Exponential tail bounds for loop-erased random walk in two dimensions

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Abstract

Let M_n be the number of steps of the loop-erasure of a simple random walk on $\mathbb{Z}^2$ from the origin to the circle of radius n. We relate the moments of M_n to $\mathrm{Es}(n)$ – the probability that a random walk and an independent loop-erased random walk both started at the origin do not intersect up to leaving the ball of radius n. This allows us to show that there exists C such that for all n and all $k = 1, 2, \ldots$, $\mathbf{E}\left[M_n^k\right] \leq C^k k! \mathbf{E}\left[M_n\right]^k$ and hence to establish exponential moment bounds for M_n. This implies that there exists $c > 0$ such that for all n and all $\lambda \geq 0$,

$$\mathbf{P}\left\{M_n > \lambda \mathbf{E}\left[M_n\right]\right\} \leq 2e^{-c\lambda}.$$

Using similar techniques, we then establish a second moment result for a specific conditioned random walk, which enables us to prove that for any $\alpha < 4/5$, there exist C and $c' > 0$ such that for all n and $\lambda > 0$,

$$\mathbf{P}\left\{M_n < \lambda^{-1} \mathbf{E}\left[M_n\right]\right\} \leq Ce^{-c'\lambda^\alpha}.$$

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1 Introduction

The loop-erased random walk (LERW) is a process obtained by chronologically erasing loops from a random walk on a graph. Since its introduction by Lawler [4], this process has played a prominent role in the statistical physics literature. It is closely related other models in statistical physics, and in particular to the uniform spanning tree (UST). Pemantle [10] proved that the unique path between any two vertices u and v on the UST has the same distribution as a LERW from u to v, and Wilson [12] devised a powerful algorithm to construct the UST using LERWs. The existence of a scaling limit of LERW on $\mathbb{Z}^d$ is now known for all d. For $d \geq 4$, Lawler

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[5, 6] showed that LERW scales to Brownian motion. For $d = 2$, Lawler, Schramm and Werner [8] proved that LERW has a conformally invariant scaling limit, SLE_2 – indeed, LERW was the prototype for the definition of SLE by Schramm [11]. Most recently, for $d = 3$, Kozma [3] proved that the scaling limit exists and is invariant under rotations and dilations.

Let $S[0, \sigma_n]$ be simple random walk on $\mathbb{Z}^2$ started at the origin and stopped at σ_n, the first time S exits B_n, the ball of radius n with center the origin. Let M_n be the number of steps of $\mathrm{L}(S[0, \sigma_n])$, the loop-erasure of $S[0, \sigma_n]$. In [2], using domino tilings, Kenyon proved for simple random walk on $\mathbb{Z}^2$ that

$$\lim_{n\to\infty} \frac{\log \mathbf{E}[M_n]}{\log n} = \frac{5}{4}. \tag{1.1}$$

Using quite different methods, Masson [9] extended this to irreducible bounded symmetric random walks on any discrete lattice of $\mathbb{R}^2$. The quantity 5/4 is called the *growth exponent* for planar loop-erased random walk. We remark that while SLE_2 has Hausdorff dimension 5/4 almost surely (see [1]), there is no direct proof of (1.1) from this fact; however, unlike the arguments in [2], the approach of [9] does use the connection between the LERW and SLE_2.

In this paper, we will not be concerned with the exact value of $\mathbf{E}[M_n]$, but in obtaining tail bounds on M_n. Our results hold for more general sets than balls. Let D be a domain in $\mathbb{Z}^2$, with $D \neq \emptyset, \mathbb{Z}^2$. Write $S[0, \sigma_D]$ for simple random walk run until its first exit from D, $\mathrm{L}(S[0, \sigma_D])$ for its loop erasure, and M_D for the number of steps in $\mathrm{L}(S[0, \sigma_D])$.

Theorem 1.1. *There exists $c_0 > 0$ that the following holds. Let D be a simply connected subset of $\mathbb{Z}^2$ containing 0, such that for all $z \in D$, $\mathrm{dist}(z, D^c) \leq n$.*

1.

$$\mathbf{E}\left[e^{c_0 M_D/\mathbf{E}[M_n]}\right] \leq 2. \tag{1.2}$$

2. Consequently, for all $\lambda \geq 0$,

$$\mathbf{P}\{M_D > \lambda \mathbf{E}[M_n]\} \leq 2e^{-c_0\lambda}. \tag{1.3}$$

Theorem 1.2. *For all $\alpha < 4/5$ there exist $C_1(\alpha) < \infty$, $c_2(\alpha) > 0$ such that for all $\lambda > 0$ and all n, and all $D \supset B_n$,*

$$\mathbf{P}\left\{M_D < \lambda^{-1}\mathbf{E}[M_n]\right\} \leq C_1(\alpha)\exp(-c_2(\alpha)\lambda^\alpha). \tag{1.4}$$

Remarks 1.3. 1. We expect that these results will hold for irreducible random walks with bounded, symmetric increments on any discrete lattice of $\mathbb{R}^2$. Almost all of the proofs in this paper can be extended to this more general case without any modification. The one exception is Lemma 4.4, where we use the fact that simple random walk on $\mathbb{Z}^2$ is invariant under reflections with respect to horizontal and vertical lines. Theorem 1.1 does not depend on Lemma 4.4 and therefore should be valid in this generality. It is likely that an alternative proof of Lemma 4.4 could be found, but we do not pursue this point further here, and restrict our attention to simple random walk on $\mathbb{Z}^2$.
2. The bound (1.4) for general $D \supset B_n$ does not follow immediately from (1.4) for B_n. The reason is that, if Y is $\mathrm{L}(S[0, \sigma_D])$ run until its first exit from B_n, then Y does not in general have the same law as $\mathrm{L}(S[0, \sigma_n])$. Similar considerations apply to Theorem 1.1.
3. We also have similar bounds for the infinite loop erased walk: see Theorems 5.8 and 6.7.
4. One motivation for proving these results for general domains in $\mathbb{Z}^2$, rather than just balls, is to study the uniform spanning tree (UST) via Wilson's algorithm. In particular we are interested in the volume of balls in the intrinsic metric on the UST, and this requires estimating the number of steps of a LERW until it hits the boundary of a fairly general domain in $\mathbb{Z}^2$.

For the remainder of this introduction, we discuss the case when $D = B_n$. The proofs of Theorems 1.1 and 1.2 involve estimates of the higher moments of M_n. Building on [9], we relate $\mathbf{E}\left[M_n^k\right]$ to $\mathrm{Es}(n)$ – the probability that a LERW and an independent random walk do not intersect up to leaving the ball of radius n. We show that there exists $C < \infty$ such that

$$\mathbf{E}\left[M_n^k\right] \le C^k k!(n^2 \,\mathrm{Es}(n))^k \quad \text{(Theorem 5.6)}; \tag{1.5}$$

$$\mathbf{E}\left[M_n\right] \ge Cn^2 \,\mathrm{Es}(n) \quad \text{(Proposition 5.7)}. \tag{1.6}$$

It is not surprising that the moments of M_n are related to $\mathrm{Es}(n)$. To begin with,

$$\mathbf{E}\left[M_n^k\right] = \sum_{z_1,\ldots,z_k \in B_n} \mathbf{P}\left\{z_1,\ldots,z_k \in \mathrm{L}(S[0,\sigma_n])\right\}.$$

Furthemore, for a point z to be on $\mathrm{L}(S[0,\sigma_n])$, it must be on the random walk path $S[0,\sigma_n]$ and not be on the loops that get erased. In order for this to occur, the random walk path after z cannot intersect the loop-erasure of the random walk path up to z. Therefore, for z to be on $\mathrm{L}(S[0,\sigma_n])$, a random walk and an independent LERW must not intersect in a neighborhood of z. Generalizing this to k points, we get for each i a contribution of $\mathrm{Es}(r_i)$, where r_i is chosen small enough to give 'near independence' of events in the balls $B_{r_i}(z_i)$. Propositions 5.2 and 5.5 make this approach precise. Summing over the $C^k n^{2k}$ k-tuples of points in B_n and using facts about $\mathrm{Es}(.)$ that we establish in Section 3.2 gives (1.5).

Combining (1.5) and (1.6) yields

$$\mathbf{E}\left[M_n^k\right] \le C^k k! \mathbf{E}\left[M_n\right]^k \quad \text{(Theorem 5.8)},$$

from which Theorem 1.1 follows easily.

To establish (1.4), we prove a second moment bound for a specific conditioned random walk and combine this with an iteration argument as follows. Let $B_n(x)$ be the ball of radius n centered at $x \in \mathbb{Z}^2$ and R_n be the square $\{(x,y) \in \mathbb{Z}^2 : -n \le x, y \le n\}$. Fix a positive integer k and consider $\mathrm{L}(S[0,\sigma_{kn}])$. We first establish an upper bound for

$$\mathbf{P}\left\{M_{kn} < \mathbf{E}\left[M_n\right]\right\}.$$

Let $k' = k/\sqrt{2}$ (so that $R_{k'n} \subset B_{kn}$). Let γ_j be the restriction of $\mathrm{L}(S[0,\sigma_{kn}])$ from 0 up to the first exit of R_{jn}, $j = 0,\ldots,k'$. For $j = 0,\ldots,k'-1$, let $x_j \in \partial R_{jn}$ be the point where γ_j hits ∂R_{jn} and $B_j = B_n(x_j)$. Finally, for $j = 1,\ldots,k'$, let N_j be the number of steps of γ_j from x_{j-1} up to the first time it exits B_{j-1}. See Figure 3 in Section 6. We consider squares instead of balls to take advantage of the symmetry of simple random walk on $\mathbb{Z}^2$ with respect to vertical and horizontal lines as mentioned above. Clearly,

$$\begin{aligned}\mathbf{P}\left\{M_{kn} < \mathbf{E}\left[M_n\right]\right\} &\le \mathbf{P}\left(\bigcap_{j=1}^{k'}\{N_j < \mathbf{E}\left[M_n\right]\}\right) \\ &\le \prod_{j=1}^{k'} \max_{\gamma_{j-1}} \mathbf{P}\left\{N_j < \mathbf{E}\left[M_n\right] \mid \gamma_{j-1}\right\}.\end{aligned} \tag{1.7}$$

However, by the domain Markov property for LERW (Lemma 3.2), conditioned on γ_{j-1}, the rest of the LERW curve is obtained by running a random walk conditioned to leave B_{kn} before hitting γ_{j-1} and then erasing loops. For this reason, we will be interested in the number of steps of the loop-erasure of a random walk started on the boundary of a square and conditioned to leave some large ball before hitting a set contained in the square. Formally, we give the following definition:

Definition 1.4 (See Figure 1)**.** *Suppose that the natural numbers m, n and N are such that $\sqrt{2}m + n \leq N$, and that K is a subset of the square $R_m = [-m, m]^2$. Suppose that $x = (m, y)$, $|y| \leq m$ is any point on the right side of R_m and let X be a random walk started at x, conditioned to leave B_N before hitting K. Let α be $\mathrm{L}(X[0, \sigma_N])$ from x up to its first exit time of the ball $B_n(x)$. Then we let $M^K_{m,n,N,x}$ be the number of steps of α in $A_n(x) = \{z : n/4 \leq |z - x| \leq 3n/4, |\arg(z - x)| \leq \pi/4\}$. Note that the condition $\sqrt{2}m+n \leq N$ ensures that $B_n(x)$ is contained in B_N.*

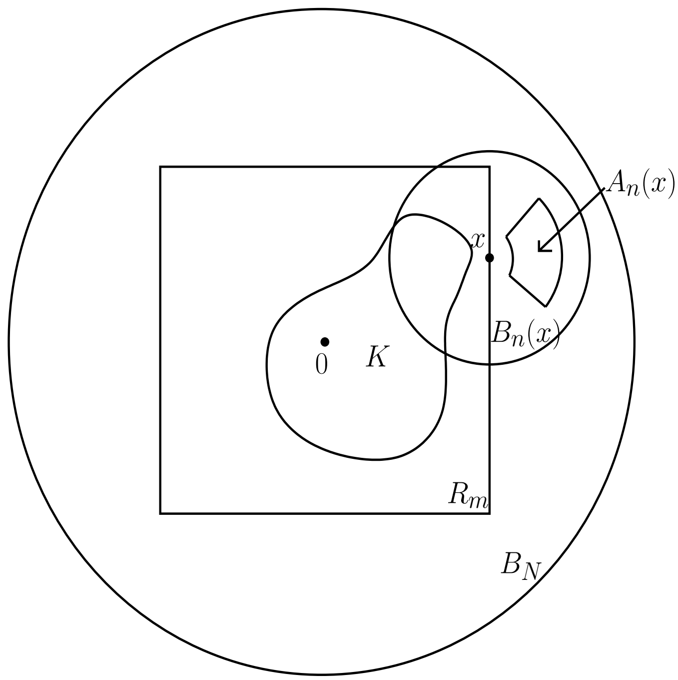

Figure 1: Setup for Definition 1.4

We look at the number of steps of the LERW in $A_n(x)$ rather than in $B_n(x)$ since the expectations of these random variables are comparable, and it is convenient not to have to worry about points that are close to x, K or $\partial B_n(x)$. We are therefore interested in estimating

$$\mathbf{P}\left\{M^K_{m,n,N,x} < \mathbf{E}\left[M_n\right]\right\}.$$

To do this, we first show that (up to a log term) $\mathbf{E}\left[M^K_{m,n,N,x}\right]$ is comparable to $n^2\,\mathrm{Es}(n)$ and therefore by (1.6), $\mathbf{E}\left[M^K_{m,n,N,x}\right]$ is comparable to $\mathbf{E}\left[M_n\right]$ (Proposition 6.2). Next, we prove that $\mathbf{E}\left[(M^K_{m,n,N,x})^2\right]$ is comparable to $\mathbf{E}\left[M^K_{m,n,N,x}\right]^2$ (again up to a log term – Proposition 6.3). By a standard second moment technique, this implies that there exist $c = c(n, N) > 0$ and $p = p(n, N) < 1$ such that

$$\mathbf{P}\left\{M^K_{m,n,N,x} < c\mathbf{E}\left[M^K_{m,n,N,x}\right]\right\} < p. \tag{1.8}$$

Using the fact that $\mathbf{E}[M_n]$ is comparable to $\mathbf{E}\left[M^K_{m,n,N,x}\right]$, we can then plug this into (1.7) to conclude that there exists $p = p(k) = 1 - c(\log k)^{-8}$ such that

$$\mathbf{P}\{M_{kn} < \mathbf{E}[M_n]\} < p^k. \tag{1.9}$$

Finally, to prove (1.4), one makes an appropriate choice of k and relates $\mathbf{E}[M_{kn}]$ to $\mathbf{E}[M_n]$. Although the logarithmic corrections in Propositions 6.2 and 6.3 mean that p in (1.8) depends on n and N, and so p in (1.9) depends on k, this correction is small enough so that (1.9) still gives a useful bound.

The paper is organized as follows. In Section 2 we fix notation and recall the basic properties of random walks that will be needed. In Section 3, we give a precise definition of the LERW and state some of its properties. Many of these properties were established in [9]. Indeed, this paper uses similar techniques to those in [9] – most notably relating the growth exponent to Es(n). It turns out that the latter quantity is often easier to analyze directly (see section 3.2).

Section 4 contains some technical lemmas involving estimates for Green's functions for random walks in various domains and for the conditioned random walks X in Definition 1.4. In Section 5 we prove Theorem 1.1 using the approach described above. Finally, in Section 6, we use the iteration outlined above to prove Theorem 1.2.

