge e, taking all of $m(e)$ still gives at most the area of the faces containing e, and since each face is counted a bounded number of times, the claim follows by summing over the edges. ∎

Corollary 4.2. *There exists a constant $c = c(D, \eta) > 0$ such that for any vertex $u \in V$ and any $r \ge r_u$ with $B_{\text{euc}}(u, r) \subset \text{carr}(G)$, and any wide cone $A = \text{Cone}(u, r, I)$ we have*

$$m\big(A \setminus B_{\text{euc}}(u, cr)\big) \ge cr^2.$$

Lemma 4.3. *There exist constants $c = c(D, \eta) > 0$ such that for any interval I with $|I| = \pi - \eta$, any vertex $u \in V$ and any $r \ge r_u$ satisfying $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ we have*

$$\mathbb{P}_u\big(\tau_S < \tau_{V \setminus V_{\text{euc}}(u, 2r)}\big) \ge c,$$

where

$$S = V \cap \text{Cone}(u, r, I) \setminus B_{\text{euc}}(u, cr).$$

Proof. Write C for the unbounded Euclidean cone $\{x \in \mathbb{R}^2 : \arg(x - u) \in I\}$ and construct an infinite simple path P from u that remains in C, as we did in the previous lemma. The existence of P implies that any edge e that intersects C must have at least one endpoint in C.

Write $c < 1$ for the smaller of the constants in Corollary 2.3 and Corollary 4.2. Let $B \subset X \cap C$ be constructed as follows: consider an edge e that intersects $C \cap B_{\text{euc}}(u, cr) \setminus B_{\text{euc}}(u, c^2 r)$ and does not contain u; if e is entirely contained in C, then we add e to B, otherwise $e = (v_1, v_2)$ where only v_1 is in C and we add to B the straight line segment between v_1 and $(v_1 + v_2)/2$ (that is, half the edge e, starting at v_1). We have that $m(B) \ge \frac{1}{2}m\big(X \cap C \cap B_{\text{euc}}(u, cr) \setminus B_{\text{euc}}(u, c^2 r)\big)$ since

for any edge e that intersects $C \cap B_{\text{euc}}(u, cr)$ we added to B at least half of $e \cap C$. Hence by Corollary 4.2 we get that $m(B) \geq c^3 r^2 / 2$.

We now appeal to Theorem 3.5 with $x_0 = u$ and $t = r^2$ and integrate over $y \in B$ to get that

$$\mathbb{P}_u\big(Z_t \in B \text{ and } t < \tau_{\partial B_{\text{euc}}(u, 2r)}\big) \geq c' > 0,$$

for some constant $c' = c'(D, \eta) > 0$. We are now ready to define our stopping time τ by

$$\tau = \min\{t \geq T : Z_t \in V\}.$$

By Corollary 2.3 we have that $B \cap V \subset X \cap C \cap V_{\text{euc}}(u, r) \setminus V_{\text{euc}}(u, c^3 r)$ and since we added either full edges or half edges, it is clear that starting from any point in B, the probability that the first vertex that we visit is in $B \cap V$ is at least $1/2$. This concludes our proof. ∎

Lemma 4.4. *For any $\varepsilon > 0$ there exists $c = c(\varepsilon, D, \eta) > 0$ such that for any vertex $u \in V$ and any $r \geq r_u$ satisfying $B_{\text{euc}}(u, r) \subset \text{carr}(G)$, and any interval I with $|I| = \pi - \eta$ we have*

$$\mathbb{P}_u(\tau_S < \tau_O) > c,$$

where

$$S = V \cap \text{Cone}(u, \infty, I) \setminus V_{\text{euc}}(u, r),$$

and

$$O = \big\{v \in V : d(v, \text{Cone}(u, \infty, I)) \geq \varepsilon r\big\}.$$

Proof. We iterate $2(c\varepsilon)^{-1}$ times Lemma 4.3 with a cone of radius $r' = \varepsilon r / 2$ and opening I. ∎

