Random walk on the incipient infinite cluster on trees

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Abstract. Let $\mathcal{G}$ be the incipient infinite cluster (IIC) for percolation on a homogeneous tree of degree n_0+1. We obtain estimates for the transition density of the the continuous time simple random walk Y on $\mathcal{G}$; the process satisfies anomalous diffusion and has spectral dimension $\frac{4}{3}$.

2000 MSC. Primary 60K37; Secondary 60J80, 60J35.
Keywords. Percolation, incipient infinite cluster, random walk, branching process, heat kernel.

1. Introduction

We recall the bond percolation model on the lattice $\mathbb{Z}^d$: each bond is open with probability $p \in (0,1)$, independently of all the others. Let $\mathcal{C}(x)$ be the open cluster containing x; then if $\theta(p) = P_p(|\mathcal{C}(x)| = +\infty)$ it is well known (see [Gm]) that there exists $p_c = p_c(d)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$.

If $d = 2$ or $d \geq 19$ (or $d > 6$ for 'spread out' models) it is known (see [Gm], [HS]) that $\theta(p_c) = 0$, and it is conjectured that this holds for all $d \geq 2$. At the critical probability $p = p_c$ it is believed that in any box of side n there exist with high probability open clusters of diameter of order n – see [BCKS]. For large n the local properties of these large finite clusters can, in certain circumstances, be captured by regarding them as subsets of an infinite cluster $\widetilde{\mathcal{C}}$, called the 'incipient infinite cluster' (IIC).

This was constructed when $d = 2$ in [Ke1], by taking the limit as $N \to \infty$ of the cluster $\mathcal{C}(0)$ conditioned to intersect the boundary of a box of side N with center at the origin. See [Ja1], [Ja2] for other constructions of the IIC in two dimensions. For large d a construction of the IIC in $\mathbb{Z}^d$ is given in [HJ], using the lace expansion. It is believed that the results there will hold for any $d > 6$. [HJ] also gives the existence and some properties of the IIC for all $d > 6$ for 'spread-out' models: these include the case when there is a bond between x and y with probability pL^{-d} whenever y is in a cube side L with center x, and the parameter L is large enough. Rather more is known about the IIC for oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}^d$ (see [HHS], [HS]), but in this discussion, which mainly concerns what is conjectured rather than what is known, we specialize to the case of $\mathbb{Z}^d$. We write $\widetilde{\mathcal{C}}_d$ for the IIC in $\mathbb{Z}^d$. It is believed that the global properties of $\widetilde{\mathcal{C}}_d$ are the same for all $d > d_c$, both for nearest neighbour and spread-out models. In [HJ] it is proved for 'spread-out' models that $\widetilde{\mathcal{C}}_d$ has one end – that is that any two paths from 0 to infinity intersect infinitely often.

[1] Research partially supported by a grant from NSERC (Canada).
[2] Research partially supported by the Grant-in-Aid for Scientific Research for Young Scientists (B) 16740052.

For large d, it is believed that the geometry of $\widetilde{\mathcal{C}}_d$ is also similar to that of the IIC when '$d = \infty$' – that is to the IIC on a regular tree; this is supported by the results in [HHS] and [HJ]. For trees the construction of the IIC is much easier than for lattices, and there is a close connection between the IIC and a critical Bienaymé-Galton-Watson branching processes conditioned on non-extinction. In [Ke2] Kesten gave the construction of the IIC $\mathcal{G}$ for critical branching processes. This is an infinite subtree, which contains only one path from the root to infinity. This tree is quite sparse, and has polynomial volume growth: in the case when the offspring distribution has finite variance, a ball $B(x,r)$ in $\mathcal{G}$ has roughly r^2 points. (This is when distance in $\mathcal{G}$ is measured using the natural graph distance).

Let $Y = (Y_t, t \geq 0)$ be the simple random walk on $\widetilde{\mathcal{C}}_d$, and $q_t(x,y)$ be its transition density (see Section 3 for a precise definition). Define the *spectral dimension* of $\widetilde{\mathcal{C}}_d$ by

$$d_s(\widetilde{\mathcal{C}}_d) = -2 \lim_{t\to\infty} \frac{\log q_t(x,x)}{\log t}, \tag{1.1}$$

(if this limit exists). Alexander and Orbach [AO] conjectured that, for any $d \geq 2$, $d_s(\widetilde{\mathcal{C}}_d) = 4/3$. While it is now thought that this is unlikely to be true for small d, the results on the geometry of $\widetilde{\mathcal{C}}_d$ in [HHS] and [HJ] are consistent with this holding for large d. (Or for any d above the critical dimension for spread-out models).

Random walks on supercritical clusters in $\mathbb{Z}^d$ are studied in [B2] (transition density estimates) and [SS] (invariance principle for $d \geq 4$). In these cases the large scale behaviour of the random walk approximates that of the random walk on $\mathbb{Z}^d$, and the unique infinite cluster has spectral dimension d.

In what follows, we will specialize to the case of critical percolation on a regular rooted tree with degree $n_0 + 1$, which we denote $\mathbb{B}$. We write 0 for the root of $\mathbb{B}$. We keep n_0 fixed, but (in view of possible future applications) wish to obtain estimates which do not depend on n_0. For bond percolation with probability p on $\mathbb{B}$, it is easy to see that if X_n is the number of vertices at level n in $\mathcal{C}(0)$, then $X = (X_n)$ is a branching process with $\texttt{Bin}(n_0, p)$ offspring distribution. Thus $p_c = 1/n_0$. For the construction of the IIC see [Ke2]: we obtain a subtree $\mathcal{G} \subset \mathbb{B}$ with law $\mathbb{P}$, on a probability space $(\Omega_1, \mathcal{F}, \mathbb{P})$. Write $\mathbb{B}_N$ for the N-th level of $\mathbb{B}$, and $\mathbb{B}_{\leq N}$ for the union of the first N levels of $\mathbb{B}$. Then the law of $\mathcal{G}$ is characterized by the fact that the law of $\mathcal{G} \cap \mathbb{B}_{\leq N}$ under $\mathbb{P}$ is the same as that of $\mathcal{C}(0)$ under P_{p_c}, conditioned on $\mathcal{C}(0)$ reaching level N.

Motivated by [AO], in [Ke2] Kesten studied the simple random walk on $\mathcal{G}(\omega)$, and also on $\widetilde{\mathcal{C}}_2$. Let $X = (X_n, n \geq 0, Q^x_\omega, x \in \mathcal{G}(\omega))$ be the simple random walk on $\mathcal{G}(\omega)$. We define the annealed law $\mathbb{P}^*$ by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times Q^0_\omega$, and the rescaled height process $Z^{(n)}$ by

$$Z^{(n)}_t = n^{-1/3} d(0, X_{\lfloor nt \rfloor}), \quad t \geq 0,$$

where $d(.,.)$ is the graph distance in $\mathcal{G}(\omega)$.

The following summarizes the main results in of [Ke2] in the tree case.

Theorem 1.1. *(a) ((1.19) in [Ke2].) Let $T_N = \min\{n : d(0, X_n) = N\}$. Then for all $\varepsilon > 0$ there exist λ_1, λ_2 such that*

$$\mathbb{P}^*(\lambda_1 \leq N^{-3} T_N \leq \lambda_2) \geq 1 - \varepsilon, \quad \textit{for all } N \geq 1.$$

(b) ((1.16) in [Ke2], full proof in [Ke3].) Under $\mathbb{P}^*$ the processes $Z^{(n)}$ converges weakly in $C[0,\infty)$ to a process Z which is not the zero process.

To understand why the $n^{-1/3}$ scaling arises in (b) it is helpful to consider the behaviour of random walks on regular deterministic graphs with a large scale fractal structure – see for example [Jo], [BB2], [HK], [GT1], [GT2] and [BCK]. Let $d_f \geq 1$ give the volume growth, so that $|B(x,r)| \sim r^{d_f}$, and suppose that the effective electrical resistance $R(x, B(x,r)^c)$ between x and the exterior of $B(x,r)$ satisfies $R(x, B(x,r)^c) \sim r^\zeta$, where $\zeta > 0$. In this 'strongly recurrent' case (see [BCK] for simple recent proofs using ideas that are also used in this paper) one finds that the mean time for X to escape from $B(x,r)$ scales as r^{d_w} where $d_w = d_f + \zeta$. While the IIC $\mathcal{G}$ is more irregular than the sets considered in these papers, it still has properties similar to regular graphs with $d_f = 2$. Further, by Proposition 2.10 below, only $O(1)$ points on $\partial B(x, r/4)$ are connected to $B(x,r)^c$ by a path outside $B(x,r/4)^c$, so one has $R(x, B(x,r)^c) \sim r$, giving $\zeta = 1$ and $d_w = 3$.

In this paper we study the simple random walk on $\mathcal{G}$, and in particular investigate both quenched and annealed properties of its transition densities. For technical convenience we work with the continuous time simple random walk on $\mathcal{G}$, which we denote $Y = (Y_t, t \in [0,\infty), P_\omega^x, x \in \mathcal{G}(\omega))$. Since we consider the law of Y with general starting points x, we need to consider the measures $\mathbb{P}_x = \mathbb{P}(\cdot | x \in \mathcal{G})$ and $\mathbb{P}_{x,y} = \mathbb{P}(\cdot | x, y \in \mathcal{G})$.

Unlike [Ke2] we restrict our attention to branching processes with a Binomial offspring distribution. Our main reason for this is to maintain good uniform control of the laws $\mathbb{P}_x$. It is clear by symmetry that $\mathbb{P}_x(|B(x,r)| > \lambda)$ is the same for any $x \in \mathbb{B}_N$, and in fact we have uniform bounds for all $x \in \mathbb{B}$. (These probabilities are not equal for all x, since a higher level x is likely to be further from the backbone of the cluster). For a general branching process, the labels of the point x may give a substantial amount of information about the size of the cluster near x.

Theorem 1.2. *(a) There exist c_0, c_1, c_2, $S(x)$ such that for each x,*

$$\mathbb{P}_x(S(x) \geq m) \leq c_0 (\log m)^{-1}, \tag{1.2}$$

and on $\{\omega : x \in \mathcal{G}(\omega)\}$

$$c_1 t^{-2/3} (\log\log t)^{-17} \leq q_t^\omega(x,x) \leq c_2 t^{-2/3} (\log\log t)^3 \text{ for all } t \geq S(x). \tag{1.3}$$

(b) $d_s(\mathcal{G}) = 4/3$ $\mathbb{P}$–a.s.

The cluster $\mathcal{G}$ contains large scale fluctuations, so that $q_t(x,x)$ does have oscillations of order $(\log\log t)^c$ as $t \to \infty$ – see Lemma 5.1.

Theorem 1.3. *(a) We have*

$$c_1 t^{1/3} \leq \mathbb{E}_x E_\omega^x d(x, Y_t) \leq \mathbb{E}_x E_\omega^x \sup_{0\leq s\leq t} d(x, Y_s) \leq c_2 t^{1/3}. \tag{1.4}$$

(b) There exists $T(x)$ with $\mathbb{P}_x(T(x) < \infty) = 1$ such that

$$c_3 t^{1/3} (\log\log t)^{-12} \leq E_\omega^x [d(x, Y_t)] \leq c_4 t^{1/3} \log t \quad \text{for all } t \geq T(x). \tag{1.5}$$

We also have (annealed) off-diagonal bounds for $q_t^\omega(x,y)$. These are of the same form as the bounds

$$ct^{-d_f/d_w}\exp(-c'(d(x,y)^{d_w}/t)^{1/(d_w-1)})$$

obtained for regular fractal graphs.