2 Definitions and background for random walks

2.1 Notation for random walks and Markov chains

Throughout the paper, when we say random walk, we will mean simple random walk on $\mathbb{Z}^2$. We will denote a random walk starting at a point $z \in \mathbb{Z}^2$ by S^z. When $z = 0$, we will omit the superscript. If we have two random walks S^z and S^w starting at two different points z and w, we assume that they are independent unless otherwise specified. We use similar notation for other Markov chains on $\mathbb{Z}^2$ (all our Markov chains are assumed to be time–homogeneous). When there is no possibility of confusion, we will also use the following standard notation. Given an event A that depends on a Markov chain X, we let $\mathbf{P}^z(A)$ denote the probability of A given that $X_0 = z$.

2.2 A note about constants

For the entirety of the paper, we will use the letters c and C to denote positive constants that will not depend on any variable but may change on each appearance. When we wish to fix a constant, we will number it with a subscript, e.g. c_0.

Given two positive functions $f(n)$ and $g(n)$, we write $f(n) \asymp g(n)$ if there exists $C < \infty$ such that for all n,

$$C^{-1}g(n) \le f(n) \le Cg(n).$$

We will say that two sequences of events $\{E_n\}$ and $\{F_n\}$ have the same probability "up to constants" if $\mathbf{P}(E_n) \asymp \mathbf{P}(F_n)$, and are independent "up to constants" if $\mathbf{P}(E_n \cap F_n) \asymp \mathbf{P}(E_n)\mathbf{P}(F_n)$. We will also use the obvious generalization for two sequences of random variables to have the same distribution "up to constants" and to be independent "up to constants".

2.3 Subsets of $\mathbb{Z}^2$

Given two points $x, y \in \mathbb{Z}^2$, we write $x \sim y$ if $|x-y| = 1$.

A sequence of points $\omega = [\omega_0, \ldots, \omega_k] \subset \mathbb{Z}^2$ is called a path if $\omega_{j-1} \sim \omega_j$ for $j = 1, \ldots, k$. We let $|\omega| = k$ be the length of the path, Θ_k be the set of paths of length k, and $\Theta = \bigcup_k \Theta_k$ denote the set of all finite paths. Also, if X is a Markov chain with transition probabilities $p^X(.,.)$ and $\omega \in \Theta_k$ we define

$$p^X(\omega) = \prod_{i=1}^{k} p^X(\omega_{i-1}, \omega_i).$$

Thus, if $X = S$ is a simple random walk, $p^S(\omega) = 4^{-k}$. A set $D \subset \mathbb{Z}^2$ is connected if for any pair of points $x, y \in D$, there exists a path $\omega \subset D$ connecting x and y, and D is simply connected if it is connected and all connected components of $\mathbb{Z}^2 \setminus D$ are infinite.

Given $z \in \mathbb{Z}^2$, let

$$B_n(z) = \{x \in \mathbb{Z}^2 : |x - z| \leq n\}$$

be the ball of radius n centered at z in $\mathbb{Z}^2$. We will write B_n for $B_n(0)$ and sometimes write $B(z; n)$ for $B_n(z)$. Also, let R_n denote the square $\{(x, y) \in \mathbb{Z}^2 : -n \leq x, y \leq n\}$.

The outer boundary of a set $D \subset \mathbb{Z}^2$ is

$$\partial D = \{x \in \mathbb{Z}^2 \setminus D : \text{ there exists } y \in D \text{ such that } x \sim y\},$$

and its inner boundary is

$$\partial_i D = \{x \in D : \text{ there exists } y \in \mathbb{Z}^2 \setminus D \text{ such that } x \sim y\}.$$

We also write $\overline{D} = D \cup \partial D$.

Given a Markov chain X on $\mathbb{Z}^2$ and a set $D \subset \mathbb{Z}^2$, let

$$\sigma_D^X = \min\{j \geq 1 : X_j \in \mathbb{Z}^2 \setminus D\}$$

be the first exit time of the set D, and

$$\xi_D^X = \min\{j \geq 1 : X_j \in D\}$$

be the first hitting time of the set D. We let $\sigma_n^X = \sigma_{B_n}^X$ and use a similar convention for ξ_n^X. If X is a random walk S^z starting at $z \in \mathbb{Z}^2$ then we let σ_D^z and ξ_D^z be the exit and hitting times for S^z. If $z = 0$, we will omit the superscripts. We will also omit superscripts when it is clear what process the stopping times refer to. For instance, we will write $X[0, \sigma_n]$ instead of $X[0, \sigma_n^X]$.

2.4 Basic facts about random walks

For a Markov chain X and $x, y \in D \subset \mathbb{Z}^2$, let

$$G_D^X(x, y) = \mathbf{E}^x \left[\sum_{j=0}^{\sigma_D^X - 1} \mathbb{1}\{X_j = y\} \right]$$

denote the Green's function for X in D. We will sometimes write $G^X(x, y; D)$ for $G_D^X(x, y)$. We will write $G_n^X(x, y)$ for $G_{B_n}^X(x, y)$, and when $X = S$ is a random walk, we will omit the superscript S.

Recall that a function f defined on $\overline{D} \subset \mathbb{Z}^2$ is discrete harmonic on D if for all $z \in D$,

$$\mathcal{L}f(z) := -f(z) + \frac{1}{4}\sum_{x\sim z} f(x) = 0.$$

For any two disjoint subsets K_1 and K_2 of $\mathbb{Z}^2$, it is easy to verify that the function

$$h(z) = \mathbf{P}^z\left\{\xi_{K_1} < \xi_{K_2}\right\}$$

is discrete harmonic on $\mathbb{Z}^2 \setminus (K_1 \cup K_2)$. Furthermore, if we let X be a random walk conditioned to hit K_1 before K_2 then X is a reversible Markov chain on $\mathbb{Z}^2 \setminus (K_1 \cup K_2)$ with transition probabilities

$$p^X(x,y) = \frac{1}{4}\frac{h(y)}{h(x)}.$$

Therefore, if $\omega = [\omega_0, \ldots, \omega_k]$ is a path in $\mathbb{Z}^2 \setminus (K_1 \cup K_2)$, then

$$p^X(\omega) = \frac{h(\omega_k)}{h(\omega_0)} 4^{-|\omega|}. \tag{2.1}$$

Using this fact, the following lemma follows readily.

Lemma 2.1. *Suppose that X is a random walk conditioned to hit K_1 before K_2, and let D be such that $D \subset \mathbb{Z}^2 \setminus (K_1 \cup K_2)$. Then for any $x, y \in D$,*

$$G_D^X(x,y) = \frac{h(y)}{h(x)} G_D(x,y).$$

In particular, $G_D^X(x,x) = G_D(x,x)$.

Using a last-exit decomposition, one can also express $h(x)$ in terms of Green's functions:

Lemma 2.2. *Let $K_1, K_2 \subset \mathbb{Z}^2$ be disjoint and $x \in \mathbb{Z}^2 \setminus (K_1 \cup K_2)$. Then,*

$$\begin{aligned}&\mathbf{P}^x\left\{\xi_{K_1} < \xi_{K_2}\right\}\\ &= \frac{G\left(x,x;\mathbb{Z}^2 \setminus (K_1 \cup K_2)\right)}{G(x,x;\mathbb{Z}^2 \setminus K_1)} \sum_{y \in \partial_i K_1} \mathbf{P}^y\left\{\xi_x < \xi_{K_2} \mid \xi_x < \xi_{K_1}\right\} \mathbf{P}^x\left\{S(\xi_{K_1}) = y\right\}.\end{aligned}$$

The following proposition was proved in [9] and will be used frequently in the paper.

Proposition 2.3. *There exists $c > 0$ such that for all n and all $K \subset \{z \in \mathbb{Z}^2 : \mathrm{Re}(z) \leq 0\}$,*

$$\mathbf{P}\left\{\arg(S(\sigma_n)) \in [-\frac{\pi}{4}, \frac{\pi}{4}] \;\middle|\; \sigma_n < \xi_K\right\} \geq c.$$

We conclude this section with a list of standard potential theory results that will be used throughout the paper, often without referring back to this proposition. The proofs of these results can all be found in [7, Chapter 6].

Proposition 2.4.

1. *Discrete Harnack Principle. Let U be a connected open subset of $\mathbb{R}^2$ and A a compact subset of U. Then there exists a constant $C(U,A)$ such that for all n and all positive harmonic functions f on $nU \cap \mathbb{Z}^2$*

$$f(x) \leq C(U,A) f(y)$$

for all $x, y \in nA \cap \mathbb{Z}^2$.

2. There exists $c > 0$ *such that for all* n *and all paths* α *connecting* B_n *to* $\mathbb{Z}^2 \setminus B_{2n}$,

$$\mathbf{P}^z\{\xi_\alpha < \sigma_{2n}\} \geq c \quad \text{for all } z \in B_n;$$
$$\mathbf{P}^z\{\xi_\alpha < \xi_n\} \geq c \quad \text{for all } z \in \partial B_{2n}.$$

3. If $m < |z| < n$, *then*

$$\mathbf{P}^z\{\xi_m < \sigma_n\} = \frac{\ln n - \ln |z| + O(m^{-1})}{\ln n - \ln m}.$$

4. If $z \in B_n$, *then*

$$\mathbf{P}^z\{\xi_0 < \sigma_n\} = \left(1 - \frac{\ln |z|}{\ln n}\right)\left[1 + O\left(\frac{1}{\ln n}\right)\right].$$

5. If $z \in B_n \setminus \{0\}$, *then*

$$G_n(0, z) \asymp \ln \frac{n}{|z|}.$$

6.

$$G_n(0, 0) \asymp \ln n.$$

3 Loop-erased random walks

3.1 Definition

We now describe the loop-erasing procedure and define the loop-erased random walk. Given a path $\lambda = [\lambda_0, \ldots, \lambda_m]$ in $\mathbb{Z}^2$, let $\mathrm{L}(\lambda) = [\hat{\lambda}_0, \ldots, \hat{\lambda}_n]$ denote its chronological loop-erasure. More precisely, we let

$$s_0 = \sup\{j : \lambda(j) = \lambda(0)\},$$

and for $i > 0$,

$$s_i = \sup\{j : \lambda(j) = \lambda(s_{i-1} + 1)\}.$$

Let

$$n = \inf\{i : s_i = m\}.$$

Then

$$\mathrm{L}(\lambda) = [\lambda(s_0), \lambda(s_1), \ldots, \lambda(s_n)].$$

One may obtain a different result if one performs the loop-erasing procedure backwards instead of forwards. In other words, if we let $\lambda^R = [\lambda_m, \ldots, \lambda_0]$ be the time reversal of λ, then in general,

$$\mathrm{L}^R(\lambda) := \left(\mathrm{L}(\lambda^R)\right)^R \neq \mathrm{L}(\lambda).$$

However, the following lemma shows that if λ is distributed according to a Markov chain, then $\mathrm{L}^R(\lambda)$ has the same distribution as $\mathrm{L}(\lambda)$. Recall that Θ denotes the set of all finite paths in $\mathbb{Z}^2$.

Lemma 3.1 (Lawler [5])**.** *There exists a bijection* $T : \Theta \to \Theta$ *such that*

$$\mathrm{L}^R(\lambda) = \mathrm{L}(T\lambda).$$

Furthermore, $T\lambda$ *and* λ *visit the same edges in* $\mathbb{Z}^2$ *in the same directions with the same multiplicities so that for any Markov chain* X *on* $\mathbb{Z}^2$, $p^X(T\lambda) = p^X(\lambda)$.

A fundamental fact about LERWs is the following "Domain Markov property."

Lemma 3.2 (Domain Markov property [5])**.** *Let $D \subset \Lambda$ and $\omega = [\omega_0, \omega_1, \dots, \omega_k]$ be a path in D. Let Y be a random walk started at ω_k conditioned to exit D before hitting ω. Suppose that $\omega' = [\omega'_0 \dots, \omega'_{k'}]$ is such that*

$$\omega \oplus \omega' := [\omega_0, \dots, \omega_k, \omega'_0, \dots, \omega'_{k'}]$$

is a path from ω_0 to ∂D. Then if we let α be the first k steps of $\mathrm{L}(S[0, \sigma_D])$*,*

$$\mathbf{P}\left\{\mathrm{L}(S[0, \sigma_D]) = \omega \oplus \omega' \;\middle|\; \alpha = \omega\right\} = \mathbf{P}\left\{\mathrm{L}(Y[0, \sigma_D]) = \omega'\right\}.$$

Suppose that l is a positive integer and D is a proper subset of $\mathbb{Z}^2$ with $B_l \subset D$. Let Ω_l be the set of paths $\omega = [0, \omega_1, \dots, \omega_k] \subset \mathbb{Z}^2$ such that $\omega_j \in B_l$, $j = 1, \dots, k-1$ and $\omega_k \in \partial B_l$. Define the measure $\mu_{l,D}$ on Ω_l to be the distribution on Ω_l obtained by restricting $\mathrm{L}(S[0, \sigma_D])$ to the part of the path from 0 to the first exit of B_l.

Two different sets D_1 and D_2 will produce different measures. However, the following proposition [9] shows that as $\mathbb{Z}^2 \setminus D_1$ and $\mathbb{Z}^2 \setminus D_2$ get further away from B_l, the measures μ_{l,D_1} and μ_{l,D_2} approach each other.

Proposition 3.3. *There exists $C < \infty$ such that the following holds. Suppose that $n \geq 4$, D_1 and D_2 are such that $B_{nl} \subset D_1$ and $B_{nl} \subset D_2$, and that $\omega \in \Omega_l$. Then*

$$1 - \frac{C}{\log n} \leq \frac{\mu_{l,D_1}(\omega)}{\mu_{l,D_2}(\omega)} \leq 1 + \frac{C}{\log n}.$$

The previous proposition shows that for a fixed l, the sequence $\mu_{l,n}(\omega) := \mu_{l,B_n}(\omega)$ is Cauchy. Therefore, there exists a limiting measure μ_l such that

$$\lim_{n\to\infty} \mu_{l,n}(\omega) = \mu_l(\omega).$$

The μ_l are consistent and therefore there exists a measure μ on infinite self-avoiding paths. We call the associated process the infinite LERW and denote it by $\widehat{S}$. We denote the exit time of a set D for $\widehat{S}$ by $\widehat{\sigma}_D$. An immediate corollary of the previous proposition and the definition of $\widehat{S}$ is the following.

Corollary 3.4. *Suppose that $B_{4l} \subset D$, and $\omega \in \Omega_l$. Then,*

$$\mathbf{P}\left\{\widehat{S}[0, \widehat{\sigma}_l] = \omega\right\} \asymp \mu_{l,D}(\omega).$$

The following result follows immediately from Corollary 3.4 and [9, Proposition 4.2].

Corollary 3.5. *Suppose that $B_{4l} \subset D_1$ and $B_{4l} \subset D_2$, and let X be a random walk conditioned to leave D_1 before D_2. Let α be* $\mathrm{L}([X[0, \sigma_{D_1}])$ *from* 0 *up to its first exit of B_l. Then for $\omega \in \Omega_l$,*

$$\mathbf{P}\{\alpha = \omega\} \asymp \mathbf{P}\left\{\widehat{S}[0, \widehat{\sigma}_l] = \omega\right\}.$$

We conclude this section with a "separation lemma" for random walks and LERWs. It states the intuitive fact that, conditioned on the event that a random walk S and an independent infinite LERW $\widehat{S}$ do not intersect up to leaving B_n, the probability that they are further than some fixed distance apart from each other on ∂B_n is bounded from below by $p > 0$.