Lemma 4.5. *For any $\varepsilon > 0$ there exists $c = c(\varepsilon, D, \eta) > 0$ such that for any vertex $u \in V$, any $r \geq r_u$ satisfying $B_{\text{euc}}(u, r) \subset \text{carr}(G)$, any interval I with $|I| = \pi - \eta$, and any vertex v such that $\varepsilon r \leq |u - v| \leq (1 - \varepsilon)r$, and $\arg(v - u) \in I$ we have the following. Let*
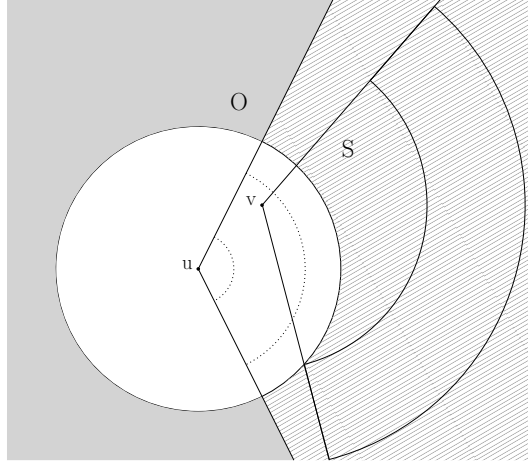
$$S = V \cap \text{Cone}(u, \infty, I) \setminus V_{\text{euc}}(u, r)$$

and

$$Q = V \setminus (\text{Cone}(u, \infty, I) \cup V_{\text{euc}}(u, r)),$$

then $\mathbb{P}_v(\tau_S < \tau_Q) \geq c$.

Proof. Write C for the Euclidean cone $\{x \in \mathbb{R}^2 : \arg(x - u) \in I\}$. If the distance of v from $\mathbb{R}^2 \setminus C$ is at least εr, then we apply Lemma 4.4 on the cone parallel to C emanating from v and the assertion follows. Assume now the opposite, and write

Figure 3. Illustration of Lemma 4.5. The probability from v of hitting S before Q cannot be too small. Also shown: the possible locations for v, the rotated cone C' and the set likely to be hit from v by Lemma 4.4.

R_1, R_2 for the two rays of the cone C so that R_1 is before R_2 clockwise and assume without loss of generality that v is closer to R_1. Let C' be the cone

$$C' = \{x \in \mathbb{R}^2 : \arg(x - v) \in I + \alpha\},$$

where $\alpha = \alpha(\varepsilon) > 0$ is the largest number so that $d(u, C' \setminus C) \geq (2c^{-1} + 2)r$ where $c > 0$ is the constant from Corollary 2.3 (see Figure 3). Define the set O' by

$$O' = \big\{v \in V : d(v, V \cap C' \cap V_{\mathrm{euc}}(v, 2r)) \geq \varepsilon' r\big\},$$

where $\varepsilon' = \varepsilon'(\varepsilon, \alpha) > 0$ is chosen so that $(V \setminus O') \setminus V_{\mathrm{euc}}(u, r) \subset C$.

We now apply Lemma 4.4 with ε', v and C' to obtain that with probability uniformly bounded below we visit $(V \cap C') \setminus V_{\mathrm{euc}}(v, 2r)$ before visiting O'. By Corollary 2.3, when this event occurs the length of the last edge traversed has length at most $2c^{-1}$. Hence, by our choice of α in this last step we find ourselves in S and by our choice of ε' we have not stepped outside of $C \cup V_{\mathrm{euc}}(u, r)$, as required.   ∎

Proof of Theorem 1.4. Let $\varepsilon = \varepsilon(D, \eta) > 0$ be a fixed small number to be chosen later. Let us consider several cases. Recall that by condition (a) there always exist an edge (u, v) such that $\arg(v - u) \in I$. Firstly, if there exists such an edge (u, v) with $|u - v| > r$, then with probability at least D^{-3} we take this edge in the first step and we are done. Secondly, if there is such an edge so that $(1 - \varepsilon)r \leq |u - v| \leq r$,

then as long as ε is small with respect to D, then by condition (b) v has a neighbor w such that $\arg(w - u) \in I$ and $|w - u| > r$, so with probability at least D^{-6} we take two steps from u to w and we are done. Thirdly, if there exists such an edge so that $\varepsilon r \le |u - v| \le (1 - \varepsilon)r$ then $\mathbb{P}_u(X_{T_r} \in S) \ge D^{-3}\mathbb{P}_v(X_{T_r} \in S)$ where S is defined in Lemma 4.5, and by that Lemma the last quantity is uniformly bounded below and we are done. Lastly, if all neighbors v of u satisfy $|u - v| \le \varepsilon r$, then we apply Lemma 4.4 with radius εr and obtain that with probability uniformly bounded from below we visit $V \cap \mathrm{Cone}(u, \varepsilon r, I) \setminus V_{\mathrm{euc}}(u, \varepsilon r)$ before visiting $O = \{v \in V : d(v, \mathrm{Cone}(u, \varepsilon r, I)) \ge \varepsilon^2 r\}$. When this occurs, the last edge taken by the random walk has length at most $c^{-1}\varepsilon r$ by Corollary 2.3, where $c > 0$ is the constant of that lemma. Hence, if ε is chosen so that $\varepsilon \le (c^{-1} + 2)^{-1}$ we get that at that hitting time we are at a vertex v such that $\varepsilon r \le |u - v| \le (1 - \varepsilon)r$ and $\arg(v - u) \in I$. The assertion of the theorem now follows by another application of Lemma 4.5, concluding the proof. ∎