Theorem 1.4. *(a) Let $x,y\in\mathbb{B}$. Then*

$$\mathbb{E}_{x,y}q_t^\omega(x,y)\le c_1t^{-2/3}\exp\big(-c_2(\frac{d(x,y)^3}{t})^{1/2}\big). \tag{1.6}$$

(b) Let $x,y\in\mathbb{B}$, with $d(x,y)=R$, and $c_3R\le t$. Then

$$\mathbb{E}_{x,y}q_t^\omega(x,y)\ge c_4t^{-2/3}\exp(-c_5(R^3/t)^{1/2}). \tag{1.7}$$

Define the continuous time rescaled height process

$$\widetilde{Z}_t^{(n)}=n^{-1/3}d(0,Y_{nt}),\quad t\ge 0.$$

By Theorem 1.3(a) the processes $(\widetilde{Z}^{(n)},n\ge 1)$ are tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^*=\mathbb{P}\times P_\omega^0$. (This is much easier to prove than the full convergence given in Theorem 1.1(b).) However, the large scale fluctuations in $\mathcal{G}$ mean that we do not have quenched tightness.

Theorem 1.5. $\mathbb{P}$-a.s., *the processes $(\widetilde{Z}^{(n)},n\ge 1)$ are not tight with respect to P_ω^0.*

In Section 2 we recall various properties of branching processes, and obtain the geometrical properties of $\mathcal{G}$ that we will require. In particular we show that, with high probability, balls $B(x,r)\subset\mathcal{G}$ have roughly r^2 points, and $O(1)$ disjoint paths between $B(x,r/4)$ and $B(x,r)^c$. Based on this, we define various types of possible 'good' behaviour of a ball $B(x,r)$, and the cluster in a neighbourhood of the path between points $x,y\in\mathcal{G}$. In Section 3 we review some general properties of random walks on graphs. Our main estimates are given in Section 4, for the random walk on a deterministic subset $\mathcal{G}$ of $\mathbb{B}$ for which balls and paths are 'good' in the ways given in Section 2. Finally, in Section 5 we tie together the results of Sections 2 and 4, and prove Theorems 1.2–1.5.

Throughout this article, $f_n\sim g_n$ means that $\lim_{n\to\infty}f_n/g_n=1$. We use c, c' and c'' to denote strictly positive finite constants whose values are not significant and may change from line to line. We write c_i for positive constants whose values are fixed within each theorem, lemma etc. When we cite a constant c_1 in Lemma 2.2, say, we denote it as $c_{2.2.1}$. None of these constants depend on the degree n_0 of the tree.

2. The incipient infinite cluster

We begin with some estimates for the critical Bienaymé-Galton-Watson branching processes X_n, $n\ge 0$, with $X_0=1$ and offspring distribution $\mathtt{Bin}(n_0,1/n_0)$ where $n_0\ge 2$. These are quite well known, but as we did not find them anywhere in exactly the form we needed, we give the proofs (which are quite short) here.

Let f be the generator of the offspring distribution, so that

$$f(s) = E(s^{X_1}) = n_0^{-n_0}(s + n_0 - 1)^{n_0}. \tag{2.1}$$

From [Har] p. 21 we have

$$P(X_n > 0) \sim \frac{2}{nf''(1)} = \frac{2n_0}{(n_0 - 1)n}. \tag{2.2}$$

Let

$$Y_n = \sum_{k=0}^{n} X_k, \qquad g_n(s) = E(s^{Y_n}), \qquad f_n(s) = Es^{X_n}.$$

Then conditioning on X_1 we obtain that $f_{n+1}(s) = f(f_n(s))$, and

$$g_{n+1}(s) = sf(g_n(s)) = \frac{s}{n_0^{n_0}}(g_n(s) + n_0 - 1)^{n_0}.$$

Set

$$h_n(\theta) = \log g_n(e^\theta), \qquad k_n(\theta) = \log f_n(e^\theta).$$

Lemma 2.1. *(a) Let $1 < \alpha \le 2$. Then*

$$h_n(\theta) \le (1 + \alpha n)\theta, \qquad \textit{provided } 0 \le \theta \le \frac{\alpha - 1}{(1 + \alpha n)^2}. \tag{2.3}$$

(b)

$$k_n(\theta) \le \theta + 2n\theta^2, \qquad \textit{provided } 0 < \theta \le \frac{1}{6n}. \tag{2.4}$$

Proof. Note that h_n and k_n are continuous, strictly increasing and $h_n(0) = k_n(0) = 0$.

For (a) we have

$$h_{n+1}(\theta) = \log\left(\frac{e^\theta}{n_0^{n_0}}(e^{h_n(\theta)} + n_0 - 1)^{n_0}\right) = \theta + n_0 \log \frac{1}{n_0}(e^{h_n(\theta)} + n_0 - 1).$$

Let $a_n = \min\{\theta : h_n(\theta) = 1\}$. Then since $e^x \le 1 + x + x^2$ on $[0, 1]$, on $[0, a_n]$,

$$h_{n+1}(\theta) \le \theta + n_0 \log(1 + \frac{1}{n_0}h_n(\theta) + \frac{1}{n_0}h_n(\theta)^2) \le \theta + h_n(\theta) + h_n(\theta)^2. \tag{2.5}$$

We verify (2.3) by induction. Since $h_0(\theta) = \theta$, (2.3) holds for $n = 0$. Writing $b_n(\alpha) = (\alpha - 1)/(1 + \alpha n)^2$, we have $h_n(\theta) \le 1$ for $\theta \in [0, b_n(\alpha)]$. So, using (2.5) and (2.3) for n

$$h_{n+1}(\theta) \le (1 + \alpha(n + 1))\theta + (1 + \alpha n)^2\theta^2 - (\alpha - 1)\theta \le (1 + \alpha(n + 1))\theta,$$

proving (2.3) for $n + 1$.

(b) Similarly, provided $k_n(\theta) \le 1$,

$$k_{n+1}(\theta) = n_0 \log\left(1 + \frac{e^{k_n(\theta)} - 1}{n_0}\right) \le k_n(\theta) + k_n(\theta)^2. \tag{2.6}$$

Using (2.4) for n we obtain, since $\theta + 2n\theta^2 \le 4\theta/3$,

$$k_{n+1}(\theta) \le (\theta + 2n\theta^2) + (\theta + 2n\theta^2)^2 \le (\theta + 2n\theta^2) + 16\theta^2/9 \le (\theta + 2(n+1)\theta^2),$$

proving (2.4) for $n+1$. □

Notation. Let ξ be a random variable. We write $\lambda\xi[n]$ for a r.v. with the distribution of $\lambda \sum_1^n \xi_i$, where ξ_i are i.i.d. with $\xi_i \stackrel{(d)}{=} \xi$. We also write $\texttt{Ber}(p)$ and $\texttt{Bin}(n,p)$ for the Bernoulli and Binomial distributions respectively. Using this notation we have for example $(\xi[n])[m] = \xi[nm]$, and $\texttt{Bin}(n,p) \stackrel{(d)}{=} \texttt{Ber}(p)[n]$. We write $\succcurlyeq$ for stochastic domination.

Lemma 2.2. *For any $\lambda > 0$*

$$P(X_n[n] \ge \lambda n) \le c_1 e^{-\lambda/6}, \tag{2.7}$$

$$P(Y_n[n] \ge \lambda n^2) \le c_2 e^{-\lambda/5}. \tag{2.8}$$

Proof. Let $\theta = 1/6n$. Using (2.4)

$$\begin{aligned} \log P(X_n[n] \ge \lambda n) &\le -\theta\lambda n + n k_n(\theta) \\ &\le -n\theta(\lambda - 2) = -(\lambda-2)/6, \end{aligned}$$

proving (2.7).

Let If $\theta \le b_n(\alpha)$ then

$$\begin{aligned} P(Y_n[n] \ge \lambda n^2) = P(e^{\theta Y_n[n]} \ge e^{\theta\lambda n^2}) &\le e^{-\theta\lambda n^2} E e^{\theta Y_n[n]} \\ &= \exp(-\theta\lambda n^2 + n h_n(\theta)) \le \exp(-\theta\lambda n^2 + (1+2n)n\theta). \end{aligned}$$

So taking $\alpha = 2$ and $\theta = b_n(2) = (1+2n)^{-2}$

$$\log P(Y_n[n] \ge \lambda n^2) \le -\frac{n^2(\lambda-2)}{(1+2n)^2} + \frac{n}{(1+2n)^2} \sim -\tfrac{1}{5}\lambda + c_3.$$

□

Lemma 2.3. *(a) There exist $c_0 > 0$, $p_0 > 0$ such that*

$$P(Y_n > c_0 n^2) \ge \frac{p_0}{n}.$$

(b) If $\eta_n \overset{(d)}{=} \mathtt{Bin}(n, p_0/n)$ *then* $Y_n[n] \succcurlyeq c_0 n^2 \eta_n$.

Proof. (a) This should be in literature, but is also easy to prove directly. Let $A_n = \{X_{n/2} > 0\}$, and $a_n = P(A_n)$. Then by (2.2) $a_n \sim (2n_0/(n_0-1))n^{-1}$. We have $EY_n = n+1$ and $EY_n^2 \le c_1 n^3$, where c_1 does not depend on n_0. On A^c we have $Y_{n/2} = Y_n$, so

$$n+1 = EY_n = E(Y_n; A_n) + E(Y_n; A_n^c) \le E(Y_n|A_n)P(A_n) + EY_{n/2}.$$

It follows that

$$E(Y_n|A_n) \ge \frac{n/2}{a_n} \ge c_2 n^2.$$

Also,

$$E(Y_n^2|A_n) \le P(A_n)^{-1} E(Y_n^2; A_n) \le c_3 n^4.$$

Using the 'Backwards Chebyshev' inequality $P(\xi \ge \frac{1}{2}E\xi) \ge (E\xi)^2/(4E\xi^2)$ with respect to $P(\cdot|A_n)$ then gives

$$P(Y_n > \tfrac{1}{2}c_2 n^2 | A_n) \ge P(Y_n > \tfrac{1}{2}E(Y_n|A_n)|A_n) \ge \frac{c_2^2 n^4}{4c_3 n^4} = c_4.$$

So $P(Y_n > \frac{1}{2}c_2 n^2) \ge P(Y_n > c_2 n^2|A_n)P(A_n) \ge c_4 a_n \ge c_5 n^{-1}$, and taking $c_0 = \frac{1}{2}c_2$, $p_0 = c_5$, this proves (a).
(b) Let now $Y_n^{(j)}$ be i.i.d. copies of Y_n, and $F_j = \{Y_n^{(j)} > c_0 n^2\}$. Then if $\xi_j = 1_{F_j}$, by (a) we have $P(\xi_j = 1) \ge p_0/n$. So,

$$Y_n[n] = \sum_{j=1}^n Y_n^{(j)} \succcurlyeq \sum_{j=1}^n c_0 n^2 \xi_j \succcurlyeq c_0 n^2 \eta_n,$$

proving (b). □

Lemma 2.4. *For* $0 < \lambda < 1$,

$$\exp(-c_1/\lambda) \le P(Y_n[n] \le \lambda n^2) \le \exp(-c_2/\lambda^{1/2}). \tag{2.9}$$

Proof. To prove the upper bound let $c_0 = c_{2.3.0}$, and $m = (\lambda/c_0)^{1/2}n$. Using Lemma 2.3 we have

$$Y_n[n] = \sum_{i=1}^n Y_m^{(i)} \succcurlyeq \sum_{i=1}^n c_0 m^2 \xi_i = \lambda n^2 \sum_{i=1}^n \xi_i;$$

here ξ_i are i.i.d. $\mathtt{Ber}(p_0/m)$ r.v. So

$$P(Y_n[n] < \lambda n^2) \le P(\sum_{i=1}^n \xi_i < 1) = (1-p_0/m)^n \le \exp(-p_0 n/m) = \exp(-c_0^{1/2} p_0/\lambda^{1/2}).$$

For the lower bound let $k \geq 1$ and $m = n/k$. Let $G_j = \{X_m^{(j)} = 0\}$, and $G = \cap_{1 \leq j \leq n} G_j$. Then $P(G_j) \geq (1 - c/m)^n$ so

$$\begin{aligned} P(Y_n[n] < \lambda n^2) &\geq P(Y_n[n] < \lambda n^2 | G) P(G) \\ &\geq (1 - c/m)^n \Big(1 - P(Y_n[n] > \lambda n^2 | G)\Big) \\ &\geq c' e^{-c'' k} \Big(1 - P(Y_n[n] > \lambda n^2 | G)\Big). \end{aligned}$$

On G we have $Y_n[n] = \sum_{j=1}^n Y_m^{(j)}$, so

$$P(Y_n[n] > \lambda n^2 | G) \leq \frac{E(\sum_{j=1}^n Y_m^{(j)} | G)}{\lambda n^2} = \frac{n E(Y_m^{(1)} | G_1)}{\lambda n^2} \leq \frac{E Y_m^{(1)}}{\lambda n P(G_1)} \leq \frac{c}{k\lambda}.$$

Taking k such that $c/(k\lambda) = \frac{1}{2}$ completes the proof. □

We will need to consider the following modified branching process. Let $\widetilde{X} = (\widetilde{X}_n, n \geq 0)$ be a branching process with $\widetilde{X}_0 = 1$ and the same $\mathtt{Bin}(n_0, 1/n_0)$ offspring distribution as X, except that at the first generation we have $\widetilde{X}_1 \stackrel{(d)}{=} \mathtt{Bin}(n_0 - 1, 1/n_0)$.