Proposition 3.6 (Separation Lemma [9])**.** *There exist $c, p > 0$ such that for all n the following holds. Let S and $\widehat{S}$ be independent and let*

$$d_n = \mathrm{dist}(S(\sigma_n), \widehat{S}[0, \widehat{\sigma}_n]) \wedge \mathrm{dist}(\widehat{S}(\widehat{\sigma}_n), S[0, \sigma_n]).$$

Then,

$$\mathbf{P}\left\{d_n \geq cn \;\middle|\; S[1, \sigma_n] \cap \widehat{S}[0, \widehat{\sigma}_n] = \emptyset\right\} \geq p. \tag{3.1}$$

3.2 Escape probabilities for LERW

Definition 3.7. *For a set D containing* 0*, we let M_D be the number of steps of* $\mathrm{L}(S[0,\sigma_D])$ *and let $M_n = M_{B_n}$. We also let $\widehat{M}_D$ be the number of steps of $\widehat{S}[0,\widehat{\sigma}_D]$ and $\widehat{M}_n = \widehat{M}_{B_n}$.*

As described in the introduction, one of the goals of the paper is to relate the moments of M_D and $\widehat{M}_D$ to escape probabilities which we now define.

Definition 3.8. *Let S and S' be two independent random walks started at* 0*. For $m \leq n$, let* $\mathrm{L}(S'[0,\sigma_n]) = \eta = [0, \eta_1, \ldots, \eta_k]$*,* $k_0 = \max\{j \geq 1 : \eta_j \in B_m\}$ *and* $\eta_{m,n}(S') = [\eta_{k_0}, \ldots, \eta_k]$*. Then we define*

$$\begin{aligned}
\mathrm{Es}(m,n) &= \mathbf{P}\left\{S[1,\sigma_n] \cap \eta_{m,n}(S') = \emptyset\right\};\\
\mathrm{Es}(n) &= \mathbf{P}\left\{S[1,\sigma_n] \cap \mathrm{L}(S'[0,\sigma_n]) = \emptyset\right\};\\
\widehat{\mathrm{Es}}(n) &= \mathbf{P}\left\{S[1,\sigma_n] \cap \widehat{S}[0,\widehat{\sigma}_n] = \emptyset\right\}.
\end{aligned}$$

We also let $\mathrm{Es}(0) = 1$*.*

Thus, $\mathrm{Es}(m,n)$ is the probability that a random walk from the origin to ∂B_n and the terminal part of an independent LERW from m to n do not intersect. $\mathrm{Es}(n)$ is the probability that a random walk from the origin to ∂B_n and the loop-erasure of an independent random walk from the origin to ∂B_n do not intersect. $\widehat{\mathrm{Es}}(n)$ is the corresponding escape probability for an *infinite* LERW from the origin to ∂B_n.

The following was proved in [9] – see Lemma 5.1, Propositions 5.2 and 5.3, and Theorem 5.6.

Theorem 3.9. *There exists $C < \infty$ such that the following holds.*

1. $$C^{-1}\,\mathrm{Es}(n) \leq \widehat{\mathrm{Es}}(n) \leq C\,\mathrm{Es}(n).$$

2. *For all $k \geq 1$, there exists $N = N(k)$ such that for $n \geq N$,*

$$C^{-1}k^{-3/4} \leq \mathrm{Es}(n,kn) \leq Ck^{-3/4}.$$

3. *For all $l \leq m \leq n$,*

$$C^{-1}\,\mathrm{Es}(n) \leq \mathrm{Es}(m)\,\mathrm{Es}(m,n) \leq C\,\mathrm{Es}(n),$$

and

$$C^{-1}\,\mathrm{Es}(l,n) \leq \mathrm{Es}(l,m)\,\mathrm{Es}(m,n) \leq C\,\mathrm{Es}(l,n).$$

We conclude this section with some easy consequences of this theorem.

Lemma 3.10. *For all $k \geq 1$, there exists $c(k) > 0$ such that for all $n = 1, 2, \ldots$,*

$$\mathrm{Es}(kn) \geq c(k)\,\mathrm{Es}(n).$$

Proof. By Theorem 3.9 parts 2 and 3, there exists $N(k)$ such that for $n \geq N(k)$,

$$\mathrm{Es}(kn) \geq c\,\mathrm{Es}(n,kn)\,\mathrm{Es}(n) \geq ck^{-3/4}\,\mathrm{Es}(n) = c(k)\,\mathrm{Es}(n).$$

Since there are only finitely many $n \leq N(k)$, the result holds. □

Lemma 3.11. *There exists $C < \infty$ such that for all $l \leq m \leq n$,*

$$\mathrm{Es}(n) \leq C\,\mathrm{Es}(m), \tag{3.2}$$

and

$$\mathrm{Es}(l,n) \leq C\,\mathrm{Es}(l,m). \tag{3.3}$$

Proof. Using Theorem 3.9 part 3 and the fact that $\mathrm{Es}(m,n) \leq 1$, one obtains that

$$\mathrm{Es}(n) \leq C\,\mathrm{Es}(m)\,\mathrm{Es}(m,n) \leq C\,\mathrm{Es}(m),$$

and

$$\mathrm{Es}(l,n) \leq C\,\mathrm{Es}(l,m)\,\mathrm{Es}(m,n) \leq C\,\mathrm{Es}(l,m).$$

□

Lemma 3.12. *For all $\varepsilon > 0$, there exist $C(\varepsilon) < \infty$ and $N(\varepsilon)$ such that for all $N(\varepsilon) \leq m \leq n$,*

$$C(\varepsilon)^{-1}\left(\frac{n}{m}\right)^{-3/4-\varepsilon} \leq \mathrm{Es}(m,n) \leq C(\varepsilon)\left(\frac{n}{m}\right)^{-3/4+\varepsilon}.$$

Proof. Fix $\varepsilon > 0$. Let C_1 be the largest of the constants in the statements of Theorem 3.9 and Lemma 3.11, and let j be any integer greater than $C_1^{2/\varepsilon}$. By Theorem 3.9 part 2, there exists N such that for all $n \geq N$,

$$C_1^{-1} j^{-3/4} \leq \mathrm{Es}(n, jn) \leq C_1 j^{-3/4}.$$

We will show that the conclusion of the lemma holds with this choice of N.

Let m and n be such that $N \leq m \leq n$, and let k be the unique integer such that

$$j^k \leq \frac{n}{m} < j^{k+1}.$$

It follows from Theorem 3.9 part 3 and Lemma 3.11 that

$$\begin{aligned}
\mathrm{Es}(m,n) &\leq C\,\mathrm{Es}(m, j^k m) \\
&\leq C_1^{k+1} \prod_{i=0}^{k-1} \mathrm{Es}(j^i m, j^{i+1} m) \\
&\leq C_1^{2k+1} (j^{-3/4})^k \\
&\leq C_1 j^{\varepsilon k} j^{3/4} \left(\frac{n}{m}\right)^{-3/4} \\
&\leq C_1 j^{3/4} \left(\frac{n}{m}\right)^{\varepsilon} \left(\frac{n}{m}\right)^{-3/4}.
\end{aligned}$$

This proves the upper bound with $C(\varepsilon) = C_1 j^{3/4}$; the lower bound is proved in exactly the same fashion. □

Lemma 3.13. *For all $\varepsilon > 0$, there exists $C(\varepsilon) < \infty$ such that for all $m \leq n$,*

$$m^{3/4+\varepsilon}\,\mathrm{Es}(m) \leq C(\varepsilon) n^{3/4+\varepsilon}\,\mathrm{Es}(n).$$

Proof. Fix $\varepsilon > 0$. Applying Lemma 3.12, we get that there exist $c > 0$ and N such that for all $N \leq m \leq n$,

$$\operatorname{Es}(m,n) \geq c\left(\frac{n}{m}\right)^{-3/4-\varepsilon}.$$

Therefore, if $N \leq m \leq n$, then by Theorem 3.9 part 3,

$$n^{3/4+\varepsilon}\operatorname{Es}(n) \geq cn^{3/4+\varepsilon}\operatorname{Es}(m)\operatorname{Es}(m,n) \geq cn^{3/4+\varepsilon}\operatorname{Es}(m)\left(\frac{n}{m}\right)^{-3/4-\varepsilon} = cm^{3/4+\varepsilon}\operatorname{Es}(m).$$

Since there are only finitely many pairs (m,n) such that $m \leq n \leq N$, there exists C such that $m^{3/4+\varepsilon}\operatorname{Es}(m) \leq Cn^{3/4+\varepsilon}\operatorname{Es}(n)$ for all such pairs (m,n). Finally, if $m \leq N \leq n$, then since $m^{3/4+\varepsilon}\operatorname{Es}(m) \leq CN^{3/4+\varepsilon}\operatorname{Es}(N)$ and $N^{3/4+\varepsilon}\operatorname{Es}(N) \leq Cn^{3/4+\varepsilon}\operatorname{Es}(n)$, the result holds in this case too. □

In Sections 5 and 6 we will have to handle various sums involving $\operatorname{Es}(n)$, and we will use the following many times.

Corollary 3.14. *Let $\gamma > 0$, $\beta > 0$ and $1+\alpha-3\gamma/4 > 0$. Then there exists $C < \infty$ (depending on α, β, γ) such that for all n,*

$$\sum_{j=1}^{n} j^{\alpha}\left(\ln\frac{n}{j}\right)^{\beta}\operatorname{Es}(j)^{\gamma} \leq Cn^{\alpha+1}\operatorname{Es}(n)^{\gamma}.$$

Proof. Choose $\varepsilon > 0$ such that $1+\alpha-3\gamma/4-(\beta+\gamma)\varepsilon > 0$. Then using Lemma 3.13,

$$\begin{aligned}
\sum_{j=1}^{n} j^{\alpha}\left(\ln\frac{n}{j}\right)^{\beta}\operatorname{Es}(j)^{\gamma} &= \sum_{j=1}^{n} j^{\alpha-3\gamma/4-\gamma\varepsilon}\left(\ln\frac{n}{j}\right)^{\beta}\left(j^{3/4+\varepsilon}\operatorname{Es}(j)\right)^{\gamma} \\
&\leq C\left(n^{3/4+\varepsilon}\operatorname{Es}(n)\right)^{\gamma}\sum_{j=1}^{n} j^{\alpha-3\gamma/4-\gamma\varepsilon}(n/j)^{\varepsilon\beta} \\
&\leq Cn^{3\gamma/4+\varepsilon\gamma+\varepsilon\beta}\operatorname{Es}(n)^{\gamma}\sum_{j=1}^{n} j^{\alpha-3\gamma/4-\gamma\varepsilon-\beta\varepsilon} \\
&\leq Cn^{1+\alpha}\operatorname{Es}(n)^{\gamma}.
\end{aligned}$$

□

4 Green's function estimates

Lemma 4.1. *There exists $C < \infty$ such that the following holds. Let $D \subset \mathbb{Z}^2$ be simply connected, and for $z \in D$, write $\operatorname{dist}(z, D^c)$ for the distance between z and D^c. Let*

$$D_n = \{z \in D : \operatorname{dist}(z, D^c) \leq n\}.$$

Then for any $w \in D$,

$$\sum_{z \in D_n} G_D(w,z) \leq Cn^2. \tag{4.1}$$

In particular, if $\operatorname{dist}(z, D^c) \leq n$ for all $z \in D$, then for all $w \in D$,

$$\sum_{z \in D} G_D(w,z) \leq Cn^2.$$

Proof. Fix $n \geq 1$ and define stopping times (T_j), (U_j) as follows:

$$\begin{aligned} T_1 &= \min\{i \geq 0 : S_i \in D_n\}, \\ U_j &= \min\{i \geq T_j : |S_i - S_{T_j}| \geq 2n\}, \\ T_{j+1} &= \min\{U_j \leq i < \sigma_D : S_i \in D_n\}. \end{aligned} \tag{4.2}$$

Here, as usual, we take $T_{j+1} = \infty$ if the set in (4.2) is empty. On the event that $T_j < \infty$, $\mathbf{E}^{S_{T_j}}[U_j - T_j] \leq Cn^2$, and thus

$$\begin{aligned} \sum_{z \in D_n} G_D(w, z) &= \mathbf{E}^w \left[\sum_{j=1}^{\sigma_D - 1} \mathbb{1}\{X_j \in D_n\} \right] \\ &\leq \mathbf{E}^w \left[\sum_{j=1}^{\infty} (U_j - T_j) \right] \\ &\leq Cn^2 \sum_{j=1}^{\infty} \mathbf{P}^w \{T_j < \infty\}. \end{aligned} \tag{4.3}$$

Since D is simply connected, for each $z \in D_n$ there is a path in D^c connecting $B(z; n+1)$ and $\partial B(z; 2n)$. Therefore, by Proposition 2.4 part 2, there exists $p > 0$ such that for any $z \in D_n$,

$$\mathbf{P}^z \left\{ \sigma_D < \sigma_{B(z,2n)} \right\} > p.$$

Consequently $\mathbf{P}^w \{T_{j+1} < \infty | T_j < \infty\} < 1 - p$, and so $\mathbf{P}^w \{T_j < \infty\} < (1-p)^{j-1}$. Therefore, summing the series in (4.3) yields (4.1). □

Lemma 4.2. *There exist $C < \infty$ and $c > 0$ such that the following holds. Suppose that D is simply connected, $w \in D$ and $\mathrm{dist}(w, D^c) = n$.*

1. *For all $z \in B_{n/2}(w)$,*

$$\mathbf{P}^z \{\xi_w < \sigma_D\} \leq C\mathbf{P}^z \left\{ \xi_w < \sigma_D \wedge \sigma_{B_{2n}(w)} \right\}; \tag{4.4}$$

2. *For all $z \in B_n(w)$ and $l \leq |z - w|$,*

$$\mathbf{P}^z \left\{ \sigma_D < \xi_{B_l(w)} \right\} \geq c\mathbf{P}^z \left\{ \sigma_D \wedge \sigma_{B_{2n}(w)} < \xi_{B_l(w)} \right\}. \tag{4.5}$$

Proof. We can take $w = 0$ so that $\sigma_{B_{2n}(w)} = \sigma_{2n}$ and $\xi_{B_l(w)} = \xi_l$. We begin with (4.4). Let $z_0 \in \partial B_{n/2}$ be such that

$$\mathbf{P}^{z_0} \{\xi_0 < \sigma_D\} = \max_{z \in \partial B_{n/2}} \mathbf{P}^z \{\xi_0 < \sigma_D\}.$$

Then,

$$\begin{aligned} \mathbf{P}^{z_0} \{\xi_0 < \sigma_D\} &\leq \mathbf{P}^{z_0} \{\xi_0 < \sigma_D \wedge \sigma_{2n}\} + \mathbf{P}^{z_0} \{\sigma_{2n} < \sigma_D\} \max_{y \in \partial B_{2n}} \mathbf{P}^y \{\xi_0 < \sigma_D\} \\ &\leq \mathbf{P}^{z_0} \{\xi_0 < \sigma_D \wedge \sigma_{2n}\} + \mathbf{P}^{z_0} \{\sigma_{2n} < \sigma_D\} \mathbf{P}^{z_0} \{\xi_0 < \sigma_D\}. \end{aligned} \tag{4.6}$$

However, because D is simply connected and $\mathrm{dist}(0, D^c) = n$, there exists a path in D^c connecting ∂B_n to ∂B_{2n}. Therefore, by Proposition 2.4 part 2, there exists $c > 0$ such that

$$\mathbf{P}^{z_0} \{\sigma_{2n} < \sigma_D\} \leq 1 - c.$$

Thus, inserting this in (4.6) yields

$$\mathbf{P}^{z_0}\{\xi_0 < \sigma_D\} \leq C\mathbf{P}^{z_0}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\}.$$

Hence if z is any point in $\partial B_{n/2}$, then

$$\mathbf{P}^{z}\{\xi_0 < \sigma_D\} \leq \mathbf{P}^{z_0}\{\xi_0 < \sigma_D\} \leq C\mathbf{P}^{z_0}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\} \leq C\mathbf{P}^{z}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\}, \tag{4.7}$$

where the last inequality follows from the discrete Harnack inequality.