Proof of Theorem 1.5. Let T_r^Z denote the exit time from $B_{\mathrm{euc}}(u, r)$ of the cable process and T_r the exit time from $V_{\mathrm{euc}}(u, r)$ for the simple random walk. Our first goal is to prove that $\mathbb{E}_u T_r^Z \asymp r^2$. We begin with the lower bound. To that aim, let $C_{3.5}$ be the constant from Theorem 3.5 and let $A = B_{\mathrm{euc}}(u, C_{3.5}r)$ and assume that $A \subset \mathrm{carr}(G)$. We apply Theorem 3.5 with $t = r^2$ and integrate over $y \in B_{\mathrm{euc}}(u, r)$ to get that $\mathbb{P}(T_{C_{3.5}r}^Z \ge r^2) \ge c$, hence $\mathbb{E}_u T_{C_{3.5}r}^Z \ge cr^2$ and so $\mathbb{E}_u T_r^Z \ge c'r^2$ for some constant $c' > 0$.

To show the upper bound, Lemma 3.2 immediately implies that there exists some constant $C_{3.2} > 0$ such that for any $r \ge r_u$ and any $x \in X \cap B_{\mathrm{euc}}(u, r)$ we have

$$m\big(X \cap B_{\mathrm{euc}}(x, C_{3.2}r) \setminus B_{\mathrm{euc}}(u, r)\big) \ge r^2.$$

We prove the theorem with $C = C_{3.2}C_{3.5} + 1$. We apply Theorem 3.5 with $A = B_{\mathrm{euc}}(x, C_{3.2}C_{3.5}r)$ (so that $A \subset \mathrm{carr}(G)$) and $t = r^2$ and integrate over $y \in X \cap B_{\mathrm{euc}}(x, C_{3.2}r) \setminus B_{\mathrm{euc}}(u, r)$ to get that for any $x \in X \cap B_{\mathrm{euc}}(u, r)$ we have $\mathbb{P}_x(T_r \ge r^2) \le 1 - c$, for some constant $c > 0$. Hence, $\mathbb{E}_u T_r^Z \le C'r^2$ for some $C' > 0$.

We got that $\mathbb{E}_u(T_r^Z) \asymp r^2$. Recall that the trace of the cable process along the vertices is distributed as the discrete weighted random walk. Hence, by writing T_r^Z as a the sum of possible random walks paths and the time it takes the cable process to traverse between vertices we obtain using (3.2) that

$$\mathbb{E}_u(T_r^Z) \asymp \mathbb{E} \sum_{t=0}^{T_r} r_{X_t}^2,$$

where $\{X_t\}$ is the discrete weighted random walk. ∎

5   The Martin boundary

It will be convenient to approximate the graph G, embedded in the plane with carrier U by finite subgraphs G_ε. For $\varepsilon > 0$ consider the subgraph G_ε induced by the vertices V_ε where

$$V_\varepsilon = \{v \in V : |v| \leq 1 - \varepsilon\}.$$

For two vertices a, z in a finite weighted graph we write $R_{\text{eff}}(a, z)$ for the effective electrical resistance between a and z (for a definition and introduction to electrical resistance, see [19]). For disjoint sets A, Z of vertices we write $R_{\text{eff}}(A, Z)$ for the electrical resistance between A and Z in the graph obtained by contracting A and Z to two vertices.

Lemma 5.1. *There exists $c = c(D, \eta) > 0$ such that for any $r > 0$ and $\varepsilon \leq r/10$ and any $\xi \in \partial U$ we have the resistance bound*

$$R_{\text{eff}}^{(\varepsilon)}\big(V_{\text{euc}}(\xi, r), V_{\text{euc}}(\xi, 2r)^c\big) \geq c,$$

where $R_{\text{eff}}^{(\varepsilon)}$ denote the resistance is in the graph G_ε.