Lemma 2.5. *(a) For any $\lambda > 0$*

$$P(\widetilde{X}_n[n] \geq \lambda n) \leq c_1 e^{-c_2 \lambda}, \tag{2.10}$$

$$P(\widetilde{Y}_n[n] \geq \lambda n^2) \leq c_3 e^{-c_4 \lambda}. \tag{2.11}$$

(b) For $0 < \lambda < 1$,

$$\exp(-c_5/\lambda) \leq P(\widetilde{Y}_n[n] \leq \lambda n^2) \leq \exp(-c_6/\lambda^{1/2}). \tag{2.12}$$

(c) There exists $p_1 > 0$ such that $\widetilde{Y}_n[n] \succcurlyeq c_7 n^2 \mathtt{Bin}(n, p_1/n)$.

Proof. (a) and the lower bound in (b) are immediate from Lemmas 2.2 and 2.4, since $\widetilde{X}_n \preccurlyeq X_n$ and $\widetilde{Y}_n \preccurlyeq Y_n$.

For the upper bound in (b), we can write

$$\widetilde{Y}_n[n] = n + \sum_{i=1}^{M} Y_{n-1}^{(i)},$$

where $M \stackrel{(d)}{=} \mathtt{Bin}(n(n_0 - 1), 1/n_0)$, and $Y^{(i)}$ are independent copies of Y. Similarly,

$$Y_m[m] = m + \sum_{i=1}^{M'} Y_{m-1}^{(i)},$$

where $M' \stackrel{(d)}{=} \mathtt{Bin}(nn_0, 1/n_0)$. So if $m = n(n_0-1)/n_0$ then

$$\widetilde{Y}_n[n] = n + \sum_{i=1}^{M} Y_{n-1}^{(i)} \geq m + \sum_{i=1}^{M} Y_{m-1}^{(i)} = Y_m[m]. \tag{2.13}$$

(2.12) now follows from Lemma 2.4, since $\frac{1}{2}n \leq m \leq n$.
(c) We have $\mathtt{Ber}(p) \succcurlyeq \frac{1}{2}\mathtt{Ber}(p/2)[2]$. So, using (2.13), with m as in (b),

$$\begin{aligned}\widetilde{Y}_n[n] \succcurlyeq Y_m[m] &\succcurlyeq c_0 m^2 \mathtt{Bin}(m, p_0/m) \\ &\succcurlyeq \tfrac{1}{2} c_0 m^2 \mathtt{Bin}(2m, p_0/2m) \\ &\succcurlyeq \tfrac{1}{2} c_0 m^2 \mathtt{Bin}(n, p_0/2m) \succcurlyeq c_1 n^2 \mathtt{Bin}(n, p_1/n).\end{aligned}$$

□

We now define the random graph $\mathcal{G}$ we will be working with. We could regard this either as critical percolation on the n_0-ary tree $\mathbb{B}$, conditioned on the cluster containing the root 0 being infinite, or as the (critical) Bienaymé-Galton-Watson process with $\mathtt{Bin}(n_0, 1/n_0)$ offspring distribution, conditioned on non-extinction.

Let $\mathbb{B}$ be the n_0-ary tree, and let 0 be the root. A point x in the nth generation (or level) is written $x = (0, l_1, \cdots, l_n)$, where $l_i \in \{1, 2, \cdots, n_0\}$. Let $\mathbb{B}_n$ be the set of n_0^n points in the nth generation, and let $\mathbb{B}_{\leq n} = \cup_{i=0}^n \mathbb{B}_i$. If $x \in \mathbb{B}_k$ we write $|x| = k$. If $x = (0, l_1, \cdots, l_n) \in \mathbb{B}_n$, let $a(x, r) = (0, l_1, \cdots, l_{n-r})$ be the ancestor of x at level $|x| - r$.

We regard $\mathbb{B}$ as a graph (in fact a tree) with edge set $E(\mathbb{B}) = \big\{\{x, a(x,1)\}, x \in \mathbb{B} - \{0\}\big\}$. Let η_e, $e \in E(\mathbb{B})$, be i.i.d. Bernoulli $1/n_0$ r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$. If $\eta_e = 1$ we say the edge e is *open*. Let

$$\mathcal{C}(0) = \{x \in \mathbb{B} : \text{ there exists an } \eta\text{–open path from 0 to } x\}$$

be the open cluster containing 0. It is clear that $Z_n = |\mathcal{C}(0) \cap \mathbb{B}_n|$ is a critical GW process with $\mathtt{Bin}(n_0, 1/n_0)$ offspring distribution. Here and in the following, $|A|$ is a cardinality of the set A. As Z has extinction probability 1, the cluster $\mathcal{C}(0)$ is P–a.s. finite.

We have

Lemma 2.6. *([Ke2, Lemma 1.14]). Let $A \subset \mathbb{B}_{\leq k}$. Then*

$$\lim_{n\to\infty} P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A | Z_n \neq 0) = |A \cap \mathbb{B}_k| P(\mathcal{C}(0) \cap \mathbb{B}_{\leq k} = A), \tag{2.14}$$

and writing $\mathbb{P}_0(A) = |A \cap \mathbb{B}_k| P(\mathcal{C}_{\leq k} = A)$, $\mathbb{P}_0$ has a unique extension to a probability measure $\mathbb{P}$ on the set of infinite connected subsets of $\mathbb{B}$ containing 0.

Let $\mathcal{G}'$ be a rooted labeled tree chosen with the distribution $\mathbb{P}$: we call this the *incipient infinite cluster* (IIC) on $\mathbb{B}$. For more information on $\mathcal{G}'$ see [Ke2] and [vH] but we remark that $\mathbb{P}$–a.s. $\mathcal{G}'$ has exactly one infinite descending path from 0, which we call the *backbone*, and denote H.

It will be useful to give another construction of the IIC, obtained by modifying the cluster $\mathcal{C}(0)$ rather than its law. We can suppose the probability space $(\Omega, \mathcal{F}, P)$ carries i.i.d.r.v. ξ_i, $i \geq 1$ uniformly distributed on $\{1, 2, \cdots, n_0\}$, and independent of (η_e). For $n \geq 0$ let $\Xi_n = (0, \xi_1, \ldots, \xi_n)$, and let

$$\widetilde{\eta}_e = \begin{cases} 1 & \text{if } e = \{\Xi_n, \Xi_{n+1}\} \text{ for some } n \geq 0, \\ \eta_e & \text{otherwise.} \end{cases}$$

Then (see [vH]) if

$$\mathcal{G} = \{x \in \mathbb{B} : \text{ there exists a } \widetilde{\eta}\text{–open path from 0 to } x\},$$

$\mathcal{G}$ has law $\mathbb{P}$. It is clear that the backbone of $\mathcal{G}$ is the set $H = \{\Xi_n, n \geq 0\}$.

For $x, y \in \mathbb{B}$ let

$$\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}), \qquad \mathbb{P}_{xy}(\cdot) = \mathbb{P}(\cdot | x, y \in \mathcal{G}),$$

and let $\mathbb{E}_x$ and $\mathbb{E}_{xy}$ denote expectation with respect to $\mathbb{P}_x$ and $\mathbb{P}_{xy}$ respectively. Given a descending path $b = \{0, b_1, b_2, \ldots\}$, (which we call a *possible backbone*) let

$$\mathbb{P}_{x,b}(\cdot) = \mathbb{P}(\cdot | x \in \mathcal{G}, H = b),$$

and define $\mathbb{P}_{x,y,b}$ analogously.

For each x, $y \in \mathbb{B}$, let $\gamma(x, y)$ be the unique geodesic path connecting x and y. We say that z is a *middle point* of $\gamma(x, y)$ if $z \in \gamma(x, y)$ and $|d(x, z) - \frac{1}{2} d(x, y)| \leq \frac{1}{2}$. We remark that the construction of $\mathcal{G}$ makes it clear that $\mathbb{P}_{x,y,b}(\eta_e = 1) = 1$ if the edge e lies in any of the paths b, $\gamma(0, x)$ and $\gamma(0, y)$, and that under $\mathbb{P}_{x,y,b}$ the r.v. η_e, $e \notin b \cup \gamma(0, x) \cup \gamma(0, y)$ are i.i.d. with $\mathbb{P}_{x,y,b}(\eta_e = 1) = 1/n_0$.

Notation. We consider the tree $\mathcal{G} = \mathcal{G}(\omega)$. Let $d(x, y)$ be the graph distance between x and y, and

$$B(x, r) = \{y \in \mathcal{G} : d(x, y) \leq r\}.$$

We write $D(x)$ for the set of descendants of x. More precisely, $y \in D(x)$ if and only if $x \in \gamma(0, y)$. Note that $x \in D(x)$. If $y \in D(x)$ we call x an *ancestor* of y and y a *decedent* of x. We set

$$D_r(x) = \{y \in D(x) : d(x, y) = r\}, \qquad D_{\leq r}(x) = \cup_{i=0}^r D_i(x).$$

We also set

$$D(x; z) = \{y \in D(x) : \gamma(x, y) \cap \gamma(x, z) = \{x\}\},$$

and write $D_r(x; z) = D_r(x) \cap D(x; z)$, $D_{\leq r}(x; z) = D_{\leq r}(x) \cap D(x; z)$. Thus if $z \in D(x)$ then $y \in D(x; z)$ if and only if the lines of descent from x to y and z are disjoint, except for x. (Note that $D(x; x) = D(x)$.) For any $A \subset \mathcal{G}$ we write

$$\partial A = \{y \in \mathcal{G} - A : y \sim x \text{ for some } x \in A\}.$$

The estimates at the beginning of this Section lead to volume growth estimates for $\mathcal{G}$. For $x \in \mathcal{G}$ let μ_x be the degree of x, and for $A \subset \mathcal{G}$ set $\mu(A) = \sum_{x \in A} \mu_x$. We write

$$V(x,r) = \mu(B(x,r)).$$

Note that as $\mathcal{G}$ is a tree, we have

$$|B(x,r)| \le V(x,r) \le 2|B(x,r+1)|. \tag{2.15}$$

Proposition 2.7. *(a) Let $\lambda > 0$, $r \ge 1$ and $x, y \in \mathbb{B}$, and b be a possible backbone. Then*

$$\mathbb{P}_{x,y,b}(V(x,r) > \lambda r^2) \le c_0 \exp(-c_1 \lambda), \tag{2.16}$$

and

$$\mathbb{P}_{x,y,b}(V(x,r) < \lambda r^2) \le c_2 \exp(-c_3/\sqrt{\lambda}). \tag{2.17}$$

(b) The bounds (2.16) and (2.17) also hold for the laws $\mathbb{P}_{x,b}$, $\mathbb{P}_{x,y}$, and $\mathbb{P}_x$.