Now suppose that z is any point in $B_{n/2}$. Then using (4.7),

$$\begin{aligned}
\mathbf{P}^{z}\{\xi_0 < \sigma_D\} &= \mathbf{P}^{z}\{\xi_0 < \sigma_n\} + \sum_{y\in\partial B_n} \mathbf{P}^{y}\{\xi_0 < \sigma_D\}\mathbf{P}^{z}\{\sigma_n < \xi_0; S(\sigma_n) = y\} \\
&\leq \mathbf{P}^{z}\{\xi_0 < \sigma_n\} + C\sum_{y\in\partial B_n} \mathbf{P}^{y}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\}\mathbf{P}^{z}\{\sigma_n < \xi_0; S(\sigma_n) = y\} \\
&\leq C\left(\mathbf{P}^{z}\{\xi_0 < \sigma_n\} + \sum_{y\in\partial B_n} \mathbf{P}^{y}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\}\mathbf{P}^{z}\{\sigma_n < \xi_0; S(\sigma_n) = y\}\right) \\
&= C\mathbf{P}^{z}\{\xi_0 < \sigma_D \wedge \sigma_{2n}\}.
\end{aligned}$$

This proves (4.4).

The proof of (4.5) is simpler. By Proposition 2.4 part 2,

$$\begin{aligned}
\mathbf{P}^{z}\{\sigma_D < \xi_l\} &\geq \mathbf{P}^{z}\{\sigma_D < \xi_l \wedge \sigma_{2n}\} + \mathbf{P}^{z}\{\sigma_{2n} < \xi_l \wedge \sigma_D\} \min_{y\in\partial B_{2n}} \mathbf{P}^{y}\{\sigma_D < \xi_n\} \\
&\geq \mathbf{P}^{z}\{\sigma_D < \xi_l \wedge \sigma_{2n}\} + c\mathbf{P}^{z}\{\sigma_{2n} < \xi_l \wedge \sigma_D\} \\
&\geq c\left(\mathbf{P}^{z}\{\sigma_D < \xi_l \wedge \sigma_{2n}\} + \mathbf{P}^{z}\{\sigma_{2n} < \xi_l \wedge \sigma_D\}\right) \\
&= c\mathbf{P}^{z}\{\sigma_D \wedge \sigma_{2n} < \xi_l\}.
\end{aligned}$$

□

Lemma 4.3. *There exists $C < \infty$ such that the following holds. Suppose $D \subset \mathbb{Z}^2$ is simply connected, $w \in D$ and $\mathrm{dist}(w, D^c) = n$. Then for any $z \in B_{n/2}(w)$,*

$$G_D(w,z) \leq CG_{B_{2n}(w)\cap D}(w,z). \tag{4.8}$$

Proof. This follows immediately from Lemma 4.2 and the fact that

$$G_D(w,z) = \mathbf{P}^{z}\{\xi_w < \sigma_D\}\mathbf{P}^{w}\{\sigma_D < \xi_w\}^{-1},$$

and

$$G_{B_{2n}(w)\cap D}(w,z) = \mathbf{P}^{z}\left\{\xi_w < \sigma_D \wedge \sigma_{B_{2n}(w)}\right\}\mathbf{P}^{w}\left\{\sigma_D \wedge \sigma_{B_{2n}(w)} < \xi_w\right\}^{-1}.$$

□

Given $D \subset \mathbb{Z}^2$, let $D_+ = \{z \in D : \mathrm{Re}(z) > 0\}$ and $D_- = \{z \in D : \mathrm{Re}(z) < 0\}$. If $z = (z_1, z_2) \in \mathbb{Z}^2$, then we let $\overline{z} = (-z_1, z_2)$ be the reflection of z with respect to the y-axis ℓ, and $\overline{D} = \{\overline{z} : z \in D\}$ be the reflection of the set D.

Lemma 4.4. *Suppose that $K \subset D \subset \mathbb{Z}^2$ are such that $D_+ \subset \overline{D_-}$ and $K_+ \subset \overline{K_-}$. Then for all $z \in D_-$,*

$$\mathbf{P}^{z}\{\sigma_D < \xi_K\} \leq \mathbf{P}^{\overline{z}}\{\sigma_D < \xi_K\}.$$

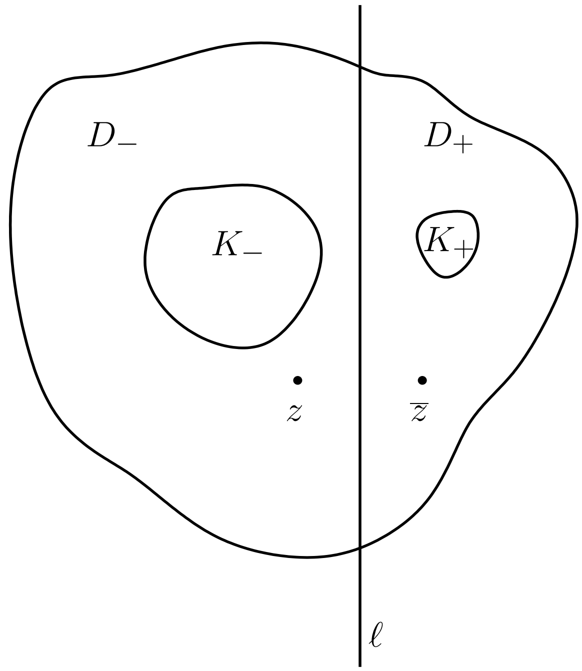

Figure 2: The setup for Lemma 4.4.

Proof. The proof uses a simple reflection argument. For a random walk started at $z \in D_-$ to escape D before hitting K, either it escapes D before hitting K while staying to the left of ℓ or it hits ℓ before hitting K and then escapes D before hitting K. In the first case, the reflected random walk path will be a random path starting at $\overline{z}$, escaping D before hitting K. In the second case, the reflection of the path up to the first time it hits ℓ will avoid K and hit ℓ at the same point. By the Markov property, the distribution of the paths after this point will be the same.

More precisely, using the fact that the reflection of a simple random walk across ℓ is again a simple random walk, it follows that for $z \in D_-$,

$$\mathbf{P}^z \left\{ \sigma_D < \xi_K \right\} = \mathbf{P}^{\overline{z}} \left\{ \sigma_{\overline{D}} < \xi_{\overline{K}} \right\}.$$

However, since $D_+ \subset \overline{D_-}$ and $K_+ \subset \overline{K_-}$,

$$\begin{aligned} \mathbf{P}^{\overline{z}} \left\{ \sigma_{\overline{D}} < \xi_{\overline{K}} \right\} &= \sum_{x \in \partial D_+} \mathbf{P}^x \left\{ \sigma_{\overline{D}} < \xi_{\overline{K}} \right\} \mathbf{P}^{\overline{z}} \left\{ \sigma_{D_+} < \xi_{\overline{K_-}} \wedge \xi_\ell ; S(\sigma_{D+}) = x \right\} \\ &\quad + \sum_{y \in \ell} \mathbf{P}^y \left\{ \sigma_{\overline{D}} < \xi_{\overline{K}} \right\} \mathbf{P}^{\overline{z}} \left\{ \xi_\ell < \xi_{\overline{K_-}} \wedge \sigma_{D_+} ; S(\xi_\ell) = y \right\} \\ &\leq \mathbf{P}^{\overline{z}} \left\{ \sigma_{D_+} < \xi_{K_+} \wedge \xi_\ell \right\} \\ &\quad + \sum_{y \in \ell} \mathbf{P}^y \left\{ \sigma_D < \xi_K \right\} \mathbf{P}^{\overline{z}} \left\{ \xi_\ell < \xi_{K_+} \wedge \sigma_{D_+} ; S(\xi_\ell) = y \right\} \\ &= \mathbf{P}^{\overline{z}} \left\{ \sigma_D < \xi_K \right\}. \end{aligned}$$

□

Corollary 4.5. *There exists $C < \infty$ such that the following holds. Suppose that m, n, N, K and x are as in Definition 1.4. Then for all $z \in A_n(x)$,*

$$\max_{w \in \partial B_{n/8}(x)} \mathbf{P}^w \{\sigma_N < \xi_K\} \leq C \mathbf{P}^z \{\sigma_N < \xi_K\}.$$

Proof. We apply Lemma 4.4 with $\ell = \{(m,k) : k \in \mathbb{Z}\}$ replacing the y-axis to conclude that

$$\max_{w \in \partial B_{n/8}(x)} \mathbf{P}^w \{\sigma_N < \xi_K\} = \max_{\substack{w \in \partial B_{n/8}(x) \\ \mathrm{Re}(w) \geq m}} \mathbf{P}^w \{\sigma_N < \xi_K\}.$$

If $x = (m, x_2)$, let

$$D_n(x) = \{(w_1, w_2) \in \mathbb{Z}^2 : n/16 \leq w_1 - m \leq n/8, |w_2 - x_2| \leq n/8\}.$$

Then by applying Lemma 4.4 again, this time with $\ell = \{(m + n/16, k) : k \in \mathbb{Z}\}$,

$$\max_{\substack{w \in \partial B_{n/8}(x) \\ \mathrm{Re}(w) \geq m}} \mathbf{P}^w \{\sigma_N < \xi_K\} = \max_{w \in D_n(x)} \mathbf{P}^w \{\sigma_N < \xi_K\},$$

However, by the discrete Harnack inequality, there exists $C < \infty$ such that for all $z \in A_n(x)$ and all $w \in D_n(x)$,

$$\mathbf{P}^w \{\sigma_N < \xi_K\} \leq C \mathbf{P}^z \{\sigma_N > \xi_K\}.$$

□

Lemma 4.6. *There exists $C < \infty$ such that the following holds. Suppose that m, n, N, K, x and X are as in Definition 1.4. Then for any $z \in A_n(x)$,*

$$C^{-1} \leq G_N^X(x,z) \leq C \ln \frac{N}{n}.$$

Proof. By Lemma 2.1,

$$\begin{aligned} G_N^X(x,z) &= G_{B_N \setminus K}(x,z) \frac{\mathbf{P}^z \{\sigma_N < \xi_K\}}{\mathbf{P}^x \{\sigma_N < \xi_K\}} \\ &= G_{B_N \setminus K}(z,z) \frac{\mathbf{P}^x \{\xi_z < \sigma_N \wedge \xi_K\} \mathbf{P}^z \{\sigma_N < \xi_K\}}{\mathbf{P}^x \{\sigma_N < \xi_K\}}. \end{aligned} \tag{4.9}$$

To begin with,

$$\ln n \asymp G_{B_{n/8}(z)}(z,z) \leq G_{B_N \setminus K}(z,z) \leq G_{B_{2N}(z)}(z,z) \asymp \ln N. \tag{4.10}$$

Next,

$$\begin{aligned} &\mathbf{P}^x \{\xi_z < \sigma_N \wedge \xi_K\} \\ &= \sum_{y \in \partial_i B_{n/8}(z)} \mathbf{P}^y \{\xi_z < \sigma_N \wedge \xi_K\} \mathbf{P}^x \left\{ S(\xi_{B_{n/8}(z)}) = y; \xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K \right\}. \end{aligned}$$

Furthermore, for any $y \in \partial_i B_{n/8}(z)$,

$$\mathbf{P}^y \{\xi_z < \sigma_N \wedge \xi_K\} \leq \mathbf{P}^y \left\{ \xi_z < \sigma_{B_{2N}(z)} \right\} \leq C \frac{\ln(N/n)}{\ln N},$$

and

$$\mathbf{P}^y\left\{\xi_z < \sigma_N \wedge \xi_K\right\} \ge \mathbf{P}^y\left\{\xi_z < \sigma_{B_{n/4}(z)}\right\} \ge \frac{c}{\ln n}.$$

Thus,

$$\frac{c}{\ln n} \le \frac{\mathbf{P}^x\left\{\xi_z < \sigma_N \wedge \xi_K\right\}}{\mathbf{P}^x\left\{\xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K\right\}} \le C\frac{\ln(N/n)}{\ln N}. \tag{4.11}$$

Next, on the one hand,

$$\mathbf{P}^x\left\{\sigma_N < \xi_K\right\} \ge \sum_{y\in\partial_i B_{n/8}(z)} \mathbf{P}^y\left\{\sigma_N < \xi_K\right\}\mathbf{P}^x\left\{S(\xi_{B_{n/8}(z)}) = y; \xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K\right\}.$$

By the discrete Harnack inequality, there exists C such that for any $y \in \partial_i B_{n/8}(z)$,

$$\mathbf{P}^z\left\{\sigma_N < \xi_K\right\} \le C\mathbf{P}^y\left\{\sigma_N < \xi_K\right\}.$$

Therefore,

$$\mathbf{P}^x\left\{\sigma_N < \xi_K\right\} \ge c\mathbf{P}^z\left\{\sigma_N < \xi_K\right\}\mathbf{P}^x\left\{\xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K\right\}.$$

On the other hand,

$$\mathbf{P}^x\left\{\sigma_N < \xi_K\right\} = \sum_{w\in\partial B_{n/8}(x)} \mathbf{P}^w\left\{\sigma_N < \xi_K\right\}\mathbf{P}^x\left\{S(\sigma_{B_{n/8}(x)}) = w; \sigma_{B_{n/8}(x)} < \xi_K\right\}.$$

By Corollary 4.5 for any $w \in \partial B_{n/8}(x)$,

$$\mathbf{P}^w\left\{\sigma_N < \xi_K\right\} \le C\mathbf{P}^z\left\{\sigma_N < \xi_K\right\}.$$

Therefore,

$$\mathbf{P}^x\left\{\sigma_N < \xi_K\right\} \le C\mathbf{P}^z\left\{\sigma_N < \xi_K\right\}\mathbf{P}^x\left\{\sigma_{B_{n/8}(x)} < \xi_K\right\}.$$

Finally, by Proposition 2.3,

$$\begin{aligned}\mathbf{P}^x\left\{\sigma_{B_{n/8}(x)} < \xi_K\right\} &\le C\mathbf{P}^x\left\{\sigma_{B_{n/8}(x)} < \xi_K; \left|\arg(S(\sigma_{B_{n/8}(x)}) - x)\right| \le \frac{\pi}{4}\right\}\\ &\le C\mathbf{P}^x\left\{\xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K\right\}.\end{aligned}$$

Thus,

$$\mathbf{P}^x\left\{\sigma_N < \xi_K\right\} \asymp \mathbf{P}^z\left\{\sigma_N < \xi_K\right\}\mathbf{P}^x\left\{\xi_{B_{n/8}(z)} < \sigma_N \wedge \xi_K\right\}. \tag{4.12}$$

The result then follows by combining (4.9), (4.10), (4.11) and (4.12).

□

5 Exponential moments for M_D and $\widehat{M}_D$

To reduce the size of our expressions, we use the following notation. For this section only, we will use the symbol $\not\pitchfork$ to denote the disjoint intersection relation. Thus, if K_1 and K_2 are two subsets of $\mathbb{Z}^2$, we will write $K_1 \not\pitchfork K_2$ to mean $K_1 \cap K_2 = \emptyset$.