Proof. We use the variational formula for resistance. Define a function $f : V_\varepsilon \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } |x - \xi| \leq r \\ \frac{|x-\xi|-r}{r} & \text{if } |x - \xi| \in [r, 2r] \\ 1 & \text{if } |x - \xi| \geq 2r, \end{cases}$$

and let us estimate the Dirichlet energy of the function. Note that f is r^{-1}-Lipschitz, so that for any edge (x, y) we have $|f(x) - f(y)| \leq |x - y|/r$. All edges (x, y) such that $x, y \in V_{\text{euc}}(\xi, r)$ or $x, y \notin V_{\text{euc}}(\xi, 2r)$ contribute 0 to the energy. Any other edge (x, y) contributes at most $|x - y|^2/r^2$ to the energy. Since $|x - y|^2$ is proportional to the area of the faces adjacent to the edge (x, y), all these faces are contained in $V_{\text{euc}}(\xi, Cr)$ for some $C = C(D, \eta) < \infty$, and each face has degree at most C, we get that the energy is bounded by some constant and the result follows. ∎

Corollary 5.2. *There exist a constants $K = K(D, \eta) < \infty$ and $c = c(D, \eta) > 0$ such that for any R, r satisfying $0 < Kr \leq R$, any $\varepsilon \leq r/10$ and any $\xi \in \partial U$ we have the resistance bound*

$$R_{\text{eff}}^{(\varepsilon)}\big(V_{\text{euc}}(\xi, r), V_\varepsilon \setminus V_{\text{euc}}(\xi, R)\big) \geq c \log \frac{R}{r},$$

where $R_{\text{eff}}^{(\varepsilon)}$ denotes the resistance is in the graph G_ε.

Proof. Let $K \geq 2$ be such that there are no edges (x, y) such that $|x - \xi| \leq r$ and $|y - \xi| \geq Kr$. Such a choice is possible by Lemma 2.2 and the fact that ξ is an accumulation point of vertices.

Suppose first that $R = K^{2m-1}r$ for some integer $m \geq 1$. Define sets $A_0 = \{v : |v - \xi| < r\}$, and $A_i = \{v : |v - \xi| \in [K^{i-1}r, K^i r]\}$. By Lemma 2.2 there are no edges connecting A_i to A_j for $|i - j| > 1$. By Lemma 5.1 we have that $R_{\text{eff}}^{(\varepsilon)}(A_i, A_{i+2}) \geq c$. Contracting all edges in A_{2i} for each i and using the series law for resistance we find

$$R_{\text{eff}}^{(\varepsilon)}\big(V_{\text{euc}}(\xi, r), V_\varepsilon \setminus V_{\text{euc}}(\xi, R)\big) \geq cm \geq c' \log \frac{R}{r}.$$

For general $R > Kr$ the claim follows by monotonicity in R. ∎

Proposition 5.3 (RW Convergence). *Let X_n be the simple random walk on G, then X_n converges a.s. to a limit $X_\infty \in \partial U$. Furthermore, the law of X_∞ has no atoms.*

Consequently, for any starting point X_0, we may define the harmonic measure ω on ∂U to be the law of X_∞.

Proof. By Lemma 2.4, for each edge $e = (u, v)$ we have that $|v - u|^2$ is bounded by a constant times the area of either faces adjacent to e. Since each face has degree at most $2\pi/\eta$, this immediately gives that the Dirichlet energy of the Euclidean location function, i.e. $\sum_{e=(u,v)} |u - v|^2$, is bounded by some constant. By [3, Theorem 1.1] this implies that X_n converges almost surely. (The theorem is stated for real valued functions, so we apply it to each coordinate separately.) It is trivial that the limit cannot be a vertex of G, so must be in ∂U.

Let us now fix X_0, and show that for any $\xi \in \partial U$ we have $\mathbb{P}(X_\infty = \xi) = 0$. We have that $R_{\text{eff}}(X_0, \partial V_\varepsilon) \leq C$ for some C, since G is transient. By Corollary 5.2 with $R = |X_0 - \xi|$, for r small enough and any $\varepsilon < r/10$ we have $R_{\text{eff}}^{(\varepsilon)}(X_0, V_{\text{euc}}(\xi, r)) \geq c|\log r|$, and therefore

$$\mathbb{P}\big(\{X_n\} \text{ visits } V_{\text{euc}}(\xi, r) \text{ before } \partial V_\varepsilon\big) \leq \frac{C}{|\log r|}.$$

Since this estimate is uniform in ε we learn that the probability that $\{X_n\}$ ever visits $V_{\text{euc}}(\xi, r)$ is at most $C|\log r|^{-1}$. This bound is uniform in ε, and so holds also for the random walk on G. Finally, $X_\infty = \xi$ implies that $V_{\text{euc}}(\xi, r)$ is visited for all r, and so $\mathbb{P}(X_\infty = \xi) \leq \inf_r C|\log r|^{-1} = 0$. ∎

We now state two variations of the Harnack principle that apply to well embedded graphs.