Proof. It is enough to prove (a), since the bounds for $\mathbb{P}_{x,b}$ follow by taking $y = 0$, and those for $\mathbb{P}_{x,y}$ and $\mathbb{P}_x$ then follow on integrating over b. Also, using (2.15), it is enough to bound $|B(x,r)|$.

We will assume that $|x| > r$; if not we can use the same arguments with minor modifications. Let $x_i = a(x,i)$ for $0 \le i \le r$. If the backbone intersects $B(x,r)$ then let s be the smallest i such that $x_i \in H$, and let $v_0 = x_s$ and v_i, $i \ge 1$ be the backbone descending from the point v_0. Similarly if $\gamma(0,y)$ intersects $B(x,r)$ then let t be the smallest j such that $y_j \in \gamma(0,y)$, and let $w_0 = y_t$ and w_i, $1 \le i \le t$ be the path $\gamma(w_0, y)$.

Then we have

$$B(x,r) \subset \big(\cup_{i=0}^{r} D_{\le r}(x_i; x) \big) \cup \big(\cup_{i=1}^{r} D_{\le r}(v_i; v_{3r}) \big) \cup \big(\cup_{i=1}^{r \wedge t} D_{\le r}(w_i; y) \big).$$

Under $\mathbb{P}_{x,y,b}$ the r.v. $|D_{\le r}(\cdot;\cdot)|$ above are i.i.d., with the same law as $\widetilde{Y}_r$. Thus $|B(x,r)| \preccurlyeq \widetilde{Y}_r[r][3]$, and by Lemma 2.5(a),

$$\mathbb{P}_{x,y,b}(|B(x,r)| > \lambda r^2) \le c \exp(-c' \lambda).$$

The proof of (2.17) is very similar. We have $\cup_{i=0}^{r/2} D_{\le r/2}(x_i; x) \subset B(x,r)$, so that $|B(x,r)| \succcurlyeq \widetilde{Y}_{r/2}[r/2]$, and using Lemma 2.5(b) leads to (2.17). □

We also wish to show that oscillations in $n^{-2} V(0,n)$ exist. If $W \overset{(d)}{=} \mathtt{Bin}(n, p/n)$ then straightforward calculations give that

$$P(W = k) \ge c_0 e^{-k \log(k/p)}, \quad 0 \le k \le n^{1/2}. \tag{2.18}$$

Proposition 2.8. *(a) For any $\varepsilon > 0$*

$$\limsup_{n \to \infty} \frac{V(0,n)}{n^2 (\log \log n)^{1-\varepsilon}} = \infty, \quad \mathbb{P} - a.s.$$

(b) There exists $c_0 < \infty$ such that

$$\liminf_{n\to\infty} \frac{(\log\log n) V(0,n)}{n^2} \le c_0. \quad \mathbb{P} - a.s.$$

Proof. It is enough to prove these for the law $\mathbb{P}_b$, for any fixed possible backbone $b = \{0, y_1, y_2, \ldots\}$.
(a) Let

$$Z_n = |\{x : x \in D(y_i; y_{i+1}),\ d(x, y_i) \le 2^{n-2},\ 2^{n-1} \le i \le 2^{n-1} + 2^{n-2}\}|.$$

Thus Z_n is the number of descendants off the backbone, to level 2^{n-2}, of points y on the backbone between levels 2^{n-1} and $2^{n-1} + 2^{n-2}$. So $|B(0, 2^n)| \ge Z_n$, the r.v. Z_n are independent, and $Z_n \overset{(d)}{=} \widetilde{Y}_{2^{n-2}}[2^{n-2}]$. Using Lemma 2.5(c) we have, if $a_n = (\log n)^{1-\varepsilon}$, and $\eta_n \overset{(d)}{=} \mathtt{Bin}(n, p_1/n)$,

$$\begin{aligned}\mathbb{P}_b(|B(0, 2^n)| \ge a_n 4^n) &\ge \mathbb{P}_b(Z_n \ge a_n 4^n)\\ &\ge P(\widetilde{Y}_{2^{n-2}}[2^{n-2}] \ge a_n 4^n)\\ &\ge P(\eta_{2^{n-2}} \ge a_n) \ge c e^{-a_n \log a_n}.\end{aligned}$$

As Z_n are independent, (a) follows by the second Borel-Cantelli Lemma.

(b) Let $n_k = \exp(2k \log k)$, so that $k^2 n_{k-1} \le n_k$, and let

$$W_k = \bigcup_{i=0}^{n_k - 1} D(y_i; y_{n_k}), \quad V_k = D_{\le n_k - n_{k-1}}(y_{n_{k-1}}).$$

Then the r.v. $|V_k|$ are independent and $B(0, n_k) \subset W_{k-1} \cup V_k$.
Fix $0 < \varepsilon < 1/3$ and let

$$F(i,k) = \{D_{k^{1+\varepsilon} n_k}(y_i; y_{i+1}) = \emptyset\}.$$

Then since $X_n \succcurlyeq \widetilde{X}_n$

$$\mathbb{P}(F(i,k)) = P(\widetilde{X}_{k^{1+\varepsilon} n_k} = 0) \ge P(X_{k^{1+\varepsilon} n_k} = 0) \ge 1 - \frac{c}{k^{1+\varepsilon} n_k}.$$

Let $G_k = \cap_{i=0}^{n_k - 1} F(i,k)$; we have

$$\mathbb{P}(G_k^c) \le c/k^{1+\varepsilon}.$$

On the event G_k we have that $|W_k|$ is stochastically dominated by $\sum_{i=1}^{n_k} Y^{(i)}_{k^{1+\varepsilon} n_k}$, so

$$\begin{aligned}\mathbb{P}(|W_k| \ge k^3 n_k^2) &\le \mathbb{P}(G_k^c) + P(Y_{k^{1+\varepsilon} n_k}[k^{1+\varepsilon} n_k] \ge k^{1-2\varepsilon}(k^{1+\varepsilon} n_k)^2)\\ &\le c k^{-2} + e^{-c' k^{1-2\varepsilon}} \le c'' k^{-2}.\end{aligned}$$

Thus $|W_k| \le k^3 n_k^2$ for all large k. Now $|V_k| \preccurlyeq Y_{n_k}[n_k]$, so

$$\mathbb{P}(|V_k| < c_1(\log k)^{-1}n_k^2) \ge P(Y_{n_k}[n_k] < c_1(\log k)^{-1}n_k^2) \ge e^{-c\log k} \ge k^{-1}$$

if c_1 is chosen large enough. As the r.v. $|V_k|$ are independent, we deduce that $|V_k| < c_1(\log k)^{-1}n_k^2$ for all k in an infinite set J. For all large $k \in J$,

$$|B(0,n_k)| \le |V_k| + (k-1)^3 n_{k-1}^2 \le (c_1(\log k)^{-1} + k^{-1})n_k^2 \le \frac{2c_1 n_k^2}{\log\log n_k}.$$

□

Remark. Let $\mathcal{C}_\infty$ denote the unique infinite cluster for supercritical bond percolation (i.e. $p > p_c$) in $\mathbb{Z}^d$. Then writing $Q(x,N)$ for the box side N and center x

$$\frac{|\mathcal{C}_\infty \cap Q(x,N)|}{|Q(x,N)|} \to \theta(p).$$

Propositions 2.7 and 2.8 show that one does not get this kind of convergence for $\mathcal{G}$, which is a much more irregular set than the clusters considered in [B2].

Definition 2.9. Let $x \in \mathcal{G}$, $r \ge 1$. Let $M(x,r)$ be the smallest number m such that there exists a set $A = \{z_1, \dots, z_m\}$ with $d(x,z_i) \in [r/4, 3r/4]$ for each i, such that any path γ from x to $B(x,r)^c$ must pass through the set A. (Since $\mathcal{G}$ is a tree, the best choice of such a set A will in fact have the points at a distance $r/4$ from x, but we will not need this.)

Proposition 2.10. *There exist $c_1, c_2 > 0$ such that for each $r \ge 1$ and each $x, y \in \mathbb{B}$, and possible backbone b*

$$\mathbb{P}_{x,y,b}(M(x,r) \ge m) \le c_1 e^{-c_2 m}.$$

Similar bounds hold for $\mathbb{P}_{x,y}$, $\mathbb{P}_{x,b}$ and $\mathbb{P}_x$.

Proof. We just consider the case $y = 0$; the general case is similar but a little more complicated since we would also need to consider offspring on the branch $\gamma(0,y)$. Let $w_0 = a(x, r/3)$. If $w_0 \in b$ then let w_1 be the point in the backbone at level $|x| + r/3$, otherwise let $w_1 = w_0$. Let

$$A_1 = \cup_{z\in\gamma(w_0,x), z\notin b} D_{r/4}(z;x), \qquad A_2 = \cup_{z\in\gamma(w_0,w_1), z\ne w_1} D_{r/4}(z;w_1).$$

Let $N_i = |A_i|$; we have $N_1 \preccurlyeq X_{r/4}[1 + r/4]$ and $N_2 \preccurlyeq X_{r/4}[r/2]$. Now let

$$A_i^* = \{z \in A_i : D_{r/4}(z) \ne \emptyset\}.$$

Then any path from x to $B(x,r)^c$ must pass through $A_1^* \cup A_2^* \cup \{w_0, w_1\}$, so $M = M(x,r) \le 2 + |A_1^*| + |A_2^*|$.

Let $p_r = P(z \in A_i^* | z \in A_i) = P(X_{r/4} > 0)$, so that $p_r \le c/r$. So, if κ_i are i.i.d. $\mathtt{Ber}(p_r)$ r.v. independent of N_i, we have

$$|A_i^*| \overset{(d)}{=} \sum_{j=1}^{N_i} \kappa_j.$$

Let

$$W_n = \sum_{i=1}^{n} (\kappa_i - p_r);$$

then $W = \{W_n\}$ is a martingale, $W_n - W_{n-1} \le 1$, $\langle W \rangle_n = np_r(1-p_r)$, and $|A_i^*| \overset{(d)}{=} W_{N_i} + N_i p_r$. Choose r large enough so that $p_r < \frac{1}{2}$. Then

$$\mathbb{P}_{x,b}(|A_i^*| \ge m) \le \mathbb{P}_{x,b}(W_{N_i} + N_i p \ge m, N_i p \le m/2) + \mathbb{P}_{x,b}(N_i p > m/2). \tag{2.19}$$

For the first term in (2.19) we have

$$\begin{aligned} \mathbb{P}_{x,b}(W_{N_i} + N_i p \ge m, N_i p \le m/2) &\le \mathbb{P}_{x,b}(W_{N_i} \ge m/2, \langle W \rangle_{N_i} \le m(1-p)/2) \\ &\le \exp\left(-\frac{(m/2)^2}{2((m/2) + m(1-p)/2)}\right) \le e^{-cm}, \end{aligned}$$

where we used an exponential martingale inequality – see (1.6) in [F]. For the second term, note that $N_i \preccurlyeq (X_{r/4}[r/4])[2]$ and so using Lemma 2.2 we deduce that

$$\mathbb{P}_{x,b}(N_i p > m/2) \le ce^{-c_3 m}.$$

Combining these bounds completes the proof. □

Definition 2.11. Let $x \in \mathbb{B}$, $r \ge 1$, $\lambda \ge 64$. We say that $B(x,r)$ is λ*–good* if:
(a) $x \in \mathcal{G}$
(b) $r^2\lambda^{-2} \le V(x,r) \le r^2\lambda$.
(c) $M(x,r) \le \frac{1}{64}\lambda$.
(d) $V(x, r/\lambda) \ge r^2\lambda^{-4}$.
(e) $V(x, r/\lambda^2) \ge r^2\lambda^{-6}$.