Definition 5.1. *Suppose that $z_0, z_1, \ldots, z_k$ are distinct points in a domain $D \subset \mathbb{Z}^2$, and X is a Markov chain on $\mathbb{Z}^2$ with $\mathbf{P}^{z_0}\left\{\sigma_D^X < \infty\right\} = 1$. Then we let $E^X_{z_0,\ldots,z_k}$ be the event that $\mathrm{L}(X^{z_0}[0, \sigma_D])$ visits $z_1, z_2, \ldots, z_k$ in order.*

Proposition 5.2. *Suppose that $z_0, z_1, \ldots, z_k$ are distinct points in a domain $D \subset \mathbb{Z}^2$, and X is a Markov chain on $\mathbb{Z}^2$ with $\mathbf{P}^{z_0}\left\{\sigma_D^X < \infty\right\} = 1$. Define z_{k+1} to be ∂D and for $i = 0, \ldots, k$, let X^i be independent versions of X started at z_i and Y^i be X^i conditioned on the event $\{\xi_{z_{i+1}}^{X_i} \leq \sigma_D^{X_i}\}$. Let $\tau^i = \max\{l \leq \sigma_D^{Y^i} : Y_l^i = z_{i+1}\}$. Then,*

$$\mathbf{P}\left(E^X_{z_0,\ldots,z_k}\right) = \prod_{i=1}^k G_D^X(z_{i-1}, z_i)\mathbf{P}\left(\bigcap_{i=1}^k \left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]) \not\cap \bigcup_{j=i}^k Y^j[1,\tau^j]\right\}\right).$$

Proof. We will write the exit times $\sigma_D^{X^j}$ as σ_D^j and the hitting times $\xi_{z_i}^{X^j}$ as ξ_i^j, $i, j = 0, \ldots, k$. For $i, j = 0, \ldots, k$, we also let

$$T_i^j = \begin{cases} \max\{l \leq \sigma_D^j : X_l^j = z_i\} & \text{if } \xi_i^j < \sigma_D^j; \\ \sigma_D^j & \text{if } \sigma_D^j \leq \xi_i^j. \end{cases}$$

For $i = 0, \ldots, k-1$, let

$$F_i = \{T_{i+1}^i < \ldots < T_k^i < \sigma_D^i\},$$

and for $i = 0, \ldots, k-2$, let

$$G_i = \bigcap_{j=i+2}^k \{\mathrm{L}(X^i[T_{j-1}^i, T_j^i]) \not\cap X^i(T_j^i, \sigma_D^i]\}.$$

Then by the definition of the loop-erasing procedure,

$$\mathbf{P}\left(E^X_{z_1,\ldots,z_k}\right) = \mathbf{P}\left\{F_0; \mathrm{L}(X^0[0,T_1^0] \not\cap X^0(T_1^0, \sigma_D^0]; G_0\right\}. \tag{5.1}$$

Conditioned on $\{T_1^0 < \sigma_D^0\}$, $X^0[1,T_1^0]$ and $X^0[T_1^0, \sigma_D^0]$ are independent. $X^0[1,T_1^0]$ has the same distribution as $Y^0[0,\tau^0]$ and $X^0[T_1^0, \sigma_D^0]$ has the same distribution as X^1 conditioned to leave D before returning to z_1.

The event $\{T_1^0 < \sigma_D^0\}$ is the same as $\{\xi_1^0 < \sigma_D^0\}$. Therefore,

$$\begin{aligned}
&\mathbf{P}\left(E^X_{z_0,\ldots,z_k}\right) \\
=&\mathbf{P}\left\{\xi_1^0 < \sigma_D^0\right\}\mathbf{P}\left\{F_1; \mathrm{L}(Y^0[0,\tau^0]) \not\cap X^1[1,\sigma_D^1]; \mathrm{L}(X^1[0,T_2^1]) \not\cap X^1(T_2^1,\sigma_D^1]; G_1 \mid \sigma_D^1 < \xi_1^1\right\} \\
=&\frac{\mathbf{P}\left\{\xi_1^0 < \sigma_D^0\right\}}{\mathbf{P}\left\{\sigma_D^1 < \xi_1^1\right\}}\mathbf{P}\left\{F_1; \mathrm{L}(Y^0[0,\tau^0]) \not\cap X^1[1,\sigma_D^1]; \mathrm{L}(X^1[0,T_2^1]) \not\cap X^1(T_2^1,\sigma_D^1]; G_1\right\} \\
=&G_D^X(z_0,z_1) \\
&\times \mathbf{P}\left\{F_1; \mathrm{L}(Y^0[0,\tau^0]) \not\cap \left(X^1[1,T_2^1] \cup X^1(T_2^1,\sigma_D^1]\right); \mathrm{L}(X^1[0,T_2^1]) \not\cap X^1(T_2^1,\sigma_D^1]; G_1\right\}.
\end{aligned}$$

By repeating the previous argument $k-1$ times with X^1 through X^{k-1}, one obtains the desired result. □

Suppose now that $D' \subset D$ and let β be $\mathrm{L}(X^{z_0}[0,\sigma_D])$ from z_0 up to the first exit time of D'. It is possible to generalize the previous formula to the probability that β hits $z_1, \ldots, z_k$ in order. However, we will only require this for the case where $k = 1$ and therefore to avoid introducing any new notation, we will only state the result in this case.

Lemma 5.3. *Suppose that $D' \subset D$, z and w are distinct points in D', and X is a Markov chain started at w. Suppose further that $\mathbf{P}^w\{\sigma_D < \infty\} = 1$. Let Y be X conditioned to hit z before leaving D and let τ be the last time that Y visits z before leaving D. Then if β is $\mathrm{L}(X[0,\sigma_D])$ from w up to the first exit time of D',*

$$\mathbf{P}\{z \in \beta\} = G_D^X(w,z)\mathbf{P}\left\{\mathrm{L}(Y[0,\tau]) \not\cap X^z[1,\sigma_D]; \mathrm{L}(Y[0,\tau]) \subset D'\right\}.$$

Proof. As in the proof of Proposition 5.2, let

$$T_z = \begin{cases} \max\{l \le \sigma_D^X : X_l = z\} & \text{if } \xi_z^X < \sigma_D^X; \\ \sigma_D^X & \text{if } \sigma_D^X \le \xi_z^X. \end{cases}$$

Then,

$$\mathbf{P}\{z \in \beta\} = \mathbf{P}\left\{T_z < \sigma_D^X; \mathrm{L}(X[0,T_z]) \not\cap X[T_z+1,\sigma_D]; \mathrm{L}(X[0,T_z]) \subset D'\right\}.$$

The proof is then identical to that of Proposition 5.2. □

Definition 5.4. *Suppose that $z_0, z_1, \ldots, z_k$ are any points (not necessarily distinct) in a domain $D \subsetneq \mathbb{Z}^2$ and let $\mathbf{z} = (z_0, \ldots, z_k)$. Then we define z_{k+1} to be ∂D, let $d(z_i) = \mathrm{dist}(z_i, D^c)$, and let*

$$r_i^{\mathbf{z}} = d(z_i) \wedge |z_i - z_{i-1}| \wedge |z_i - z_{i+1}| \quad i = 1, 2, \ldots, k.$$

In addition, if π is an element of the symmetric group $\mathfrak{S}_k$ on $\{1,\ldots,k\}$, then we let $\pi(0) = 0$ and $\pi(\mathbf{z}) = (z_0, z_{\pi(1)}, \ldots, z_{\pi(k)})$.

Proposition 5.5. *There exists $C < \infty$ such that the following holds. Suppose that either*

1. *$z_0, z_1, \ldots, z_k$ are any points in a domain $D \subsetneq \mathbb{Z}^2$ and X is a random walk S started at z_0, or*
2. *m, n, N, K, x and X are as in Definition 1.4, $z_0 = x$, $D = B_N$ and $z_1, \ldots, z_k$ are in $A_n(x)$.*

Then, letting $\mathbf{z} = (z_0, \ldots, z_k)$ and $r_i^{\mathbf{z}}$ be as in Definition 5.4,

$$\mathbf{P}\{z_1, \ldots, z_k \in \mathrm{L}(X[0,\sigma_D])\} \le C^k \sum_{\pi \in \mathfrak{S}_k} \prod_{i=1}^k G_D^X(z_{\pi(i-1)}, z_{\pi(i)}) \,\mathrm{Es}\left(r_{\pi(i)}^{\pi(\mathbf{z})}\right). \tag{5.2}$$

Proof. The proofs of the two cases are almost identical and we will prove them both at the same time.

Suppose first that $z_0, \ldots, z_k$ **are distinct.** Recall the definition of $E^X_{z_0,\ldots,z_k}$ from Definition 5.1. Then,

$$\mathbf{P}\{z_1, \ldots, z_k \in \mathrm{L}(X[0,\sigma_D])\} = \sum_{\pi \in \mathfrak{S}_k} E^X_{z_0, z_{\pi(1)}, \ldots, z_{\pi(k)}}.$$

Therefore, if we let $Y^0, \ldots, Y^k$ be as in Proposition 5.2, then it suffices to show that

$$\mathbf{P}\left(\bigcap_{i=1}^k \left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]) \not\cap \bigcup_{j=i}^k Y^j[1,\tau^j]\right\}\right) \le C^k \prod_{i=1}^k \mathrm{Es}\left(r_i^{\mathbf{z}}\right).$$

For $i = 1, \ldots, k$, let $B_i = B(z_i; r_i^{\mathbf{z}}/4)$. Then,

$$\mathbf{P}\left(\bigcap_{i=1}^k \left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]) \not\cap \bigcup_{j=i}^k Y^j[1,\tau^j]\right\}\right) \le \mathbf{P}\left(\bigcap_{i=1}^k \left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]) \not\cap Y^i[1,\tau^i]\right\}\right).$$

Let $T : \Theta \to \Theta$ be the bijection given in Lemma 3.1. For all $\lambda \in \Theta$, $p^X(T(\lambda)) = p^X(\lambda)$ and $T\lambda$ visits the same points as λ. Thus,

$$\begin{aligned}\mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}])\not\cap Y^i[1,\tau^i]\right\}\right) &= \mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\mathrm{L}(T\circ Y^{i-1}[0,\tau^{i-1}])\not\cap(T\circ Y^i[1,\tau^i])\right\}\right)\\ &= \mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]^R)\not\cap Y^i[1,\tau^i]\right\}\right).\end{aligned}$$

For $i = 1,\ldots,k$, let β^i be the restriction of $\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]^R)$ from z_i to the first exit of B_i. Then,

$$\mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\mathrm{L}(Y^{i-1}[0,\tau^{i-1}]^R)\not\cap Y^i[1,\tau^i]\right\}\right) \le \mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\beta^i\not\cap Y^i[1,\sigma_{B_i}]\right\}\right).$$

Furthermore, by the domain Markov property (Lemma 3.2), conditioned on $\beta^i = [\beta^i_0,\ldots,\beta^i_m]$, $Y^{i-1}[0,\tau^{i-1}]$ is, in case 1 a random walk started at z_{i-1} and conditioned to hit β^i_m before $\partial D \cup \{\beta^i_0,\ldots,\beta^i_{m-1}\}$ and in case 2 a random walk started at z_{i-1} and conditioned to hit β^i_m before $K\cup\partial D\cup\{\beta^i_0,\ldots,\beta^i_{m-1}\}$. In either case, by the Harnack principle, $Y^{i-1}[0,\sigma_{B_{i-1}}]$ and β^i are independent "up to constants" and thus

$$\mathbf{P}\left(\bigcap_{i=1}^{k}\left\{\beta^i\not\cap Y^i[1,\sigma_{B_i}]\right\}\right) \le C^k\prod_{i=1}^{k}\mathbf{P}\left\{\beta^i\not\cap Y^i[1,\sigma_{B_i}]\right\}.$$

By another application of the Harnack principle, $Y^i[0,\sigma_{B_i}]$ has the same distribution up to constants as a random walk started at z_i and stopped at its first exit of B_i. Furthermore, by Corollary 3.4, β^i has the same distribution up to constants as an infinite LERW started at z_i and stopped at the first exit of B_i. Therefore, for $i = 1,\ldots,k$,

$$\mathbf{P}\left\{\beta^i\not\cap Y^i[1,\sigma_{B_i}]\right\} \le C\widehat{\mathrm{Es}}(r_i^{\mathbf{z}}/4).$$

Finally, by Theorem 3.9 part 1 and Lemma 3.10, $\widehat{\mathrm{Es}}(r_i^{\mathbf{z}}/4) \le C\,\mathrm{Es}(r_i^{\mathbf{z}})$.

Suppose now that $z_0,\ldots,z_k$ are any points in D. Let

$$p(\mathbf{z}) = \prod_{i=1}^{k} G_D^X(z_{i-1},z_i)\,\mathrm{Es}\left(r_i^{\mathbf{z}}\right).$$

We will establish (5.2) by induction on k. We have already proved that (5.2) holds for $k = 1$. Now suppose that (5.2) holds for $k-1$ and suppose that $z_0,\ldots,z_k$ are not distinct. Since (5.2) involves a sum over all possible permutations of the entries of $\mathbf{z}$, we may assume without loss of generality that $z_j = z_{j+1}$ for some j. Let $\mathbf{z}^{(\mathbf{j})}$ be $\mathbf{z}$ with the jth entry deleted, and indexed by $\{0,\ldots,k\}\setminus\{j\}$ (so that $z_i = z_i^{(j)}$ for all $i \neq j$). Then since $r_i^{\mathbf{z}^{(\mathbf{j})}} = r_i^{\mathbf{z}}$ for all $i \neq j$, $i \neq j+1$,

$$p(\mathbf{z}) = p(\mathbf{z}^{(\mathbf{j})})\cdot G_D^X(z_j,z_j)\,\mathrm{Es}\left(r_j^{\mathbf{z}}\right)\mathrm{Es}\left(r_{j+1}^{\mathbf{z}}\right)\mathrm{Es}\left(r_{j+1}^{\mathbf{z}^{(\mathbf{j})}}\right)^{-1}.$$

Since $z_j = z_{j+1}$, $r_j^{\mathbf{z}} = r_{j+1}^{\mathbf{z}} = 0$ and therefore, $\mathrm{Es}(r_j^{\mathbf{z}}) = \mathrm{Es}(r_{j+1}^{\mathbf{z}}) = 1$. Also, $G_D^X(z_j,z_j) \ge 1$. Therefore, $p(\mathbf{z}) \ge p(\mathbf{z}^{(\mathbf{j})})$.