Theorem 5.4 (Elliptic Harnack inequality)**.** *For any $A > 1$ there exists $C = C(D, \eta, A) > 0$ such that for any $x \in X$ and $r > 0$ such that $d_0(x, \partial U) > Ar$, and any positive, harmonic function h on $B_{d_0}(x, Ar)$ we have*

$$\max_{y \in B_{d_0}(x,r)} h(y) \le C \min_{y \in B_{d_0}(x,r)} h(y).$$

Theorem 5.5 (Boundary Harnack principle)**.** *There exists positive constants A_0, A_1 and R, depending only on D and η, such that for any $\xi \in \partial U$, any $r \in (0, R)$ and any two functions $h_1, h_2 : X \to \mathbb{R}$ that are positive, harmonic, bounded on $B_{d_0}(\xi, A_0 r)$, and almost surely $h_i(X_n) \to 0$ as $n \to \infty$ for $i = 1, 2$, we have*

$$A_1^{-1} \le \frac{h_1(x)/h_2(x)}{h_1(y)/h_2(y)} \le A_1 \qquad \forall \, x, y \in B_{d_0}(\xi, r) \cap X.$$

Theorem 5.4 follows from Theorem 3.5 of [25] and Theorem 5.5 follows from [18, Theorem 4.2]. The conditions are provided by Theorem 3.3 and Lemmas 2.6 and 3.2.

We now proceed to the proof of Theorem 1.2. We closely follow the methodology of Aikawa [1] who proved a boundary Harnack principle for Brownian motion on uniform domains and, more relevant to us, demonstrated how to deduce from that a characterization of the Martin boundary. The methods in [1] are robust enough to apply in our setting as well. For convenience we consider the Martin kernels as a function of the first coordinate, that is, we denote $M_y(\cdot) = G(\cdot, y)/G(x_0, y)$. Let $\mathscr{H}^+$ denote the set of positive harmonic functions h on X, normalized to have $h(x_0) = 1$. Note that on any locally finite connected graph, $\mathscr{H}^+$ is compact w.r.t. the product (pointwise) topology. Of those, we let $\mathscr{H}_0^+$ denote the set of functions so that $h(X_n) \xrightarrow[n \to \infty]{a.s.} 0$ for any starting point X_0. By the martingale convergence theorem [9], a.s. convergence holds for any positive harmonic function; it is of course enough to assume the limit is a.s. 0 for a single starting point. Finally, for $\xi \in \partial U$ let us denote by $\mathscr{H}_\xi$ those functions $h \in \mathscr{H}_0^+$ which are bounded on $X \setminus B_{d_0}(\xi, r)$ for any $r > 0$. Our immediate goal is the following.

Proposition 5.6. *For any $\xi \in \partial U$ the set $\mathscr{H}_\xi$ is a singleton.*

We first prove that $\mathscr{H}_\xi$ is not empty.

Lemma 5.7. *Let y_n be a sequence of vertices and suppose $y_n \to \xi \in \partial U$. Then there exists a subsequence y_{n_k} such that $M_{y_{n_k}}$ converges pointwise to some $h \in \mathscr{H}_\xi$.*

Proof. Since $\mathscr{H}^+$ is compact, there exist a subsequence y_{n_k} such that $M_{y_{n_k}}$ converges pointwise. For clarity we pass to the subsequence. Let M_ξ be the limit. Let

us now prove that $M_\xi \in H_\xi$. It is clear that M_ξ is harmonic and $M_\xi(x_0) = 1$ since these are local constraints and are immediately satisfied by the limiting procedure, so we need to show that $M_\xi(X_n) \xrightarrow[n\to\infty]{a.s.} 0$ and that M_ξ is bounded outside any neighbourhood of ξ.