Corollary 2.12. *For $x \in \mathbb{B}$ and any possible backbone b*

$$\mathbb{P}_{x,b}(B(x,r) \text{ is not } \lambda\text{–good}) \le c_1 e^{-c_2\lambda}. \tag{2.20}$$

Proof. By Propositions 2.7 and 2.10 the probability of each of conditions (a)–(d) above failing is bounded by $\exp(-c\lambda)$. □

We now need to introduce some more complicated conditions on the tree $\mathcal{G}$, and will prove that these hold with high probability. These conditions describe various kinds of 'good' behaviour of balls with centers on a path $\gamma(x,y)$, and will be used when we consider off-diagonal bounds on the transition probabilities of the random walk in Sections 4 and 5.

Fix $\lambda_1 \ge 64$ large enough so that the right hand side of (2.20) is less than $\frac{1}{4}$. For $x, y \in \mathbb{B}$ and $k \in \mathbb{N}$, define the event

$$\begin{aligned} F_1(x,y,r,k) = \{&x, y \in \mathcal{G} \text{ and there exist at least } k \text{ disjoint balls} \\ &B(z, r/2) \text{ with } z \in \gamma(x,y) \text{ and which are } \lambda_1\text{–good.}\} \end{aligned}$$

For $x, y \in \mathbb{B}$, let z_0 be a middle point of $\gamma(x,y)$. Define the events

$$A_*(z,r,N) = \{z \in \mathcal{G} \text{ and } B(z,r) \text{ is } N\text{–good.}\},$$
$$F_*(x,y,R,k;r,N) = F_1(x,z_0,R,k/2) \cap F_1(z_0,y,R,k/2)$$
$$\cap A_*(x,r,N) \cap A_*(z_0,r,N) \cap A_*(y,r,N).$$

Definition 2.13. The vertex $x \in \mathbb{B}$ satisfies the condition $G_2(N,R)$ if:
(a) $x \in \mathcal{G}$,
(b) For every $z \in \partial B(x,NR)$ the event $F_1(x,z,R,\frac{1}{8}N)$ holds.

Proposition 2.14. *Let $x_0, y_0 \in \mathbb{B}$, and b be a possible backbone.*

(a) For $R \geq 1$, $N \geq 8$,

$$\mathbb{P}_{x_0,y_0,b}\big(x_0 \text{ satisfies the condition } G_2(N,R)\big) \geq 1 - c_1 \exp(-c_2 N).$$

(b) The same bounds as in (a) hold for the laws $\mathbb{P}_{x_0,b}$, $\mathbb{P}_{x_0,y_0}$, and $\mathbb{P}_{x_0}$.
(c) For $x_0, y_0 \in \mathbb{B}$, $8 \leq N < d(x_0,y_0)/8$, $r \geq 1$,

$$\mathbb{P}_{x_0,y_0,b}\Big(F_*(x_0,y_0,\frac{d(x_0,y_0)}{N},\tfrac{1}{8}N;r,N)\Big) \geq 1 - c_3 \exp(-c_4 N).$$

Proof. (a) We prove this for $y_0 = 0$; as in Proposition 2.10 the general case is handled by a similar argument.

Let
$$F_0(y,s) = \{y \in \mathcal{G} \text{ and } B(y,s) \text{ is } \lambda_1\text{–good.}\},$$
and write $v_i = a(x,i)$, $R' = RN/4$. We assume that $|x| \geq NR$ and $v_{R'}$ is on the backbone b: the other cases can be handled by minor modifications to the arguments below. Let w_0 be the highest level point in both b and $\gamma(0,x)$, and w_i, $i \geq 1$ be the backbone b from w_0 on.

Under $\mathbb{P}_{x,b}$ the events $F_0(v_{Rj},\frac{R}{2})$, $1 \leq j \leq N$ are independent, and $\mathbb{P}_{x,b}(F_0(v_{Rj},\frac{R}{2})^c) \leq \frac{1}{4}$. So standard exponential bounds give

$$\mathbb{P}_{x,b}(F_1(x,v_{R'},R,N/8)^c) \leq c \exp(-c'N). \tag{2.21}$$

Similarly
$$\mathbb{P}_{x,b}(F_1(w_0,w_{R'},R,N/8)^c) \leq c \exp(-c'N).$$

Now let $A_1 = \{v_i, 0 \leq i \leq R'\} \cup \{w_i, 0 \leq i \leq R'\}$; note that under $\mathbb{P}_{x,b}$ this set is non-random. Let

$$A_2 = \big\{y \in \mathbb{B} : a(y,R') \in A_1, \gamma(y,a(y,R')) \cap A_1 = \{a(y,R')\}\big\}.$$

For $y \in A_2$ let
$$H_1(y) = F_1(a(y,R),a(y,R'),R,N/8)^c,$$
$$H_2(y) = \{y \in \mathcal{G},\ D_{R'}(y) \neq \emptyset\}.$$

Then

$$\mathbb{P}_{x,b}(\bigcup_{y\in A_2} H_1(y)\cap H_2(y)) \le \sum_{y\in A_2} \mathbb{P}_{x,y,b}(H_1(y)\cap H_2(y))\mathbb{P}_{x,b}(y\in\mathcal{G}).$$

Under $\mathbb{P}_{x,y,b}$ the events $H_1(y)$ and $H_2(y)$ are independent, and as in (2.21) we obtain $\mathbb{P}_{x,y,b}(H_1(y)) \le c\exp(-c'N)$. So,

$$\begin{aligned}\mathbb{P}_{x,b}(\bigcup_{y\in A_2} H_1(y)\cap H_2(y)) &\le ce^{-c'N}\sum_{y\in A_2}\mathbb{P}_{x,y,b}(H_2(y))\mathbb{P}_{x,b}(y\in\mathcal{G})\\ &= ce^{-c'N}\sum_{y\in A_2}\mathbb{P}_{x,b}(H_2(y))\\ &= ce^{-c'N}\mathbb{E}_{x,b}\sum_{y\in A_2} 1_{H_2(y)}.\end{aligned}$$

The final sum above is bounded by a constant c' by the same argument as in Proposition 2.10.

Finally, we have

$$\begin{aligned}&\{\, G_2(N,R) \text{ fails for } x\} \subset\\ &F_1(x,v_{R'},R,N/8)^c \cup F_1(w_0,w_{R'},R,N/8)^c \cup \bigcup_{y\in A_2}(H_1(y)\cap H_2(y)),\end{aligned}$$

so combining the bounds above completes the proof. (b) follows on integrating the bounds in (a).

For (c), we first note that, by the argument for (2.21),

$$\mathbb{P}_{x,y,b}\big(F_1(x,y,\frac{d(x,y)}{N},\frac{1}{16}N)^c\big) \le c'\exp(-cN).$$

So, using Corollary 2.12, we have

$$\begin{aligned}\mathbb{P}_{x,y,b}\big(F_*^c\big) \le&\mathbb{P}_{x,y,b}\big(F_1(x,z_0,\frac{d(x,y)}{N},\frac{1}{16}N)^c\big) + \mathbb{P}_{x,y,b}\big(F_1(z_0,y,\frac{d(x,y)}{N},\frac{1}{16}N)^c\big)\\ &+\sum_{w=x,z_0,y}\mathbb{P}_{x,y,b}\big(A_*(w,r,N)^c\big)\\ \le& 2c'\exp(-cN) + 3c'\exp(-cN) = 5c'\exp(-cN).\end{aligned}$$

□

Definition 2.15. Let $x,y\in\mathbb{B}$, $m,\theta\in\mathbb{N}$. Define the condition $G_3(x,y,m,\kappa)$ as follows. Let $r = d(x,y)/m$, and let $z_0 = x, z_1,\dots,z_m = y$ be points on the path $\gamma(x,y)$ with $|d(z_{i-1},z_i) - r| \le 1$. (We choose these points in some fixed way – for example so that $d(z_{i-1},z_i)$ are non-decreasing.) For each $i = 1,\dots m$ let Θ_i be the smallest integer $\lambda \ge \max(64, 3c_{4.7.2}^{-1})$ such that $B(z_i,\lambda^{20}r)$ is λ–good, and $|B(z_i,r)| \ge r^2/\lambda^2$. Then $G_3(x,y,m,\kappa)$ holds if:

(a) $x,y\in\mathcal{G}$,

(b) $\sum_{i=1}^m \Theta_i^{54} \le \kappa m$.

Proposition 2.16. *For each backbone b and $x, y \in \mathbb{B}$*

$$\mathbb{P}_{x,y,b}\big(\ G_3(x,y,m,\kappa) \text{ holds}\ \big) \geq 1 - c_1\kappa^{-1}.$$

Proof. By Proposition 2.7 and Corollary 2.12, $\mathbb{P}_{x,y,b}(\Theta_i = k) \leq e^{-ck}$. Thus $\mathbb{E}_{x,y,b}\Theta_i^{54} \leq c'$, and so

$$\mathbb{P}_{x,y,b}\big(\ G_3(x,y,m,\kappa) \text{ fails}\ \big) = \mathbb{P}_{x,y,b}\big(\sum_{i=1}^m \Theta_i^{54} > \kappa m\big) \leq c'/\kappa.$$

□

3. Markov chains on weighted graphs and trees

Let Γ be a infinite connected locally finite graph. Assume that the graph Γ is endowed by a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if x and y are connected by a bond (in which case we write $x \sim y$). We call the pair (Γ, μ) a *weighted graph*. We can also regard it as an electrical network, in which the bond $\{x, y\}$ has conductance μ_{xy}. We will be mainly concerned with the case when $\mu_{xy} = 1$ if and only if $\{x, y\}$ is an edge: we call these the *natural weights* on Γ. Let $\mu_x = \sum_{y\in\Gamma} \mu_{xy}$ for each $x \in \Gamma$, and set $\mu(A) = \sum_{x\in A} \mu_x$ for each $A \subset \Gamma$, so that μ is then a measure on Γ.

We next define a quadratic form $\mathcal{E}$ on Γ by

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y\in\Gamma \\ x\sim y}} (f(x) - f(y))(g(x) - g(y))\mu_{xy},$$

and set

$$H^2 = H^2(\Gamma, \mu) = \{f \in \mathbb{R}^\Gamma : \mathcal{E}(f,f) < \infty\}.$$

For $f, g \in H^2$ we define $\mathcal{E}(f,g)$ by polarization. We sometimes abbreviate $\mathcal{E}(f,f)$ as $\mathcal{E}(f)$. Note that if $f = \min_{1\leq i\leq n} g_i$ then since

$$|f(x) - f(y)|^2 \leq \max_i |g_i(x) - g_i(y)|^2 \leq \sum_i |g_i(x) - g_i(y)|^2,$$

it follows that

$$\mathcal{E}(f,f) \leq \sum_{i=1}^n \mathcal{E}(g_i, g_i). \tag{3.1}$$

Let $Y = \{Y_t\}_{t\geq 0}$ be the continuous time random walk on Γ associated with $\mathcal{E}$ and the measure μ. When the natural weights are given on Γ, Y is called the simple random walk on Γ. Y is the Markov process with generator

$$\mathcal{L}f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy}(f(y) - f(x));$$

Y waits at x for an exponential mean 1 random time and then moves to a neighbour y of x with probability proportional to μ_{xy}. We define the transition density (heat kernel density) of Y with respect to μ by

$$q_t(x,y) = \mathbb{P}^x(Y_t = y)/\mu_y. \tag{3.2}$$

If $A \subset \Gamma$ we write

$$T_A = \inf\{t \ge 0 : Y_t \in A\}, \qquad \tau_A = T_{A^c}.$$

The natural metric on the graph, obtained by counting the number of steps in the shortest path between points, is written $d(x,y)$ for $x,y \in \Gamma$. As before, we write

$$B(x,r) = \{y : d(x,y) \le r\}, \quad V(x,r) = \mu(B(x,r)).$$

Let A, B be disjoint subsets of Γ. The effective resistance between A and B is defined by:

$$R(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{3.3}$$

Let $R(x,y) = R(\{x\},\{y\})$, and $R(x,x) = 0$. In general R is a metric on Γ – see [Kig] Section 2.3. If (Γ,μ) has natural weights then $R(x,y) \le d(x,y)$, and if in addition Γ is a tree then $R(x,y) = d(x,y)$.