Now let $\mathfrak{S}_A$ be the symmetric group on the set $A = \{1,\ldots,k\}\setminus\{j\}$. Then there is an obvious bijection between $\mathfrak{S}_A$ and

$$\mathfrak{B} = \{\pi \in \mathfrak{S}_k : \pi^{-1}(j+1) = \pi^{-1}(j)+1\}.$$

Therefore, by our induction hypothesis,

$$\begin{aligned}\mathbf{P}\{z_1,\ldots,z_k\in \mathrm{L}(X[0,\sigma_D])\} &\le C^{k-1}\sum_{\pi\in\mathfrak{S}_A} p(\pi(\mathbf{z}^{(\mathbf{j})}))\\ &\le C^k\sum_{\pi\in\mathfrak{S}_A} p(\pi(\mathbf{z}^{(\mathbf{j})}))\\ &\le C^k\sum_{\pi\in\mathfrak{B}} p(\pi(\mathbf{z}))\\ &\le C^k\sum_{\pi\in\mathfrak{S}_k} p(\pi(\mathbf{z})).\end{aligned}$$

□

Theorem 5.6. *There exists $C<\infty$ such that the following holds. Suppose that D is a simply connected subset of $\mathbb{Z}^2$ containing 0, and $D'\subset D$ is such that for all $z\in D'$, $\mathrm{dist}(z,D^c)\le n$. Then for all $k=1,2,\ldots$,*

$$\mathbf{E}\left[M_D^k;\mathrm{L}(S[0,\sigma_D])\subset D'\right]\le C^k k!(n^2\,\mathrm{Es}(n))^k.$$

In particular, if D is simply connected, contains 0 and for all $z\in D$, $\mathrm{dist}(z,D^c)\le n$ then

$$\mathbf{E}\left[M_D^k\right]\le C^k k!(n^2\,\mathrm{Es}(n))^k.$$

Proof. Let $\mathfrak{S}_k$ denote the symmetric group on k elements and recall the definition of $r_i^{\mathbf{z}}$ given in Definition 5.4 (here $z_0=0$). Then by Proposition 5.5,

$$\begin{aligned}\mathbf{E}\left[M_D^k;\mathrm{L}(S[0,\sigma_D])\subset D'\right] &= \mathbf{E}\left[\left(\sum_{z\in D}\mathbb{1}\{z\in \mathrm{L}(S[0,\sigma_D]);\mathrm{L}(S[0,\sigma_D])\subset D'\}\right)^k\right]\\ &=\sum_{z_1\in D}\cdots\sum_{z_k\in D}\mathbf{P}\left\{z_1,\ldots,z_k\in\mathrm{L}(S[0,\sigma_D]);\mathrm{L}(S[0,\sigma_D])\subset D'\right\}\\ &\le \sum_{z_1\in D'}\cdots\sum_{z_k\in D'}\mathbf{P}\{z_1,\ldots,z_k\in\mathrm{L}(S[0,\sigma_D])\}\\ &\le C^k\sum_{\pi\in\mathfrak{S}_k}\sum_{z_1\in D'}\cdots\sum_{z_k\in D'}\prod_{i=1}^k G_D(z_{\pi(i-1)},z_{\pi(i)})\,\mathrm{Es}(r_{\pi(i)}^{\pi(\mathbf{z})})\\ &= C^k k!\sum_{z_1\in D'}\cdots\sum_{z_k\in D'}\prod_{i=1}^k G_D(z_{i-1},z_i)\,\mathrm{Es}(r_i^{\mathbf{z}}).\end{aligned}$$

Therefore, it suffices to show that

$$\sum_{z_1\in D'}\cdots\sum_{z_k\in D'}\prod_{i=1}^k G_D(z_{i-1},z_i)\,\mathrm{Es}(r_i^{\mathbf{z}})\le C^k(n^2\,\mathrm{Es}(n))^k. \tag{5.3}$$

Let $f_i=G_D(z_{i-1},z_i)\,\mathrm{Es}(r_i^{\mathbf{z}})$ and $F_j=\prod_{i=1}^j f_i$. Then if $d(z)=\mathrm{dist}(z,D^c)$,

$$\prod_{i=1}^k G_D(z_{i-1},z_i)\,\mathrm{Es}(r_i^{\mathbf{z}})=F_{k-1}G_D(z_{k-1},z_k)\left(\mathrm{Es}(|z_k-z_{k-1}|\wedge d(z_k))\right). \tag{5.4}$$

Then since only the terms f_k and f_{k-1} involve z_k, and $\mathrm{Es}(a \wedge b) \le \mathrm{Es}(a) + \mathrm{Es}(b)$, we have

$$\begin{aligned}
&\sum_{z_1 \in D'} \cdots \sum_{z_k \in D'} \prod_{i=1}^{k} G_D(z_{i-1}, z_i)\, \mathrm{Es}(r_i^{\mathbf{z}}) \\
&\le \sum_{z_1 \in D'} \cdots \sum_{z_{k-1} \in D'} F_{k-2} G_D(z_{k-2}, z_{k-1}) \\
&\quad \times \sum_{z_k \in D'} G_D(z_{k-1}, z_k)(\mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1})) + \mathrm{Es}(|z_{k-1} - z_k|)) \\
&\quad \times (\mathrm{Es}(|z_k - z_{k-1}|) + \mathrm{Es}(d(z_k))).
\end{aligned}$$

Multiplying out the final terms in the expression above, we need to bound the following sums:

$$S_1 = \mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1})) \sum_{z_k} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|), \tag{5.5}$$

$$S_2 = \mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1})) \sum_{z_k} G_D(z_{k-1}, z_k)\, \mathrm{Es}(d(z_k)), \tag{5.6}$$

$$S_3 = \sum_{z_k} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|)^2, \tag{5.7}$$

$$S_4 = \sum_{z_k} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|)\, \mathrm{Es}(d(z_k)). \tag{5.8}$$

Since $2ab \le a^2 + b^2$ we can bound S_4 by

$$S_4 \le S_3 + \sum_{z_k} G_D(z_{k-1}, z_k)\, \mathrm{Es}(d(z_k))^2 = S_3 + S_5. \tag{5.9}$$

We first consider S_3. Let $D_1 = D \cap B_{n/2}(z_{k-1})$ and $D_2 = D' \setminus D_1$. Then

$$S_3 \le \sum_{z_k \in D_1} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|)^2 + \sum_{z_k \in D_2} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|)^2.$$

Since $d(z_{k-1}) \le n$, then by Lemma 4.3, for all $z_k \in D_1$,

$$G_D(z_{k-1}, z_k) \le C G_{B_{2n}(z_{k-1})}(z_{k-1}, z_k) \le C \ln\left(\frac{2n}{|z_{k-1} - z_k|}\right).$$

So,

$$\begin{aligned}
\sum_{z_k \in D_1} G_D(z_{k-1}, z_k)\, \mathrm{Es}(|z_{k-1} - z_k|)^2 &\le \sum_{z_k \in D_1} \ln\left(\frac{2n}{|z_{k-1} - w|}\right) \mathrm{Es}(|z_{k-1} - z_k|)^2 \\
&\le C \sum_{z_k \in B_{2n}(z_{k-1})} \ln\left(\frac{2n}{|z_{k-1} - z_k|}\right) \mathrm{Es}(|z_{k-1} - z_k|)^2 \\
&\le C \sum_{j=1}^{2n} j \ln\left(\frac{2n}{j}\right) \mathrm{Es}(j)^2 \\
&\le C n^2\, \mathrm{Es}(n)^2,
\end{aligned}$$

where the last inequality is justified by Corollary 3.14. Furthermore, for $z_k \in D_2$, $\mathrm{Es}(|z_{k-1} - z_k|)^2 \leq C\,\mathrm{Es}(n)^2$. Therefore by Lemma 4.1,

$$\sum_{z_k \in D_2} G_D(z_{k-1}, z_k)\,\mathrm{Es}(|z_{k-1} - z_k|)^2 \leq C\,\mathrm{Es}(n)^2 \sum_{z_k \in D'} G_D(z_{k-1}, z_k) \leq Cn^2\,\mathrm{Es}(n)^2.$$

Therefore, $S_3 \leq Cn^2\,\mathrm{Es}(n)^2$. Similarly we obtain

$$S_1 \leq C\,\mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1}))n^2\,\mathrm{Es}(n). \tag{5.10}$$

Let $D_j = \{z \in D : d(z) \leq j\}$ be as in Lemma 4.1. Then by first applying Lemma 4.1 and then Lemma 3.13,

$$\begin{aligned}
S_5 &\leq \sum_{j=0}^{\lceil \log_2 n \rceil} \sum_{z_k \in D_{2^j} \setminus D_{2^{j-1}}} G_D(z_{k-1}, z_k)\,\mathrm{Es}(d(z_k))^2 \\
&\leq C \sum_{j=0}^{\lceil \log_2 n \rceil} \mathrm{Es}(2^j)^2 \sum_{z_k \in D_{2^j} \setminus D_{2^{j-1}}} G_D(z_{k-1}, z_k) \\
&\leq \sum_{j=0}^{\lceil \log_2 n \rceil} \mathrm{Es}(2^j)^2 \sum_{z_k \in D_{2^j}} G_D(z_{k-1}, z_k) \\
&\leq C \sum_{j=0}^{\lceil \log_2 n \rceil} 2^{2j}\,\mathrm{Es}(2^j)^2 \\
&\leq C \sum_{j=0}^{\lceil \log_2 n \rceil} \big((2^j)^{3/4+\varepsilon}\,\mathrm{Es}(2^j)\big)^2 (2^j)^{1/2-2\varepsilon} \\
&\leq C\big(n^{3/4+\varepsilon}\,\mathrm{Es}(n)\big)^2 \sum_{j=1}^{\lceil \log_2 n \rceil} (2^j)^{1/2-2\varepsilon} \\
&\leq Cn^2\,\mathrm{Es}(n)^2.
\end{aligned}$$

A similar calculation gives

$$S_2 \leq C\,\mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1}))n^2\,\mathrm{Es}(n). \tag{5.11}$$

Combining these bounds gives

$$\begin{aligned}
&\sum_{z_1 \in D'} \cdots \sum_{z_k \in D'} \prod_{i=1}^{k} G_D(z_{i-1}, z_i)\,\mathrm{Es}(r_i^{\mathbf{z}}) \\
&\quad \leq Cn^2\,\mathrm{Es}(n) \sum_{z_1 \in D'} \cdots \sum_{z_{k-1} \in D'} F_{k-2} G_D(z_{k-2}, z_{k-1}) \Big(\mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1})) + \mathrm{Es}(n) \Big) \\
&\quad \leq Cn^2\,\mathrm{Es}(n) \sum_{z_1 \in D'} \cdots \sum_{z_{k-1} \in D'} F_{k-2} G(z_{k-2}, z_{k-1}) \Big(\mathrm{Es}(|z_{k-1} - z_{k-2}| \wedge d(z_{k-1})) \Big).
\end{aligned}$$

Since this is of the same form as (5.4), except with only $k - 1$ terms, iterating this argument gives (5.3). □

Proposition 5.7. *There exists $c > 0$ such that for all n and all simply connected $D \supset B_n$,*

$$\mathbf{E}\left[M_D\right] \geq cn^2 \operatorname{Es}(n).$$

Proof. By Lemma 3.13, $n^2 \operatorname{Es}(n)$ is increasing (up to a constant). Therefore, we may assume that n is the largest integer such that $B_n \subset D$. Let $A_n = \{z : n/4 \leq |z| \leq 3n/4, |\arg z| \leq \pi/4\}$ be as in Definition 1.4. Then since there are on the order of n^2 points in A_n, it suffices to show that for all $z \in A_n$,

$$\mathbf{P}\left\{z \in \mathrm{L}(S[0,\sigma_D])\right\} \geq c \operatorname{Es}(n). \tag{5.12}$$

By Proposition 5.2,

$$\mathbf{P}\left\{z \in \mathrm{L}(S[0,\sigma_D])\right\} = G_D(0,z)\mathbf{P}\left\{\mathrm{L}(Y[0,\tau]) \cap S^z[1,\sigma_D] = \emptyset\right\} \tag{5.13}$$

where Y is a random walk started at 0 conditioned to hit z before leaving D and $\tau = \max\{k < \sigma_D : Y_k = z\}$. By Lemma 3.1, $\mathrm{L}(Y[0,\tau])$ has the same distribution as $\mathrm{L}(Y[0,\tau]^R)$. Furthermore, if we let Z be a random walk started at z conditioned to hit 0 before leaving D, then $Y[0,\tau]^R$ has the same distribution as $Z[0,\xi_0]$. Therefore,

$$\mathbf{P}\left\{\mathrm{L}(Y[0,\tau]) \cap S^z[1,\sigma_D] = \emptyset\right\} = \mathbf{P}\left\{\mathrm{L}(Z[0,\xi_0]) \cap S^z[1,\sigma_D] = \emptyset\right\}.$$

Furthermore,

$$G_D(0,z) \geq G_n(0,z) \geq c.$$

Therefore, in order to show (5.12), it is sufficient to prove that

$$\mathbf{P}\left\{\mathrm{L}(Z[0,\xi_0]) \cap S^z[1,\sigma_D] = \emptyset\right\} \geq c \operatorname{Es}(n). \tag{5.14}$$

Let $B = B(z; n/8)$, and let β be the restriction of $\mathrm{L}(Z[0,\xi_0])$ from z up to the first time it leaves the ball B. Then,

$$\begin{aligned}&\mathbf{P}\left\{\mathrm{L}(Z[0,\xi_0]) \cap S^z[1,\sigma_D] = \emptyset\right\}\\ &\quad= \mathbf{P}\left\{\mathrm{L}(Z[0,\xi_0]) \cap S^z[1,\sigma_D] = \emptyset \mid \beta \cap S^z[1,\sigma_B] = \emptyset\right\} \mathbf{P}\left\{\beta \cap S^z[1,\sigma_B] = \emptyset\right\}.\end{aligned}$$

By Corollary 3.4, β has the same distribution "up to constants" as an infinite LERW started at z and stopped at the first exit of B. Therefore, by Theorem 3.9 part 1 and Lemma 3.11,

$$\mathbf{P}\left\{\beta \cap S^z[1,\sigma_B] = \emptyset\right\} \geq c\widehat{\operatorname{Es}}(n/8) \geq c \operatorname{Es}(n/8) \geq c \operatorname{Es}(n).$$

By the domain Markov property (Lemma 3.2), if we condition on β, the rest of $\mathrm{L}(Z[0,\xi_0])$ is obtained by running a random walk conditioned to hit 0 before $\beta \cup \partial B_n$ and then loop-erasing. Therefore, by the separation lemma (Proposition 3.6) and Proposition 2.3, there is a probability greater than $c > 0$ that this conditioned random walk reaches $\partial B_{n/16}$ without hitting $S^z[1,\sigma_n]$ or leaving $B_{7n/8}$.

Therefore, it remains to show that for all $v \in \partial B_{n/16}$,

$$\mathbf{P}^v\left\{\xi_0 < \sigma_{B_{n/8}} \,\middle|\, \xi_0 < \sigma_D\right\} \geq c, \tag{5.15}$$

and for all $w \in \partial B_n$,

$$\mathbf{P}^w\left\{\sigma_D < \xi_{B_{7n/8}}\right\} \geq c. \tag{5.16}$$

By Lemma 4.2,

$$\mathbf{P}^v\left\{\xi_0 < \sigma_D\right\} \leq C\mathbf{P}^v\left\{\xi_0 < \sigma_{2n}\right\},$$

and

$$\mathbf{P}^w\left\{\sigma_D < \xi_{7n/8}\right\} \geq c\mathbf{P}^w\left\{\sigma_{2n} < \xi_{7n/8}\right\}.$$

By Proposition 2.4 these imply (5.15) and (5.16), □

Theorem 5.8. *There exist $C_0, C_1 < \infty$ and $c_0, c_1 > 0$ such that the following holds. Suppose that D is a simply connected subset of $\mathbb{Z}^2$ containing 0, and $D' \subset D$ is such that for all $z \in D'$, $\mathrm{dist}(z, D^c) \le n$.*

1. *For all $k = 1, 2, \ldots$,*
$$\mathbf{E}\left[M_D^k; \mathrm{L}(S[0,\sigma_D]) \subset D'\right] \le (C_0)^k k! (\mathbf{E}\left[M_n\right])^k. \tag{5.17}$$

2. *There exists $c_0 > 0$ such that*
$$\mathbf{E}\left[e^{c_0 M_D / \mathbf{E}[M_n]}; \mathrm{L}(S[0,\sigma_D]) \subset D'\right] \le 2. \tag{5.18}$$

3. *For all $\lambda \ge 0$,*
$$\mathbf{P}\left\{M_D > \lambda \mathbf{E}\left[M_n\right]; \mathrm{L}(S[0,\sigma_D]) \subset D'\right\} \le 2e^{-c_0\lambda}. \tag{5.19}$$

4. *For all n and all $\lambda \ge 0$,*
$$\mathbf{P}\left\{\widehat{M}_n > \lambda \mathbf{E}\left[\widehat{M}_n\right]\right\} \le C_1 e^{-c_1\lambda}. \tag{5.20}$$

In particular, if D is a simply connected set containing 0 and for all $z \in D$, $\mathrm{dist}(z, D^c) \le n$, then one can omit the event $\{\mathrm{L}(S[0,\sigma_D]) \subset D'\}$ in (5.17), (5.18) and (5.19).