Recall that by the reversibility of the random walk we have

$$\deg(x) \cdot G(x, y) = \deg(y) \cdot G(y, x), \tag{5.1}$$

and since degrees are bounded, $G(x, y)$ and $G(y, x)$ are equivalent up to constants. We therefore have that

$$M_{y_k}(x) \asymp \frac{G(y_k, x)}{G(y_k, x_0)}.$$

Let A_0 be the constant from Theorem 5.5, let $r > 0$ be arbitrary small such that $x_0 \notin B_{d_0}(\xi, A_0 r)$ and let x be an arbitrary vertex satisfying $x \notin B_{d_0}(\xi, A_0 r)$. Define the functions $h_0 = G(\cdot, x_0)$ and $h_1 = G(\cdot, x)$. The functions h_0, h_1 are positive, harmonic on $X \cap B_{d_0}(\xi, r)$ and bounded above by $G(x, x)$ and $G(x_0, x_0)$ respectively. Furthermore, both tend to 0 almost surely over the random walk since G is a transient graph. Hence we may apply Theorem 5.5 to them and deduce that

$$\frac{G(z, x)}{G(z, x_0)} \asymp \frac{G(v_r, x)}{G(v_r, x_0)},$$

where v_r, z are any two vertices in $B_{d_0}(\xi, r)$ and the constants in the $\asymp$ do not depend on the choice of x. Let k_0 be a number so that for all $k \geq k_0$ we have $y_k \in B_{d_0}(\xi, r)$ so by the previous approximate equality we get that for any $k \geq k_0$ we have

$$M_{y_k}(x) \asymp \frac{G(v_r, x)}{G(v_r, x_0)} \asymp \frac{G(x, v_r)}{G(x_0, v_r)}, \tag{5.2}$$

for all $x \notin B_{d_0}(\xi, A_0 r)$ and v_r a fixed vertex (its choice may depend on r). Since $G(x, v_r) \leq G(v_r, v_r)$ we learn that M_{y_k} is bounded outside of $B_{d_0}(\xi, r)$ for any $r > 0$ and $k > k_0$, and we deduce the same for M_ξ immediately.

Next, by Proposition 5.3 the probability that $\lim X_t = \xi$ is 0. We learn that almost surely there exists $r > 0$ such that $X_t \notin B_{d_0}(\xi, A_0 r)$ for all $t \geq 0$. Let k_0 be as above. By (5.2) we get that almost surely for any $t \geq 0$

$$M_{y_k}(X_t) \asymp \frac{G(v_r, X_t)}{G(v_r, x_0)},$$

and by taking a limit $k \to \infty$ we have that almost surely

$$M_\xi(X_t) \leq A \frac{G(v_r, X_t)}{G(v_r, x_0)},$$

for all $t \geq 0$ where $A = A(D, \eta) < \infty$. Since $G(v_r, X_t) \to 0$ as $t \to \infty$ almost surely, we deduce that $\lim M_\xi(X_n) = 0$ almost surely, concluding the proof. ∎

Proof of Proposition 5.6. We first prove that there exists $A = A(D, \eta) < \infty$ such that for any $h_1, h_2 \in \mathscr{H}_\xi$ we have

$$A^{-1} \leq \frac{h_1(x)}{h_2(x)} \leq A, \qquad \text{for all } x \in X. \tag{5.3}$$

Let $r > 0$ be an arbitrary small number and let $\xi_1, \xi_2 \in \partial U$ be the two boundary points so that $|\xi - \xi_1| = |\xi - \xi_2| = r$. We appeal to Lemma 2.6 and get a curve $\Gamma : (0, L) \to X$ satisfying the conditions of the lemma. We now use the curve to construct balls $B_0, \ldots, B_N$ for some $N = N(D, \eta) < \infty$ such that for some small $c \in (0, 1/2)$ the following holds:

(1) $B_0 = B_{d_0}(\xi_1, r/(2A_0))$ and $B_N = B_{d_0}(\xi_2, r/(2A_0))$,
(2) For $i = 1, \ldots, N-1$ we have $B_i = B_{d_0}(x_i, cr)$ where $x_i \in \gamma$ and $d_0(x_i, \partial U) > 2cr$,
(3) $B_i \cap B_{i+1} \neq \varnothing$ for $i = 0, \ldots, N-1$.

We apply Theorems 5.4 and 5.5 to obtain that there exists $A = A(D, \eta) < \infty$ such that

$$A^{-1} \leq \frac{h_1(x)/h_2(x)}{h_1(x')/h_2(x')} \leq A \qquad \forall\, x, x' \in \bigcup_{i=1}^{N} B_i.$$

Indeed, the assertion for $x, x' \in B_0$ and $x, x' \cup B_N$ is precisely Theorem 5.5. Moreover, Theorem 5.4 gives that the values of h_1, h_2 within $B_1 \cup \cdots \cup B_{N-1}$ change by at most a multiplicative constant.