The following is an easy consequence of (3.3).

Lemma 3.1. *For all $f \in \mathbb{R}^\Gamma$ and $x,y \in \Gamma$,*

$$|f(x) - f(y)|^2 \le R(x,y)\mathcal{E}(f,f). \tag{3.4}$$

Further, for each $x,y \in \Gamma$, there exists f so that the equality holds in (3.4).

We recall some basic properties of Green kernels. Let Y_t^B be the continuous time random walk on (Γ,μ) killed outside $B := B_R(x_0,r)$, and $q_t^B(x,y)$ be the transition density of Y_t^B. The Green kernel $g_B(x,y)$ of Y_t^B is defined by $g_B(x,y) = \int_0^\infty q_t^B(x,y)dt$. Then $g_B(\cdot,\cdot)$ has the reproducing property that

$$\mathcal{E}(g_B(x,\cdot),f) = f(x)$$

for all $f \in H^2$ such that $f|_{B^c} = 0$.

Using this and the fact that $e_{B,x}(y) := g_B(x,y)/g_B(x,x)$ is the equilibrium potential for $R(x,B^c)$, we have

$$R(x,B^c)^{-1} = \mathcal{E}(e_{B,x}, e_{B,x}) = g_B(x,x)^{-1},$$

so that

$$R(x,B^c) = g_B(x,x) = \int_0^\infty q_t^B(x,x)dt \qquad \forall x \in \Gamma, B \subset \Gamma. \tag{3.5}$$

4. Heat kernel estimates on graphs and trees

Recall that for $x \in \Gamma$ and $r \geq 0$, we denote $V(x,r) = \mu(B(x,r))$.

Theorem 4.1. *Let (Γ,μ) be a weighted graph and suppose that the edge weights satisfy $\mu_{xy} \geq 1$ for all x and y. Then*

$$q_{2rV(x,r)}(x,x) \leq \frac{2}{V(x,r)}, \quad x \in \Gamma,\, r > 0.$$

Remark. This is similar to the bound in Proposition 3.2 of [BCK], but has weaker hypotheses: in particular the bound on $q_t(x,x)$ only uses the volumes of the balls $V(x,R)$.

Proof. Fix $x_0 \in \Gamma$, write $B(r) = B(x_0,r)$ and $V(r) = V(x_0,r)$. Set $f_t(y) = q_t(x_0,y)$ and

$$\psi(t) = ||f_t||_2^2 = q_{2t}(x_0,x_0) = f_{2t}(x_0);$$

note that ψ is decreasing. Let $r > 0$; since

$$\sum_{y \in B(r)} f_t(y)\mu_y \leq 1,$$

there exists $y = y(t,r) \in B(r)$ with $f_t(y) \leq V(r)^{-1}$. Note that, since $\mu_e \geq 1$ for every edge e, it follows that $R(x,y) \leq d(x,y)$ for all x, y. Then by (3.4)

$$\begin{aligned}\tfrac{1}{2} f_t(x_0)^2 &\leq f_t(y)^2 + |f_t(x_0) - f_t(y)|^2 \\ &\leq \frac{1}{V(r)^2} + R(x_0,y)\mathcal{E}(f_t,f_t) \leq \frac{1}{V(r)^2} + r\mathcal{E}(f_t,f_t).\end{aligned}$$

Hence

$$\psi'(t) = -2\mathcal{E}(f_t,f_t) \leq \frac{2V(r)^{-2} - \psi(t/2)^2}{r}. \tag{4.1}$$

Since $-\psi(s/2) \leq -\psi(t)$ for $t \leq s \leq 2t$, integrating (4.1) from t to $2t$ we obtain

$$\psi(2t) - \psi(t) \leq 2tr^{-1}V(r)^{-2} - tr^{-1}\psi(t)^2.$$

So as $\psi(2t) > 0$,

$$tV(r)^2\psi(t)^2 \leq 2t + rV(r)^2\psi(t) \leq (4t) \vee (2rV(r)^2\psi(t)).$$

Hence

$$\psi(t) \leq \frac{2}{V(r)} \vee \frac{2r}{t}.$$

Taking r such that $t = rV(r)$ completes the proof. □

Corollary 4.2. *Let $V(x,r) \geq r^2/A$, and $t = r^3$. Then*

$$q_{2t}(x,x) \leq \frac{2(A \vee 1)}{r^2} = \frac{2(A \vee 1)}{t^{2/3}}. \tag{4.2}$$

Proof. Let $\lambda = r^{-2}V(x,r)$, so that $\lambda \geq A^{-1}$. Let $t_0 = rV(x,r) = \lambda r^3$. If $\lambda \leq 1$ then $t_0 \leq t$ and so Theorem 4.1 gives

$$q_{2t}(x,x) \leq q_{2t_0}(x,x) \leq \frac{2}{V(x,r)} = \frac{2}{\lambda r^2} = 2\lambda^{-1}t^{-2/3} \leq 2At^{-2/3}.$$

Now suppose that $\lambda \geq 1$. Let r' be such that $t = r'V(x,r')$; as $rV(x,r) = \lambda r^3 = \lambda t$, we have $r' \leq r$. So

$$q_{2t}(x,x) = q_{2r'V(r')}(x,x) \leq \frac{2}{V(x,r')} = \frac{2r'}{t} \leq \frac{2r}{t} = 2t^{-2/3} \leq 2(A \vee 1)t^{-2/3}.$$

□

Lemma 4.3. *Let $f_t(y) = q_t(x_0,y)$. Then*

$$\left| \frac{f_t(y)}{f_t(x_0)} - 1 \right|^2 \leq \frac{d(x_0,y)}{tf_t(x_0)}. \tag{4.3}$$

Proof. Let $e(t) = \mathcal{E}(f_t, f_t)$. Then e is decreasing, and

$$|f_t(x_0) - f_t(y)|^2 \leq d(x_0,y)e(t).$$

So as

$$\psi(t) - \psi(t/2) = -2\int_{t/2}^{t} e(s)ds,$$

we have

$$2e(t) \cdot t/2 \leq 2\int_{t/2}^{t} e(s)ds \leq \psi(t/2).$$

So,

$$|f_t(x_0) - f_t(y)|^2 \leq \frac{d(x_0,y)f_t(x_0)}{t},$$

and dividing by $f_t(x_0)^2$ completes the proof. □

Up to this point we have not needed to use the fact that Γ is a tree, but the following lemma relies strongly on this. From now on we take Γ to be a subgraph of $\mathbb{B}$, and define $M(x,r)$, and the conditions λ–good, $G_2(N,R)$ and $G_3(x,y,m,\kappa)$ as in Section 2.

Lemma 4.4. *Let $B = B(x_0, r)$, and $x \in B(x_0, r/8)$. Then*

$$\frac{r}{8M(x_0,r)} \le g_B(x,x) = R(x, B^c) \le 9r/8. \tag{4.4}$$

Proof. Since x is connected to $B(x,r)^c$ by a path of length $9r/8$, the upper bound is clear.

For the lower bound let $m = M(x_0, r)$ and $A = \{z_1, \dots, z_m\}$ be the set given in Definition 2.9: note that $d(x, z_i) \ge r/8$ for each i. Let h_i be the function on G such that $h_i(z_i) = 1, h_i(x) = 0$ and h_i is harmonic $G - \{x, z_i\}$. Then $h_i(y) = \mathbb{P}^y(T_{z_i} < T_x)$, and

$$\mathcal{E}(h_i, h_i) = R(x, z_i)^{-1} = d(x, z_i)^{-1} \le \frac{8}{r}.$$

If $y \in B(x,r)^c$ then since any path from y to x passes through A, we have $h_i(y) = 1$ for at least one i. So if $h = \max_i h_i$ then $h(x) = 0$ and $h = 1$ on $B(x,r)^c$. So, using (3.1),

$$R(x, B^c)^{-1} \le \mathcal{E}(h,h) \le m \max_i \mathcal{E}(h_i, h_i) \le \frac{8M(x_0,r)}{r},$$

proving the lower bound □

Lemma 4.5. *Let $B = B(x_0, r)$, $M = M(x_0, r)$.*
(a)

$$E^z \tau_B \le 2rV(x_0, r), \qquad z \in B(x_0, r). \tag{4.5}$$

(b)

$$E^x \tau_B \ge \frac{rV(x_0, r/(32M))}{32M}, \text{ for } x \in B(x_0, r/(32M)). \tag{4.6}$$

Proof. For any $z \in B$,

$$E^z \tau_B = \sum_{y \in B} g_B(z,y)\mu_y. \tag{4.7}$$

The upper bound follows easily from (4.7), since

$$\sum_{y\in B} g_B(z,y)\mu_y \le \sum_{y \in B} g_B(z,z)\mu_y = R(z, B^c)V(x,r) \le 2rV(x,r).$$

For the lower bound, let $x \in B(x_0, r/8)$, and set $p_B^x(y) = g_B(x,y)/g_B(x,x)$. Then $\mathcal{E}(p_B^x, p_B^x) = g_B(x,x)^{-1}$ and so

$$|1 - p_B^x(y)|^2 \le d(x,y)R(x,B^c)^{-1} \le d(x,y)(8M/r).$$

Let $B' = B(x_0, r/(32M))$. Then if $x, y \in B'$, $d(x,y) \le r/(16M)$ and so $p_B^x(y) \ge 1 - 2^{-1/2} \ge \frac{1}{4}$. So, using Lemma 4.4,

$$E^x \tau_B \ge \sum_{y \in B'} g_B(x,x)p_B^x(y) \ge \tfrac{1}{4}\mu(B')R(x,B^c) \ge r\mu(B')/(32M).$$

□

Proposition 4.6. *Let $r \geq 1$ and $x_0 \in \Gamma$, and $B = B(x_0, r)$. Write $M = M(x_0, r)$, $V = V(x_0, r)$ and let $V_1 = V_1(x_0, r) = V(x_0, r/(32M(x_0, r)))$. Then if $x \in B(x_0, r/(32M))$,*

$$P^x(\tau_B \leq t) \leq \Big(1 - \frac{V_1}{64MV}\Big) + \frac{t}{2rV}.$$

and

$$q_{2t}(x, x) \geq \frac{c_1 V_1(x_0, r)^2}{V(x_0, r)^3 M(x_0, r)^2} \qquad \text{for } t \leq \frac{rV_1(x_0, r)}{64M(x_0, r)}.$$

Proof. The proof is standard. By the Markov property,

$$\mathbb{E}^x[\tau_B] \leq t + \mathbb{E}^x[1_{\{\tau_B > t\}} E^{Y_t}(\tau_B)],$$

for all $t > 0$. Using this and Lemma 4.5,

$$\frac{rV_1}{32M} \leq t + P^x(\tau_B > t) 2rV,$$

and rearranging this we have

$$P^x(Y_t \in B) \geq P^x(\tau_B > t) \geq \frac{(rV_1/32M) - t}{2rV}. \tag{4.8}$$

This proves the first assertion.