Proof. The first part follows immediately from Propositions 5.6 and 5.7.

To prove the second part, let $c_0 = 1/(2C_0)$. Then,

$$\mathbf{E}\left[e^{c_0 M_D/\mathbf{E}[M_n]};; \mathrm{L}(S[0,\sigma_D]) \subset D'\right] = \sum_{k=0}^{\infty} \frac{(c_0)^k \mathbf{E}\left[M_D^k;; \mathrm{L}(S[0,\sigma_D]) \subset D'\right]}{k! \mathbf{E}\left[M_n\right]^k} \le \sum_{k=0}^{\infty} 2^{-k} = 2.$$

The third part is then immediate by Markov's inequality.

To prove the last part, we first note that by Corollary 3.4,

$$\mathbf{P}\left\{\widehat{M}_n > \lambda \mathbf{E}\left[\widehat{M}_n\right]\right\} \le C\mathbf{P}\left\{M_{4n} > \lambda \mathbf{E}\left[\widehat{M}_n\right]\right\}.$$

By Proposition 6.2 (even though it appears later in this paper, its proof does not rely on this theorem), $\mathbf{E}\left[\widehat{M}_n\right] \le Cn^2 \,\mathrm{Es}(n)$. Using Lemma 3.13 and Proposition 5.7, this implies that $\mathbf{E}\left[\widehat{M}_n\right] \le C\mathbf{E}\left[M_{4n}\right]$, and therefore

$$\mathbf{P}\left\{M_{4n} > \lambda \mathbf{E}\left[\widehat{M}_n\right]\right\} \le C\mathbf{P}\left\{M_{4n} > c\lambda \mathbf{E}\left[M_{4n}\right]\right\} \le Ce^{-c \cdot c_0 \lambda} = C_1 e^{-c_1\lambda}.$$

□

6 Estimating the lower tail of M_D and $\widehat{M}_D$

Lemma 6.1. *There exists $c > 0$ such that the following holds. Suppose that m, n, N, K, x, X and α are as in Definition 1.4. Then for any $z \in A_n(x)$,*

$$\mathbf{P}\left\{z \in \alpha\right\} \ge c\left(\ln \frac{N}{n}\right)^{-3} \mathrm{Es}(n).$$

Proof. By Lemma 5.3, if Y is a random walk started at x conditioned to hit z before hitting K or leaving B_N, and τ is the last visit of z before leaving B_N, then

$$\mathbf{P}\left\{z \in \alpha\right\} = G_N^X(x,z)\mathbf{P}\left\{\mathrm{L}(Y[0,\tau]) \cap X^z[1,\sigma_N] = \emptyset; \mathrm{L}(Y[0,\tau]) \subset B_n(x)\right\}.$$

By Lemma 4.6, $G_N^X(x,z) \geq c$. Therefore, if we imitate the proof of Proposition 5.7 up to (5.15), it is sufficient to prove that for all $v \in \partial B(x;n/16)$, $|\arg(v-x)| \leq \pi/3$,

$$\mathbf{P}^v\left\{\xi_x < \sigma_{B(x;n/8)} \;\middle|\; \xi_x < \xi_K \wedge \sigma_N\right\} \geq c\left(\ln\frac{N}{n}\right)^{-2}, \tag{6.1}$$

and for all $w \in \partial B_n(x)$, $|\arg(w-x)| \leq \pi/3$,

$$\mathbf{P}^w\left\{\sigma_N < \xi_{B(x;7n/8)} \;\middle|\; \sigma_N < \xi_K\right\} \geq c\left(\ln\frac{N}{n}\right)^{-1}. \tag{6.2}$$

We first establish (6.1).

$$\mathbf{P}^v\left\{\xi_x < \sigma_{B(x;n/8)} \;\middle|\; \xi_x < \xi_K \wedge \sigma_N\right\} = \frac{\mathbf{P}^v\left\{\xi_x < \sigma_{B(x;n/8)} \wedge \xi_K\right\}}{\mathbf{P}^v\left\{\xi_x < \xi_K \wedge \sigma_N\right\}}$$

Let $K' = K \cup \{x\}$. By Lemma 2.2,

$$\mathbf{P}^v\left\{\xi_x < \sigma_{B(x,n/8)} \wedge \xi_K\right\} = \frac{G(v,v;B(x;n/8)\setminus K')}{G(v,v;\mathbb{Z}^2\setminus\{x\})}\frac{\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\}}{\mathbf{P}^x\left\{\xi_v < \xi_x\right\}},$$

and

$$\mathbf{P}^v\left\{\xi_x < \xi_K \wedge \sigma_N\right\} = \frac{G(v,v;B_N\setminus K')}{G(v,v;\mathbb{Z}^2\setminus\{x\})}\frac{\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\}}{\mathbf{P}^x\left\{\xi_v < \xi_x\right\}}.$$

Therefore,

$$\begin{aligned}&\mathbf{P}^v\left\{\xi_x < \sigma_{B(x;n/8)} \;\middle|\; \xi_x < \xi_K \wedge \sigma_N\right\} \\ &\quad = \frac{G(v,v;B(x;n/8)\setminus K')}{G(v,v;B_N\setminus K')}\frac{\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\}}{\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\}}.\end{aligned}$$

Since $|v-x| = n/16$,

$$G(v,v;B(x;n/8)\setminus K') \geq G(v,v;B(v;n/16)) \geq c\ln n.$$

Also,

$$G(v,v;B_N\setminus K') \leq G(v,v;B(v;2N)) \leq C\ln N.$$

Therefore,

$$\frac{G(v,v;B(x;n/8)\setminus K')}{G(v,v;B_N\setminus K')} \geq c\frac{\ln n}{\ln N} \geq c\left(\ln\frac{N}{n}\right)^{-1}.$$

To prove (6.1) it therefore suffices to show that

$$\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\} \leq C\ln\frac{N}{n}\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\}.$$

Indeed,

$$\begin{aligned}
&\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\} \\
&= \mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\} \\
&\quad + \sum_{y\in\partial B(x;n/8)} \mathbf{P}^y\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\} \mathbf{P}^x\left\{S(\sigma_{B(x;n/8)}) = y; \sigma_{B(x;n/8)} < \xi_{K'} \wedge \xi_v\right\} \\
&\leq \mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\} \\
&\quad + \sum_{y\in\partial B(x;n/8)} \mathbf{P}^y\left\{\xi_v < \sigma_{B(v;2N)}\right\} \mathbf{P}^x\left\{S(\sigma_{B(x;n/8)}) = y; \sigma_{B(x;n/8)} < \xi_{K'}\right\}.
\end{aligned}$$

For all $y \in \partial B(x;n/8)$, $|y-v| > n/16$, and thus,

$$\mathbf{P}^y\left\{\xi_v < \sigma_{B(v;2N)}\right\} \leq C\frac{\ln(N/n)}{\ln N}.$$

Therefore,

$$\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\} \leq \mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\} + C\frac{\ln(N/n)}{\ln N}\mathbf{P}^x\left\{\sigma_{B(x;n/8)} < \xi_{K'}\right\}.$$

However, by Proposition 2.3,

$$\begin{aligned}
\mathbf{P}^x\left\{\sigma_{B(x;n/8)} < \xi_{K'}\right\} &\leq \mathbf{P}^x\left\{\sigma_{B(x;n/16)} < \xi_{K'}\right\} \\
&\leq C\mathbf{P}^x\left\{\sigma_{B(x;n/16)} < \xi_{K'}; \left|\arg(S(\sigma_{B(x;n/16)}) - x)\right| \leq \frac{\pi}{4}\right\} \\
&\leq C\ln n\mathbf{P}^x\left\{\xi_v < \sigma_{B(x;n/8)} \wedge \xi_{K'}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_N\right\} &\leq \left(1 + C\frac{\ln(N/n)\ln n}{\ln N}\right)\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\} \\
&\leq C\ln\frac{N}{n}\mathbf{P}^x\left\{\xi_v < \xi_{K'} \wedge \sigma_{B(x;n/8)}\right\}.
\end{aligned}$$

We now prove (6.2).

$$\mathbf{P}^w\left\{\sigma_N < \xi_{B(x;7n/8)} \;\middle|\; \sigma_N < \xi_K\right\} = \frac{\mathbf{P}^w\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\}}{\mathbf{P}^w\left\{\sigma_N < \xi_K\right\}}.$$

Let $y_0 \in \partial B_n(x)$ be such that

$$\mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\} = \max_{y\in\partial B_n(x)} \mathbf{P}^y\left\{\sigma_N < \xi_K\right\}.$$

Then,

$$\begin{aligned}
&\mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\} \\
&= \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\} \\
&\quad + \sum_{u\in\partial_i B(x;7n/8)} \mathbf{P}^u\left\{\sigma_N < \xi_K\right\} \mathbf{P}^{y_0}\left\{S(\xi_{B(x;7n/8)}) = u; \xi_{B(x;7n/8)} < \xi_K \wedge \sigma_N\right\} \\
&\leq \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\} + \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\}\mathbf{P}^{y_0}\left\{\xi_{B(x;7n/8)} < \xi_K \wedge \sigma_N\right\} \\
&\leq \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\} + \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\}\mathbf{P}^{y_0}\left\{\xi_{B(x;7n/8)} < \sigma_{B(x;2N)}\right\}.
\end{aligned}$$

However, by Proposition 2.4,

$$\mathbf{P}^{y_0}\left\{\xi_{B(x;7n/8)} < \sigma_{B(x;2N)}\right\} \leq 1 - \frac{c}{\ln(N/n)},$$

and therefore

$$\mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\} \leq C \ln \frac{N}{n} \mathbf{P}^{y_0}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\}.$$

This establishes (6.2) for the special case where $w = y_0$. However, we can apply Lemma 4.4 twice, as in Corollary 4.5 to conclude that

$$\mathbf{P}^{w}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\} \geq c \max_{y \in \partial B_n(x)} \mathbf{P}^{y}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\}.$$

Therefore,

$$\frac{\mathbf{P}^{w}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\}}{\mathbf{P}^{w}\left\{\sigma_N < \xi_K\right\}} \geq c \frac{\mathbf{P}^{y_0}\left\{\sigma_N < \xi_K \wedge \xi_{B(x;7n/8)}\right\}}{\mathbf{P}^{y_0}\left\{\sigma_N < \xi_K\right\}} \geq c \left(\ln \frac{N}{n}\right)^{-1}.$$

□

Proposition 6.2.

1. *There exists $C < \infty$ such that for any m, n, N, K and x as in Definition 1.4,*

$$C^{-1}\left(\ln \frac{N}{n}\right)^{-3} n^2 \operatorname{Es}(n) \leq \mathbf{E}\left[M^K_{m,n,N,x}\right] \leq C \left(\ln \frac{N}{n}\right) n^2 \operatorname{Es}(n).$$

2.

$$\mathbf{E}\left[M_n\right] \asymp \mathbf{E}\left[\widehat{M}_n\right] \asymp n^2 \operatorname{Es}(n).$$

Proof. We first prove part 1. Let α be as in Definition 1.4. Then by Lemma 6.1,

$$\mathbf{E}\left[M^K_{m,n,N,x}\right] = \sum_{z \in A_n(x)} \mathbf{P}\left\{z \in \alpha\right\} \geq \sum_{z \in A_n(x)} c \left(\ln \frac{N}{n}\right)^{-3} \operatorname{Es}(n) \geq c \left(\ln \frac{N}{n}\right)^{-3} n^2 \operatorname{Es}(n).$$

To prove the other direction note that by Proposition 5.5, with $k = 1$, for any $z \in \alpha$,

$$\mathbf{P}\left\{z \in \alpha\right\} \leq \mathbf{P}\left\{z \in \mathrm{L}(X[0,\sigma_N])\right\} \leq C G^X_N(x,z) \operatorname{Es}(n).$$

By Lemma 4.6, $G^X_N(x,z) \leq C \ln(N/n)$ and therefore

$$\mathbf{E}\left[M^K_{m,n,N,x}\right] = \sum_{z \in A_n(x)} \mathbf{P}\left\{z \in \alpha\right\} \leq C \left(\ln \frac{N}{n}\right) n^2 \operatorname{Es}(n).$$

We now prove part 2. The fact that $\mathbf{E}\left[M_n\right] \asymp n^2 \operatorname{Es}(n)$ follows immediately from Propositions 5.6 and 5.7.

In order to show that $\mathbf{E}\left[\widehat{M}_n\right] \asymp n^2 \operatorname{Es}(n)$, let β be $\mathrm{L}(S[0,\sigma_{4n}])$ from 0 up to its first exit of the ball B_n. By Corollary 3.4, β has the same distribution up to constants as $\widehat{S}[0,\widehat{\sigma}_n]$ and thus it suffices to show that

$$\sum_{z \in B_n} \mathbf{P}\left\{z \in \beta\right\} \asymp n^2 \operatorname{Es}(n).$$

To begin with,

$$\sum_{z\in B_n} \mathbf{P}\left\{z\in\beta\right\} \le \sum_{z\in B_{4n}} \mathbf{P}\left\{z\in \mathrm{L}(S[0,\sigma_{4n}])\right\} \asymp n^2\,\mathrm{Es}(4n).$$

By Lemma 3.11, the latter is less than a constant times $n^2\,\mathrm{Es}(n)$.

To prove the other direction, the number of steps of β is strictly larger than $M^K_{n,m,N,x}$ where $m=0$, $N=4n$, $x=0$ and $K=\emptyset$. Therefore, by part 1 and Lemma 3.10,

$$\sum_{z\in B_n} \mathbf{P}\left\{z\in\beta\right\} \ge \mathbf{E}\left[M^{\emptyset}_{n,0,4n,0}\right] \ge cn^2\,\mathrm{Es}(4n) \ge cn^2\,\mathrm{Es}(n).$$

□

Proposition 6.3. *There exists $C<\infty$ such that if m, n, N, K and x are as in Definition 1.4,*

$$\mathbf{E}\left[(M^K_{m,n,N,x})^2\right] \le C\left(\ln\frac{N}{n}\right)^2 n^4\,\mathrm{Es}(n)^2.$$

Proof. Let α be as in Definition 1.4. Then, by Proposition 5.5,

$$\begin{aligned}
\mathbf{E}\left[(M^K_{m,n,N,x})^2\right] &= \mathbf{E}\left[\left(\sum_{z\in A_n(x)} \mathbb{1}_{\{z\in\alpha\}}\right)^2\right]\\
&= \sum_{z,w\in A_n(x)} \mathbf{P}\left(z,w\in\alpha\right)\\
&\le C\sum_{z,w\in A_n(x)} G^X_N(x,z)G^X_N(z,w)\,\mathrm{Es}(r_z)\,\mathrm{Es}(r_w)
\end{aligned}$$

where $r_z = \mathrm{dist}(z,\partial B_N)\wedge|z-x|\wedge|z-w|$ and $r_w = \mathrm{dist}(z,\partial B_N)\wedge|z-w|$. However, since z and w are in $A_n(x)$, r_z and r_w are comparable to $|z-w|$. Therefore, by Lemmas 4.6 and 3.10 and the fact that

$$G^X_N(z,w) = G_{B_N\setminus K}(z,w)\frac{\mathbf{P}^w\left\{\sigma_N<\xi_K\right\}}{\mathbf{P}^z\left\{\sigma_N<\xi_K\right\}} \le CG_{B_{2N}(z)}(z,w) \le C\ln\frac{2N}{|z-w|},$$

$$\begin{aligned}
\mathbf{E}\left[(M^K_{m,n,N,x})^2\right] &\le C\ln\frac{N}{n}\sum_{z,w\in A_n(x)} \ln\frac{2N}{|z-w|}\,\mathrm{Es}(|z-w|)^2\\
&\le C\ln\frac{N}{n}\sum_{z\in A_n(x)}\sum_{w\in B_n(z)} \ln\frac{2N}{|z-w|}\,\mathrm{Es}(|z-w|)^2\\
&\le C\ln\frac{N}{n}\sum_{z\in A_n(x)}\sum_{k=1}^{n} k\ln\frac{N}{k}\,\mathrm{Es}(k)^2\\
&\le C\ln\frac{N}{n}n^2\left(\sum_{k=1}^{n} k\ln\frac{n}{k}\,\mathrm{Es}(k)^2 + \sum_{k=1}^{n} k\ln\frac{N}{n}\,\mathrm{Es}(k)^2\right).
\end{aligned}$$

By Corollary 3.14, both of the sums above are bounded by $C\ln(N/n)n^2\,\mathrm{Es}(n)^2$ which finishes the proof.