Fix $x' \in \gamma$ and note that $h_1(x)/h_2(x) \leq q$ for all $x \in \gamma$ where $q = A h_1(x')/h_2(x')$. Then $g(x) = h_1(x) - q h_2(x)$ is harmonic and non-positive on γ. The martingale $g(X_n)$ stopped when hitting γ is bounded, converges to 0 if γ is not hit, and is stopped at a negative value if γ is hit. By L^1-convergence for bounded martingales $g(x) \leq 0$ everywhere, and so

$$A^{-1} \leq \frac{h_1(x)/h_2(x)}{h_1(x')/h_2(x')} \leq A \qquad \forall\, x \in X \setminus B_{d_0}(\xi, r).$$

In particular $h_1(x')/h_2(x') \leq A h_1(x_0)/h_2(x_0) = A$ and similarly $h_1(x')/h_2(x') \geq A^{-1}$. Hence $A^{-2} \leq h_1(x)/h_2(x) \leq A^2$ for all $x \in X \setminus V_{\text{euc}}(\xi, r)$. Since $r > 0$ was arbitrary, this gives (5.3).

Next we show that in fact $A = 1$; the following argument is due to Ancona [2]. Indeed, write

$$c = \sup_{h_1, h_2 \in H_\xi, x \in X} \frac{h_1(x)}{h_2(x)},$$

so that $c \in [1, \infty)$. Assume by contradiction that $c > 1$ and let $h_1, h_2 \in H_\xi$. Then $h_3 = (ch_1 - h_2)/(c - 1)$ is a function in H_ξ so $h_2 \le ch_3$ which simplifies to $(2c - 1)h_2 \le c^2 h_1$. Since $c^2/(2c - 1) < c$ we have reached a contradiction. ∎

Proof of Theorem 1.2′. Minimality of M_ξ follows easily from Proposition 5.6, since if $0 \le h \le M_\xi$ then $h(\cdot)/h(x_0)$ is easily seen to be in $\mathcal{H}_\xi$, and so it must equal M_ξ.

Suppose $y_n \to \xi \in \partial U$ then Proposition 5.6 and Lemma 5.7 together show that $\lim_{y_n \to \xi} M_{y_n}(\cdot)$ exists and is the unique function in $\mathcal{H}_\xi$. Thus convergence of y_n implies convergence of M_{y_n}.

Next, note that if $\xi \ne \xi'$ are two points on ∂U, then $M_\xi \ne M_{\xi'}$. Indeed, M_ξ is an unbounded function, since otherwise, by the bounded martingale convergence theorem we would get that $\mathbb{E} \lim M_\xi(X_n) = M_\xi(x_0) = 1$, contradicting the fact that $M_\xi(X_n) \to 0$ almost surely. However, M_ξ is bounded away from ξ and so must be unbounded in any neighborhood of ξ. It follows that $M_\xi \ne M_{\xi'}$.

Now, suppose we have a convergent sequence $M_{y_n} \to M_\infty$ for some sequence y_n. Since U is compact, there is a convergent subsequence $y_{n_k} \to \xi$. If ξ is not in ∂U then eventually $y_{n_k} = \xi$. Otherwise, $M_{y_{n_k}} \to M_\xi$, and in either case $M_\infty = M_\xi$. Since ξ is determined by M_ξ, we have that $y_n \to \xi$, completing the proof of (1).

Finally, we show that the map $\xi \mapsto M_\xi(\cdot)$ is a homeomorphism. It is invertible, so we need continuity of the map and its inverse. Suppose $\xi_n \to \xi$ are points in ∂U. For an arbitrary x, we may find y_n so that $d(y_n, \xi_n) < \frac{1}{n}$, and also $|M_{y_n}(x) - M_{\xi_n}(x)| \le \frac{1}{n}$. We have that $y_n \to \xi$, and therefore $M_{y_n}(x) \to M_\xi(x)$, and so also $M_{\xi_n}(x) \to M_\xi(x)$. Similarly, if $M_{\xi_n} \to M_\xi$ we can diagonalize to find y_n with $d(y_n, \xi_n) < \frac{1}{n}$ so that $M_{y_n} \to M_\xi$. By (1) we have $y_n \to \xi$, and therefore $\xi_n \to \xi$. ∎