By (4.8) if $t \leq rV_1/(64M)$ then

$$P^x(Y_t \in B) \geq \frac{c_2 V_1}{VM}.$$

By Chapman-Kolmogorov and Cauchy-Schwarz

$$P^x(Y_t \in B)^2 = (\sum_{y \in B} q_t(x, y)\mu_y)^2 \leq \mu(B) \sum_{y \in B} q_t(x, y)^2 \mu_y \leq q_{2t}(x, x)V.$$

So

$$q_{2t}(x, x) \geq V^{-1} P^x(Y_t \in B)^2 \geq \frac{c_2^2 V_1^2}{V^3 M^2}. \tag{4.9}$$

□

Theorem 4.7. *Suppose that $B = B(x_0, r)$ is λ–good for $\lambda \geq 1$, and let $I = I(\lambda, r) = [r^3\lambda^{-6}, r^3\lambda^{-5}]$.*
(a) For $x \in B(x_0, r/\lambda)$,

$$c_0 \frac{r^3}{\lambda^5} \leq E^x \tau_B \leq 2\lambda r^3. \tag{4.10}$$

(b) For each $K \geq 0$

$$q_{2t}(x_0, y) \leq (1 + \sqrt{K}) t^{-2/3} \lambda^3 \qquad \text{for } t \in I, \quad y \in B(x_0, Kt^{1/3}). \tag{4.11}$$

(c) *Let* $x \in B(x_0, r/\lambda)$. *Then*

$$q_{2t}(x,y) \geq c_1 t^{-2/3}\lambda^{-17}, \quad \text{if } d(x,y) \leq c_2\lambda^{-19}r, \quad t \in I. \tag{4.12}$$

Proof. (a) Let B, V, V_1, M be as in the previous proof. As $32M \leq 64M \leq \lambda$, $V_1 \geq V(x, r/\lambda) \geq r^2\lambda^{-4}$, while $V \leq \lambda r^2$. Thus (4.10) is immediate from Lemma 4.5.
(b) Let $t_1 = (r/\lambda^2)^3$. Then by Corollary 4.2 (taking $A = \lambda^2$), if $t \in I$,

$$q_{2t}(x_0,x_0) \leq q_{2t_1}(x_0,x_0) \leq 2\lambda^2 t_1^{-2/3} \leq 2\lambda^{8/3}t^{-2/3} \leq \lambda^3 t^{-2/3}. \tag{4.13}$$

Now, for $t \in I$ and $y \in B(x_0, Kt^{1/3})$, we have, using Lemma 4.3 and (4.13),

$$\begin{aligned} q_{2t}(x_0,y) &\leq q_{2t}(x_0,x_0) + |q_{2t}(x_0,y) - q_{2t}(x_0,x_0)| \\ &\leq q_{2t}(x_0,x_0) + \sqrt{\frac{K}{2t^{2/3}} q_{2t}(x_0,x_0)} \leq (1+\sqrt{K})t^{-2/3}\lambda^3, \end{aligned}$$

proving (4.11).
(c) Let $x \in B(x_0, r/\lambda) \subset B(x_0, r/(32M))$. Then $rV_1/(64M) \geq r^3\lambda^{-5}$, so for $t \in I$ by Proposition 4.6,

$$q_{2t}(x,x) \geq c_2 V_1^2/(V^3M^2) \geq c_2 r^{-2}\lambda^{-13} \geq c_2 t^{-2/3}\lambda^{-17},$$

where $c_2 = c_{4.6.1}$. Hence, by Lemma 4.3, if $d(x,y) \leq c_2\lambda^{-19}r$,

$$\left|\frac{q_{2t}(x,y)}{q_{2t}(x,x)} - 1\right|^2 \leq \frac{d(x,y)}{2tq_{2t}(x,x)} \leq \frac{d(x,y)r^2\lambda^{13}}{2c_2 t} \leq \frac{d(x,y)\lambda^{19}}{2c_2 r} \leq \tfrac{1}{2},$$

from which (4.12) follows. □

Corollary 4.8. *Let* $\lambda \geq 64$, *and* $B(x,r)$ *and* $B(x,\lambda^{-5}r)$ *be* λ*–good. Then*

$$E^x d(x, Y_t) \geq c_1\lambda^{-12}t^{1/3}, \quad \text{for } \frac{r^3}{\lambda^6} \leq t \leq \frac{r^3}{\lambda^5}.$$

Proof. Let $I = [r^3\lambda^{-6}, r^3\lambda^{-5}]$ and $B' = B(x, r\lambda^{-5})$. Let $t \in I$, and $y \in B'$. Then since $r \leq \lambda^2 t^{1/3}$, $d(x_0,y) \leq \lambda^{-5}r \leq \lambda^{-3}t^{1/3}$, so by (4.11) (with $K=1$) we have $q_{2t}(x_0,y) \leq 2t^{-2/3}\lambda^3$. Hence since B' is λ–good,

$$P^x(Y_{2t} \in B') = \sum_{y \in B'} q_{2t}(x_0,y)\mu_y \leq \mu(B')2t^{-2/3}\lambda^3 \leq 2\lambda^{-2} \leq \tfrac{1}{2}.$$

Thus

$$E^x d(x, Y_{2t}) \geq \lambda^{-5} r P^x(Y_{2t} \notin B') = \lambda^{-5}r(1 - P^x(Y_{2t} \in B')) \geq \tfrac{1}{2}r\lambda^{-5}.$$

□

Lemma 4.9. *Suppose x satisfies $G_2(N,R)$. Then*

$$P^x(\tau_{B(x,NR)} \le t) \le e^{-c_1 N} \quad \text{provided } N \ge c_2 t/R^3.$$

Proof. We use the argument of [BB1]. Let

$$A = \{y \in G : B(y, R/2) \text{ is } \lambda_1\text{–good}\}.$$

Define stopping times (T_i), (S_i) by taking $T_0 = \min\{t : Y_t \in A\}$, and

$$\begin{aligned} S_n &= \min\{t \ge T_{n-1} : Y_t \notin B(Y_{T_{n-1}}, R/2)\}, \\ T_n &= \min\{t \ge S_n : Y_t \in A\}. \end{aligned}$$

Since x satisfies $G_2(N,R)$ we have $T_{N/8} \le \tau_{B(x,NR)}$ P^x -a.s. Let $\xi_i = S_{i+1} - T_i$, $i \ge 1$. Then by Proposition 4.6 there exists $p = p(\lambda_1) < 1$ and $c_3 = c_3(\lambda) > 0$ such that

$$P^x\big(\xi_i \le s | \sigma(Y_u, 0 \le u \le T_i)\big) \le p + c_3 R^{-3} s. \tag{4.14}$$

Lemma 1.1 of [BB1] (see also Lemma 3.14 of [B1]) gives that, writing $a = c_3/R^3$, (4.14) implies that

$$\log P^x(\sum_{i=1}^{N/8} \xi_i \le t) \le -\tfrac{1}{8} N \log(1/p) + 2\Big(\frac{aNt}{8p}\Big)^{1/2}.$$

Substituting for a we deduce that

$$\log P^x(\tau_{B(x,NR)} \le t) \le -N\Big(2c_4 - c_5(t/(R^3N))^{1/2}\Big) \le -c_4 N,$$

provided $N \ge (c_5/c_4)^2 \cdot (t/R^3)$. □

Theorem 4.10. *Let $x, y \in \mathcal{G}$, $t > 0$ be such that $N := [\sqrt{d(x,y)^3/t}] \ge 8$ and suppose the event $F_*(x, y, d(x,y)N^{-1}, \frac{1}{8}N; d(x,y)^3 t^{-2/3}, N)$ holds. Then*

$$q_t(x,y) \le c_1 t^{-2/3} \exp(-c_2 N). \tag{4.15}$$

Proof. Define $T_{z_0} = \inf\{t : Y_t = z_0\}$ and $R = d(x,y)/N$, where z_0 is a middle point in $\gamma(x,y)$. Let G_x be the set of points w in $\mathcal{G}$ such that $\gamma(x,w)$ does not contain z_0, and let $G_y = \mathcal{G} - G_x$. Then, we have

$$\begin{aligned} q_t(x,y)\mu_x\mu_y &= \mu_x P^x(Y_t = y) \\ &= \mu_x P^x(Y_{t/2} \in G_y, Y_t = y) + \mu_x P^x(Y_{t/2} \in G_x, Y_t = y) \\ &= \mu_x P^x(Y_{t/2} \in G_y, Y_t = y) + \mu_y P^y(Y_{t/2} \in G_x, Y_t = x), \end{aligned} \tag{4.16}$$

where in the last line we used the μ–symmetry of Y. The two terms in (4.16) are bounded in the same way. For the first,

$$\begin{aligned} P^x(Y_{t/2} \in G_y, Y_t = y) &\le P^x(T_{z_0} \le t/2, Y_t = y) \\ &= E^x\big(1_{(T_{z_0} \le t/2)} P^{z_0}(Y_{t-T_{z_0}} = y)\big) \\ &\le P^x(T_{z_0} \le t/2) \sup_{t/2 \le s \le t} q_s(z_0, y)\mu_y. \\ &\le \mu_y \sqrt{q_{t/2}(y,y) q_{t/2}(z_0,z_0)} P^x(T_{z_0} \le t/2) \\ &\le \mu_y N^3 t^{-2/3} P^x(T_{z_0} \le t/2), \end{aligned}$$

where we used (4.11) with $\lambda = N, r = N^2 t^{1/3}$ in the last inequality. Now, $t/R^3 \sim (d(x,y)^3/t)^{1/2} \sim N$, so $N \ge ct/R^3$. Thus, by Lemma 4.9 we have

$$P^x(T_{z_0} \le t/2) \le e^{-cN} \quad \text{and} \quad P^y(T_{z_0} \le t/2) \le e^{-cN}.$$

Combining these facts

$$q_t(x,y) \le c' N^3 t^{-2/3} e^{-cN} \le c t^{-2/3} e^{-c''N},$$

which completes the proof. □

Theorem 4.11. *Let $x, y \in \mathcal{G}$, $m \ge 1$, $\kappa \ge 1$ and suppose $G_3(x,y,m,\kappa)$ holds. Then if $T = d(x,y)^3\kappa/m^2$*

$$q_{2T}(x,y) \ge c_1 T^{-2/3} e^{-c_2(\kappa + c_3)m}. \tag{4.17}$$

Proof. Let $r = d(x,y)/m$, and (z_i), (Θ_i) be the points and integers given by the condition $G_3(x,y,m,\kappa)$ in Definition 2.15. Let $B_i = B(z_i, \Theta_i^{20} r)$, and $B_i' = B(z_i, r)$. Applying (4.12) to B_i we deduce that if $d(y,y') \le c_{4.7.2}\theta^{-19}(\Theta_i^{20} r)$, and

$$\Theta_i^{54} r^3 \le t_i \le \Theta_i^{55} r^3, \tag{4.18}$$

then

$$q_{2t_i}(y,y') \ge c_4 t_i^{-2/3} \Theta_i^{-17}. \tag{4.19}$$

If $y_i \in B_i'$ then by the choice of Θ_i

$$d(y_{i-1}, y_i) \le 3r \le c_{4.7.2} \Theta_i^{-19} (\Theta_i^{20} r),$$

and so the bound in (4.19) holds for $q_{2t_i}(y,y')$. Therefore for $y_{i-1} \in B_{i-1}'$ and t_i satisfying (4.18),

$$\int_{B_i'} q_{2t_i}(y_{i-1}, y_i)\mu(dy_i) \ge c_4 t_i^{-2/3} \Theta_i^{-17} \mu(B_i') \ge c_4 \Theta_i^{-c_5};$$

we used here the fact that $\mu(B_i') \geq \Theta_i^{-2} r^2$. So if t_i satisfy (4.18), and $s = \sum t_i$ then since $\sum \log \Theta_i \leq \sum \Theta_i^{54} \leq m\kappa$,

$$
\begin{aligned}
q_{2s}(x,y) &\geq \int_{B_1'} \cdots \int_{B_{m-1}'} q_{2t_1}(x,y_1) q_{2t_1}(y_1,y_2) \dots q_{2t_m}(y_{m-1},y) \mu(dy_1) \dots \mu(dy_{m-1}) \\
&\geq (c t_m^{-2/3} \Theta_m^{-17}) c_4^{m-1} \Pi_{i=1}^{m-1} \Theta_i^{-c_5} \geq s^{-2/3} \exp(-c_6 m - c_5 \sum \log \Theta_i) \\
&\geq s^{-2/3} e^{-(c_5\kappa + c_6)m}.
\end{aligned}
$$