□

Corollary 6.4. *There exist $C < \infty$ and $c_2, c_3 > 0$ such that if m, n, N, K and x are as in Definition 1.4,*

1.
$$\mathbf{E}\left[(M^K_{m,n,N,x})^2\right] \leq C \left(\ln \frac{N}{n}\right)^8 \mathbf{E}\left[(M^K_{m,n,N,x})\right]^2.$$

2.
$$\mathbf{P}\left\{M^K_{m,n,N,x} \leq c_2 \left(\ln \frac{N}{n}\right)^{-3} \mathbf{E}\left[M_n\right]\right\} \leq 1 - c_3 \left(\ln \frac{N}{n}\right)^{-8}.$$

Proof. The first part follows immediately from Propositions 6.2 and 6.3.

To prove the second part, by a standard second moment result (see for example [7, Lemma 12.6.1], for any $0 < r < 1$,

$$\mathbf{P}\left\{M^K_{m,n,N,x} \leq r\mathbf{E}\left[M^K_{m,n,N,x}\right]\right\} \leq 1 - \frac{(1-r)^2 \mathbf{E}\left[M^K_{m,n,N,x}\right]^2}{\mathbf{E}\left[(M^K_{m,n,N,x})^2\right]}.$$

Letting $r = 1/2$ and using part 1, one obtains that

$$\mathbf{P}\left\{M^K_{m,n,N,x} \leq \frac{1}{2}\mathbf{E}\left[M^K_{m,n,N,x}\right]\right\} \leq 1 - c_3 \left(\ln \frac{N}{n}\right)^{-8}.$$

Finally, by Proposition 6.2 again,

$$\mathbf{E}\left[M^K_{m,n,N,x}\right] \geq c \left(\ln \frac{N}{n}\right)^{-3} \mathbf{E}\left[M_n\right].$$

□

Lemma 6.5. *For all $\varepsilon > 0$, there exists $C(\varepsilon) < \infty$ and $N(\varepsilon) < \infty$ such that for all $n \geq N(\varepsilon)$ and $k \geq 1$,*

$$\mathbf{E}\left[M_{kn}\right] \leq C(\varepsilon) k^{5/4+\varepsilon} \mathbf{E}\left[M_n\right],$$

and

$$\mathbf{E}\left[\widehat{M}_{kn}\right] \leq C(\varepsilon) k^{5/4+\varepsilon} \mathbf{E}\left[\widehat{M}_n\right].$$

Remark It is possible to take $\varepsilon = 0$ in the inequality above, but in that case N has to depend on k.

Proof. The second statement follows immediately from the first by Proposition 6.2.

By Proposition 6.2 and Theorem 3.9 part 3,

$$\mathbf{E}\left[M_{kn}\right] \leq C(kn)^2 \operatorname{Es}(kn) \leq C(kn)^2 \operatorname{Es}(n) \operatorname{Es}(n, kn).$$

By Lemma 3.12, there exists $C(\varepsilon) < \infty$ and $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$,

$$\operatorname{Es}(n, kn) \leq C(\varepsilon) k^{-3/4+\varepsilon}.$$

Therefore,

$$\mathbf{E}\left[M_{kn}\right] \leq C(\varepsilon) k^{5/4+\varepsilon} n^2 \operatorname{Es}(n).$$

Finally, by a second application of Proposition 6.2,

$$n^2 \operatorname{Es}(n) \leq C\mathbf{E}\left[M_n\right].$$

□

Proposition 6.6.

1. *Let c_2 be as in Corollary 6.4. Then there exists $c_4 > 0$ such that for all n and all $k \geq 2$,*

$$\mathbf{P}\left\{M_{kn} \leq c_2 (\ln k)^{-3}\, \mathbf{E}\left[M_n\right]\right\} \leq e^{-c_4 k (\ln k)^{-8}}.$$

2. *There exist $c_5, c_6 > 0$ and $C < \infty$ such that for all n and $k \geq 2$,*

$$\mathbf{P}\left\{\widehat{M}_{kn} \leq c_5 (\ln k)^{-3}\, \mathbf{E}\left[\widehat{M}_n\right]\right\} \leq Ce^{-c_6 k (\ln k)^{-8}}.$$

Proof. We first prove part 1.

Let $k' = \lfloor k/\sqrt{2} \rfloor$. Then $R_{k'n} \subset B_{kn}$. We view the loop-erased random walk $\mathrm{L}(S[0, \sigma_{kn}])$ as a distribution on the set Ω_{kn} of self-avoiding paths γ from the origin to ∂B_{kn}. Given such a γ, let γ_j be its restriction from 0 to the first exit of R_{jn}, $j = 0, \ldots, k'$. Let $\mathcal{F}_j$ be the σ-algebra generated by the γ_j. For $j = 0, \ldots, k'-1$, let $x_j(\gamma) \in \partial R_{jn}$ be the point where γ first exits R_{jn} and $B_j = B_n(x_j)$. Finally, for $j = 1, \ldots, k'$, let $\alpha_j(\gamma)$ be γ from x_{j-1} up to the first exit of B_{j-1} and let $N_j(\gamma)$ be the number of steps of α_j in $A_n(x_{j-1})$. Note that $N_j \in \mathcal{F}_j$.

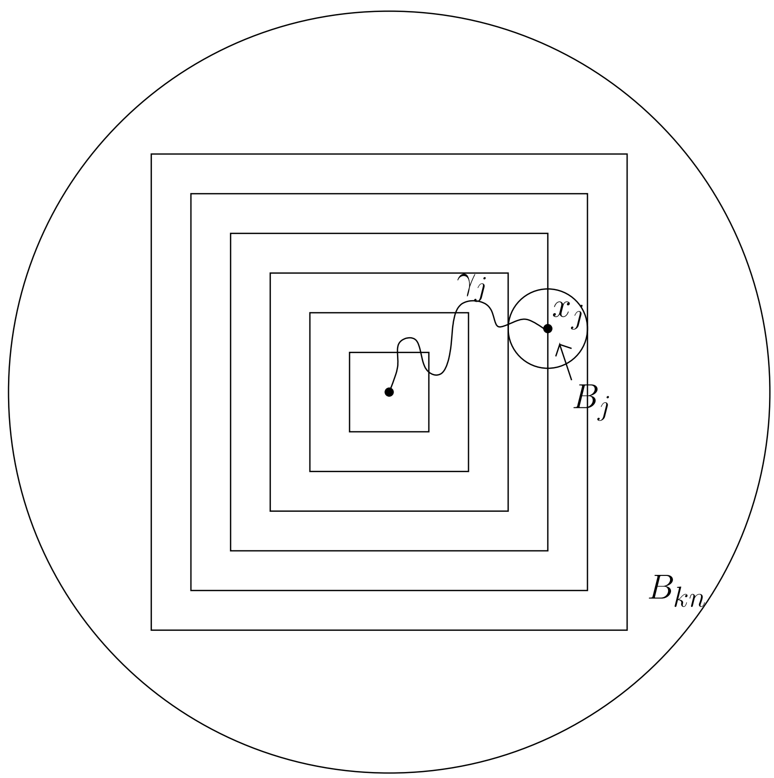

Figure 3: The setup for Proposition 6.6.

Then,

$$\begin{aligned}
&\mathbf{P}\left\{M_{kn} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \\
&\le \mathbf{P}\left\{\sum_{j=1}^{k'} N_j \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \\
&\le \mathbf{P}\left(\bigcap_{j=1}^{k'} \left\{N_j \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\}\right) \\
&= \mathbf{E}\left[\left(\prod_{j=1}^{k'-1} \mathbb{1}_{\{N_j \le c_2(\ln k)^{-3}\mathbf{E}[M_n]\}}\right) \mathbf{P}\left\{N_{k'} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n] \;\middle|\; \mathcal{F}_{k'-1}\right\}\right].
\end{aligned}$$

However, by the domain Markov property, for all $j = 1, \dots, k'$,

$$\mathbf{P}\left\{N_j \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n] \;\middle|\; \mathcal{F}_{j-1}\right\}(\gamma) = \mathbf{P}\left\{M^{\gamma_j}_{jn,n,kn,x_j(\gamma)} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\}.$$

Furthermore, by Corollary 6.4,

$$\mathbf{P}\left\{M^{\gamma_j}_{jn,n,kn,x_j(\gamma)} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \le 1 - c_3(\ln k)^{-8}.$$

Therefore, by applying the above inequality k' times,

$$\mathbf{P}\left\{M_{kn} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \le \left(1 - c_3(\ln k)^{-8}\right)^{k'} \le e^{-c_3(\ln k)^{-8}k'}$$

The proof of part 2 is entirely similar. By Proposition 6.2, it suffices to show that

$$\mathbf{P}\left\{\widehat{M}_{kn} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \le e^{-c_6 k(\ln k)^{-8}}.$$

However, by Corollary 3.4, $\widehat{S}[0, \widehat{\sigma}_{kn}]$ has the same distribution up to constants as $\mathrm{L}(S[0, \sigma_{4kn}])$ from 0 up to its first exit of the ball B_{kn}. Therefore, we can apply the previous iteration argument to obtain that

$$\mathbf{P}\left\{\widehat{M}_{kn} \le c_2 (\ln k)^{-3}\, \mathbf{E}[M_n]\right\} \le C\left(1 - c_3(\ln 4k)^{-8}\right)^{k'} \le Ce^{-c_6 k(\ln k)^{-8}}.$$

□

Theorem 6.7. *For all $\varepsilon > 0$ there exist $C_2(\varepsilon) < \infty$, $C_3(\varepsilon) < \infty$, $c_7(\varepsilon) > 0$ and $c_8(\varepsilon) > 0$ such that for all $\lambda > 0$ and all n,*

1.
$$\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \le C_2(\varepsilon)e^{-c_7(\varepsilon)\lambda^{4/5-\varepsilon}}.$$

2. *For all $D \supset B_n$, $\lambda > 0$,*

$$\mathbf{P}\left\{M_D < \lambda^{-1}\mathbf{E}[M_n]\right\} \le C_3(\varepsilon)e^{-c_8(\varepsilon)\lambda^{4/5-\varepsilon}}.$$

Proof. The second part follows from the first, since by Corollary 3.4, Proposition 6.2 and Lemma 6.5,

$$\begin{aligned}\mathbf{P}\left\{M_D < \lambda^{-1}\mathbf{E}\left[M_n\right]\right\} &\le C\mathbf{P}\left\{\widehat{M}_{n/4} < \lambda^{-1}\mathbf{E}\left[M_n\right]\right\} \\ &\le C\mathbf{P}\left\{\widehat{M}_{n/4} < C\lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \\ &\le C\mathbf{P}\left\{\widehat{M}_{n/4} < C\lambda^{-1}\mathbf{E}\left[\widehat{M}_{n/4}\right]\right\}.\end{aligned}$$

We now prove the first part. We will prove the result for all ε such that $0 < \varepsilon < 7/40$, and note that for such ε,

$$\frac{5}{4} + \varepsilon \le \frac{1}{4/5 - \varepsilon} \le \frac{5}{4} + 2\varepsilon.$$

Clearly this will imply that the result holds for all $\varepsilon > 0$.

Fix such an $\varepsilon > 0$. We will show that there exist $C < \infty$, $c_7 > 0$, λ_0 and N such that for $\lambda > \lambda_0$ and $n \ge N$,

$$\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \le Ce^{-c_7\lambda^{4/5-\varepsilon}}. \tag{6.3}$$

We claim that this implies the statement of the theorem with $C_2 = C \vee e^{4c_7(\lambda_0 \vee N)^{4/5-\varepsilon}}$. To see this, if $\lambda < \lambda_0$, then for any n,

$$\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \le 1 \le C_2 e^{-c_7\lambda^{4/5-\varepsilon}}.$$

Next, if $n \le N$, then for any λ,

$$\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \le \mathbf{P}\left\{\widehat{M}_n < 4\lambda^{-1}n^2\right\}$$

since $\mathbf{E}\left[\widehat{M}_n\right] \le |B_n| < 4n^2$. If $\lambda > 4n$ then the above probability is 0 since $\mathbf{P}\left\{\widehat{M}_n \ge n\right\} = 1$. If $\lambda < 4n \le 4N$ then

$$C_2 e^{-c_7\lambda^{4/5-\varepsilon}} \ge e^{4c_7 N^{4/5-\varepsilon}} e^{-4c_7 N^{4/5-\varepsilon}} = 1.$$

We now prove (6.3). Let c_5 be as in Proposition 6.6 and $C^* = C(\varepsilon/2)$ and $N_0 = N(\varepsilon/2)$ be as in Lemma 6.5. Let

$$k = c_5 (C^*)^{-1} \lambda^{4/5-\varepsilon/2}.$$

We choose λ_0 so that for all $\lambda > \lambda_0$, $k \ge 2$, $k^{\varepsilon/2} > (\ln k)^3$ and $k(\ln k)^{-8} \ge \lambda^{4/5-\varepsilon}$. We also choose $N = 4N_0^5$. Then for all $n \ge N$ and $\lambda > \lambda_0$,

$$\mathbf{E}\left[\widehat{M}_{kn}\right] \le C^* k^{5/4+\varepsilon/2}\mathbf{E}\left[\widehat{M}_n\right] \le c_5 k^{-\varepsilon/2}\lambda\mathbf{E}\left[\widehat{M}_n\right]. \tag{6.4}$$

Suppose first that $n/k \le N_0$. Then

$$\lambda^{-1} \le k^{-5/4} \le (N_0 n^{-1})^{5/4} \le 1/(4n)$$

and so $\lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right] \le n$. Hence since $\widehat{M}_n \ge n$ almost surely,

$$\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} \le \mathbf{P}\left\{\widehat{M}_n < n\right\} = 0.$$

If $n/k \geq N_0$, then by (6.4) and Proposition 6.6,

$$\begin{aligned}
\mathbf{P}\left\{\widehat{M}_n < \lambda^{-1}\mathbf{E}\left[\widehat{M}_n\right]\right\} &= \mathbf{P}\left\{\widehat{M}_{k(n/k)} < \lambda^{-1}\mathbf{E}\left[\widehat{M}_{k(n/k)}\right]\right\} \\
&\leq \mathbf{P}\left\{\widehat{M}_{k(n/k)} < c_5 k^{-\varepsilon/2}\mathbf{E}\left[\widehat{M}_{n/k}\right]\right\} \\
&\leq \mathbf{P}\left\{\widehat{M}_{k(n/k)} < c_5 (\ln k)^{-3}\mathbf{E}\left[\widehat{M}_{n/k}\right]\right\} \\
&\leq Ce^{-c_6 k (\ln k)^{-8}} \\
&\leq Ce^{-c_7 \lambda^{4/5-\varepsilon}}.
\end{aligned}$$

□

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