Proof of Theorem 1.1′. We appeal to general properties of the Martin boundary, see chapter 24 of [26] for a concise introduction. This theory implies that any positive harmonic function h can be represented as an integral on the Martin boundary $\mathcal{M}$ with respect to some measure. When h is bounded, this measure is absolutely continuous with respect to the exit measure on the Martin boundary, hence it can be written as

$$h(x) = \int_\mathcal{M} M(x) f(M) d\nu_{x_0}(M), \tag{5.4}$$

where ν_{x_0} is the law of $\lim_n M_{X_n}(\cdot)$ starting from x_0 and $f : \mathcal{M} \to \mathbb{R}$ is some bounded measurable function, see Theorem 24.12 in [26]. Theorem 24.10 in [26] states that the Radon-Nikodym derivative of ν_x with respect to ν_{x_0} is the function from $\mathcal{M}$ to $\mathbb{R}$ mapping each $M \in \mathcal{M}$ to $M(x)$. Hence we may rewrite (5.4) as

$$h(x) = \int_\mathcal{M} f(M) d\nu_x(M).$$

Now, apply Theorem 1.2 and let $\iota : \partial U \to \mathscr{M}$ be the homeomorphism $\xi \mapsto M_\xi$. Theorem 1.2 implies that the image under ι of the random walk's exit measure on ∂U coincides with ν_x, concluding our proof. ∎

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References

[1] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan **53** (2001), no. 1, 119–145.

[2] A. Ancona, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 169–213.

[3] A. Ancona, R. Lyons, and Y. Peres, *Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths*, Ann. Probab. **27** (1999), no. 2, 970–989.

[4] M. Barlow, A. Grigor'yan, and T. Kumagai, *On the equivalence of parabolic Harnack inequalities and heat kernel estimates*, J. Math. Soc. Japan **64** (2012), no. 4, 1091-1146.

[5] R. F. Bass and K. Burdzy, *A boundary Harnack principle in twisted Hölder domains*, Ann. of Math. (2) **134** (1991), no. 2, 253–276.

[6] I. Benjamini and O. Schramm, *Harmonic functions on planar and almost planar graphs and manifolds, via circle packings*, Invent. Math. **126** (1996), no. 3, 565–587.

[7] D. Chelkak, *Robust Discrete Complex Analysis: A Toolbox*, Preprint, available at `http://arxiv.org/abs/1212.6205`.

[8] Y. Derriennic, *Lois "zéro ou deux" pour les processus de Markov. Applications aux marches aléatoires*, Ann. Inst. H. Poincaré Sect. B (N.S.) **12** (1976), no. 2, 111–129.

[9] R. Durrett, *Probability: theory and examples*, Fourth, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010.

[10] E. B. Dynkin, *The boundary theory of Markov processes (discrete case)*, Uspehi Mat. Nauk **24** (1969), no. 2 (146), 3–42.

[11] M. Folz, *Volume growth and stochastic completeness of graphs*, Trans. Amer. Math Soc. (to appear), available at `http://arxiv.org/abs/1201.5908`.

[12] A. Georgakopoulos, *The Boundary of a Square Tiling of a Graph coincides with the Poisson Boundary*, Preprint, available at `http://arxiv.org/abs/1301.1506`.

[13] Z.-X. He and O. Schramm, *Hyperbolic and parabolic packings*, Discrete Comput. Geom. **14** (1995), no. 2, 123–149.

[14] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), no. 2, 503–523.

[15] D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147.

[16] V. A. Kaimanovich, *Measure-theoretic boundaries of Markov chains,* 0-2 *laws and entropy*, Harmonic analysis and discrete potential theory (Frascati, 1991), 1992, pp. 145–180.

[17] Koebe P., *Kontaktprobleme der Konformen Abbildung*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **88** (1936), 141–164.

[18] J. Lierl and L. Saloff-Coste, *Scale-invariant Boundary Harnack Principle in Inner Uniform Domains*, Preprint, available at `http://arxiv.org/abs/1110.2763`.

[19] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge University Press, 2014. In preparation. Current version available at `http://mypage.iu.edu/~rdlyons/prbtree/book.pdf`.

[20] R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172.

[21] B. Rodin and D. Sullivan, *The convergence of circle packings to the Riemann mapping*, J. Differential Geom. **26** (1987), no. 2, 349–360.

[22] S. Rohde, *Oded Schramm: from circle packing to SLE*, Ann. Probab. **39** (2011), no. 5, 1621–1667.

[23] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge, 2002.

[24] K. Stephenson, *Introduction to circle packing* (2005), xii+356. The theory of discrete analytic functions.

[25] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297.

[26] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000.

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