As $G_3(x,y,m,\kappa)$ holds we have $r^3 \sum \Theta_i^{54} \leq m\kappa r^3 = T$. If $T \leq r^3 \sum \Theta_i^{55}$ we can choose (t_i) satisfying (4.18) so that $s = T$. If not, let $s' = T - s$, so that $s' \leq m\kappa r^3$. Fix a j such that Θ_j is minimal and in the chaining argument above add m' extra steps (of time length t' satisfying (4.18) for $i = j$) between B_{j-1}' and B_j'. Since $c_7^{54} \leq \Theta_j^{54} \leq \kappa$, we have $c_8 r^3 \leq t' \leq \kappa r^3$. Then choose m', t' so that $m't' + s = T$; we have $m' \leq cm$. Each extra step gives a factor of $c_4 \Theta_j^{-c_5}$ in the lower bound in the chaining argument, so the total contribution multiplies the lower bound by a number greater than $e^{-c(\kappa + c')m}$. Thus (4.17) holds. □

5. Random walk on the conditioned critical GW-branching precess

In this section, we state and prove our main results on the random walk on the IIC. As in Section 2 we write $\mathcal{G}$ for the IIC on $\mathbb{B}$, and $\mathbb{P}$ for its law. Let $Y = \{Y_t\}_{t \geq 0}$ be the simple random walk on $\mathcal{G}(\omega)$ defined in Section 3; we write E_ω^x for its law of Y started at x. Let $q_t^\omega(x,y)$ be the transition density of Y.

Proof of Theorem 1.2. Fix $x \in \mathbb{B}$, and let $c_3 = c_{2.12.2}$. Let $a = 2/c_3$ and $\lambda_n = e + a \log n$, and r_n satisfy $r_n^3 \lambda_n^{-6} = e^n$. Let F_n be the event that $B(x, r_n)$ is λ_n–good. Then by Corollary 2.12

$$\mathbb{P}(F_n^c) \leq c e^{-c_3 a \log n} = c' n^{-2},$$

so by Borel-Cantelli F_n^c occurs for only finitely many n, $\mathbb{P}$–a.s. Let N be the largest m such that F_m^c occurs; then

$$\mathbb{P}(N > m) \leq \sum_{m+1}^{\infty} \mathbb{P}(F_n^c) \leq c m^{-1}.$$

Set $S(x) = e^N$. For $n \geq (\log S(x)) + 1$ we have, by (4.11) and (4.12),

$$c' t^{-2/3} \lambda_n^{-17} \leq q_{2t}(x,x) \leq c'' t^{-2/3} \lambda_n^3 \tag{5.1}$$

for $e^n \leq t \leq \lambda_n e^n$. Let $n(t)$ be the unique integer such that $\log t \in [n(t) - 1, n(t))$. Hence, if $t \geq S(x)$, $n(t) > N$ and so (5.1) holds for $n = n(t)$. Since

$$\lambda_{n(t)} = e + a \log n(t) \sim a \log \log t,$$

we obtain (5.1). □

While the powers of the terms in $\log \log t$ given in Theorem 1.2 are not the best possible, we do have oscillations in $t^{-2/3} q_t^\omega(.,.)$ of that order.

Lemma 5.1.
$$\liminf_{t\to\infty}(\log\log t)^{1/6}t^{2/3}q^{\omega}_{2t}(0,0)\le 2,\quad P^0_\omega - a.s. \tag{5.2}$$

Proof. Define a_n by $V(0,2^n)=a_n 2^{2n}$, and let $t_n=2^nV(0,2^n)=a_n2^{3n}$. Then by Theorem 4.1,
$$q^{\omega}_{2t_n}(0,0)\le \frac{2}{V(0,2^n)}=\frac{2t_n^{-2/3}}{a_n^{1/3}}.$$
By Proposition 2.8(a), $a_n>(\log n)^{1/2}$ for infinitely many n, a.s., giving (5.2). □

Proof of Theorem 1.3. (a) The lower bound in (1.4) is an immediate consequence of Corollaries 2.12 and 4.8. For the upper bound, let $Z_t=\sup_{0\le s\le t}d(x,Y_s)$, $R=t^{1/3}$ and $T_M=\tau_{B(x,MR)}$. Let $K_t(x)(\omega)$ be the largest n such that x does not satisfy $G_2(n,R)$. Then by Proposition 2.14
$$\mathbb{P}_x(K_t(x)\ge k)\le\sum_{l=k}^{\infty}\mathbb{P}_x(x\text{ does not satisfy }G_2(l,R))\le c'e^{-ck}. \tag{5.3}$$

Then $\{Z_t\ge nR\}\subset\{T_n\le t\}$, and so by Lemma 4.9,
$$\begin{aligned}E^x_\omega Z_t&\le R\sum_{n=0}^{\infty}P^x_\omega(T_n\le t)\\&\le R\Big(1+K_t(x)+\sum_{n=K_t(x)+1}^{\infty}P^x_\omega(T_n\le t)\Big)\\&\le R\Big(1+K_t(x)+\sum_{n=K_t(x)+1}^{\infty}e^{-cn}\Big)\le R(c+K_t(x)).\end{aligned} \tag{5.4}$$

Since $\mathbb{E}_xK_t(x)\le c'$ this completes the proof.
(b) Let $m(t)=\lfloor t\rfloor$; Since
$$|E^x_\omega d(x,Y_t)-E^x_\omega d(x,Y_{m(t)})|\le E^x_\omega d(Y_{m(t)},Y_t)\le c,$$
it is enough to prove (1.5) for integer t. Using (5.3) and Borel-Cantelli there exists c' such that
$$\mathbb{P}_x(K_n(x)>c'\log n\ \text{ i.o.})=0.$$
and so by (5.4)
$$E^x_\omega d(x,Y_n)\le c''n^{1/3}\log n$$
for all sufficiently large n. The lower bound in (1.5) follows from Corollary 4.8 by the same argument as in Theorem 1.2. □

Proof of Theorem 1.4. We begin with the on-diagonal case $x=y$. Let $\lambda_n=n$ and r_n be defined by $2r_n^3/\lambda_n^6=t$. Let $F_n=\{B(x,r_n)\text{ is }\lambda_n\text{–good }\}$, and $N(\omega)=\min\{n:\omega\in F_n\}$.

By Corollary 2.12 $\mathbb{P}_x(N > n) \le \mathbb{P}_x(F_n^c) \le e^{-cn}$. On F_n we have, by (4.11), $q_t^\omega(x,x) \le ct^{-2/3}n^3$, so

$$\mathbb{E}_x[q_t^\omega(x,x)] \le ct^{-2/3}\mathbb{E}_x N^3 \le c' t^{-2/3}, \tag{5.5}$$

proving the on-diagonal upper bound in (1.6).

For the on-diagonal lower bound choose m_0 such that $\mathbb{P}_x(F_{m_0}) \ge \frac{1}{2}$ and then on F_{m_0}, by the lower bound in (4.12),

$$q_t^\omega(x,x) \ge ct^{-2/3}m_0^{-17}.$$

For the off-diagonal bounds, when $d(x,y) \le 64t^{1/3}$, (1.6) can be proved similarly to (5.5) using Theorem 4.7(b). So we will assume $d(x,y) > 64t^{1/3}$. Now, let $N := [\sqrt{d(x,y)^3/t}] \ge 8$ and define $F_0 = F_*(x,y,d(x,y)N^{-1}, \frac{1}{8}N; d(x,y)^3 t^{-2/3}, N)$. Let $\lambda_0 = N$ and define $\lambda_n = N+n-1$ for $n \ge 1$. For each $n \ge 1$, set $r_n = t^{1/3}\lambda_n^2$ and let $F_n = \{B(x,r_n)$ is λ_n-good $\}$. Then, $\mathbb{P}_{x,b}(F_n^c) \le e^{-c\lambda_n}$. We now apply Theorem 4.7 (b) with $K = \lambda_n^2$ and obtain the following. (Note that we can apply the theorem because $d(x,y)/t^{1/3} \le cN^{2/3} \le c\lambda_n^2$.)

$$q_{2t}(x,y) \le c(1+\sqrt{\lambda_n^2})t^{-2/3}\lambda_n^3 \le c' t^{-2/3}\lambda_n^4. \tag{5.6}$$

Let $M(\omega) = \min\{n \ge 0 : \omega \in F_n\}$. Then, $\mathbb{P}_x(M = 0) = \mathbb{P}_x(F_0) \ge 1 - 4e^{-N}$ and $\mathbb{P}_x(M > n) \le \mathbb{P}_x(F_n^c) \le ce^{-c'\lambda_n}$. Thus, using Theorem 4.10 and (5.6), we obtain

$$\begin{aligned}\mathbb{E}_{x,y}[q_t^\omega(x,y)] =& \mathbb{E}_{x,y}[q_t^\omega(x,y) : M = 0] + \mathbb{E}_{x,y}[q_t^\omega(x,y) : M > 0] \\ \le& ct^{-2/3}\exp(-c'N) + c'' t^{-2/3}\mathbb{E}[\lambda_M^4 : M > 0].\end{aligned}$$

Since $\mathbb{E}[\lambda_M^4 : M > 0] \le c\sum_{k=1}^\infty (N+k-1)^4 e^{-c'(N+k-1)} \le ce^{-c''N}$, we obtain (1.6).

We next prove (b). Choose $\kappa = 2c_{2.16.1}$, so that $\mathbb{P}_{x,y}(G_3(x,y,m,\kappa)$ holds $) \ge \frac{1}{2}$. Now choose $m = (R^3\kappa/t)^{1/2}$; by Theorem 4.11, for ω such that $G_3(x,y,m,\kappa)$ holds,

$$q_{2t}^\omega(x,y) \ge ct^{-2/3}\exp(-c'(\kappa + c'')m).$$

Taking expectations gives (1.7). □

Let

$$\widetilde{Z}_t^{(n)} = n^{-1/3} d(0, Y_{nt}), \quad t \ge 0.$$

By Theorem 1.3(a) the process $\widetilde{Z}^{(n)}$ is tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times P_\omega^0$. (See Theorem 1.1 for the analogous result for the discrete time simple random walk.)

Proof of Theorem 1.5. Let $U_n = \sup_{0\le s\le 1} Z_s^{(n)}$. Then, by (4.5),

$$\begin{aligned}P_\omega^0(U_n \le \lambda) =& P_\omega^0(\sup_{t\le n} d(0,Y_s) \le \lambda n^{1/3}) \\ =& P_\omega^0(\tau_{B(0,\lambda n^{1/3})} \ge n) \le \frac{2\lambda n^{1/3} V(0,\lambda n^{1/3})}{n}.\end{aligned}$$

So by Proposition 2.8(b), we have, for any $\lambda > 0$, that $\liminf_{n\to\infty} P_\omega^0(U_n \leq \lambda) = 0$, which shows that the r.v. U_n (and hence the processes $Z^{(n)}$) are not tight. $\square$

Remark. This result illustrates the difference in the type of results that can arise between the quenched and annealed cases. For the case of supercritical bond percolation in $\mathbb{Z}^d$, while an invariance principle was proved in the annealed case in [DFGW] in 1989, the quenched case (for $d \geq 4$) was only proved recently in [SS].

Acknowledgment. The authors thank Antal Járai, Harry Kesten and Gordon Slade for valuable comments.

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Version 1.00, 4 March 2005

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